Path Partitions, Cycle Covers and Integer Decomposition

(Lecture Note)

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Abstract. A polyhedron P has the *integer decomposition property*, if every integer vector in kP is the sum of k integer vectors in P. We explain that the projections of polyhedra defined by totally unimodular constraint matrices have the integer decomposition property, in order to deduce the same property for coflow polyhedra defined by Cameron and Edmonds. We then apply this result to the convex hull of particular stable sets in graphs. Therebye we prove a generalization of Greene and Kleitman's well-known theorem on posets to arbitrary digraphs which implies recent and classical purely graph theoretical results on cycle covers, is closely related to conjectures of Berge and Linial on path partitions, and implies these for some particular values of the parameters.

1 Introduction

Partitioning the vertex-set of a graph by a minimum number of paths is one of the most natural problems concerning graphs. Minimizing the number of paths in such a partition contains the Hamiltonian Path problems both in the directed and undirected case.

For undirected graphs some variants involve matching theory some others the connectivity of graphs. Some results concern only particular classes of graphs. The only general result about minimum partitions of the vertex-set into paths in undirected graphs concerns intersection graphs of paths in a tree, by Monma and Wei [27], a class later generalized in [19], [20].

For digraphs, a classical theorem of Gallai and Milgram *relates the problem to the stability number of a graph.* Path partitions in digraphs have been treated both with elegant graph theory, network flows, and polyhedral combinatorics, but have not yet revealed all of their secrets:

Conjectures of Berge [4] and Linial [26] about the relation of maximum sets of vertices inducing a k-chromatic subgraph and particular path partitions resist

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through the decades. Hartman's excellent survey [22] witnesses of the variety of the methods that have been tried out with a lot of partial results but no breakthrough as far as the general conjectures are concerned. Some other conjectures are less well-known or have not yet been stated.

Led by analogies, we ask and answer in this talk more questions, and point at some connections.

Section 2 states analogous pairs of theorems on path partitions and cycle covers.

Section 3 presents the results concerning cycle covers deducing all from a general theorem proved with the help of the property of a corresponding polyhedron: the integer decomposition property.

Section 4 presents some results on path partitions, and some connections of these to cycle covers.

Notation and Terminology: Let $G = (V, E)$ a digraph. A *path* of a digraph is an ordered set $P = (v_1, \ldots, v_{|P|})$ of vertices, all different, so that $v_i v_{i+1} \in E$ $(i = 1, \ldots, |P| - 1)$. We will denote $\mathrm{ini}(P) := v_1$ the *initial* (first) vertex of a path and $\mathrm{ter}(P) := v_{|P|}$ the *terminal* (last) vertex of it. For us a path will be a vertex-set, that is, with an abuse of notation we will apply set-operations involving a path P, and in this case P is just the set of its vertices. If the first and last points are equal it is called a *cycle* which also included one element sets (even if there is no incident loop). A *subpartition* is just a family of disjoint subsets of V.

For a family of sets $\mathcal{P}$, $R(\mathcal{P}) := V \setminus \cup\mathcal{P}$. This complementation concerns the complement *with respect to the graph in which it is defined.* If we delete some vertices and the new vertex set is V', for a subpartition $\mathcal{P}'$ of V', $R(\mathcal{P}')$ is defined as $R(\mathcal{P}') := V' \setminus \cup\mathcal{P}'$. We apply this notation only when the vertex-set to which we apply it is clear. A subpartition of paths is a *path partition* if and only if $R(\mathcal{P}) = \emptyset$.

If we do not say otherwise, $G = (V, E)$ is a digraph, $n := |V|$ and $m := |E|$.

A *stable* set is a subset of vertices that does not induce any edge. The maximum size of a stable set of a graph G is denoted by $\alpha = \alpha(G)$. The maximum size of a k-chromatic induced subgraph (equivalently, the union of k stable sets) is denoted by α_k; $\alpha_1 = \alpha$. The *chromatic number*, denoted by $\chi = \chi(G)$, is the minimum of k such that $\alpha_k = n$. The minimum size of a partition into paths is denoted by $\pi = \pi(G)$, and the minimum cover by cycles is $\zeta = \zeta(G)$. By convention $\{v\}$ is also a cycle for all $v \in V$ (and it is of course a path too). So $\pi, \zeta \leq n$. The number of vertices of the longest path is denoted by $\lambda = \lambda(G)$.

Two families of sets are called *orthogonal* if taking any set of each, the intersection is always 1. A family is said to *cover* a set, if the union of its members contains the set.

The subgraph induced by a set $X \subseteq V$ will be denoted by $G(X)$, just replaced by X to avoid double parentheses, for instance $\alpha(X) := \alpha(G(X))$; "strongly connected" will sometimes be replaced by *strong*.

2 Pairs of Assertions

2.1 Tournaments ($\alpha = 1$)

A *tournament* is an oriented complete graph.

Theorem 1 (Rédei [28]). *Let G be a tournament. Then it has a Hamiltonian Path.*

Theorem 2 (Camion [12]). *Let G be a strong tournament. Then it has a Hamiltonian Cycle.*

We state "loose" and "tight" versions of some assertions. The former refers to inequalities that generalize the nontrivial inequalities of minmax theorems (of Dilworth's, of Greene-Kleitman's or of some more recent ones), and the latter generalize "complementary slackness" (the sstructure implied by the equality in these inequalities).

2.2 Stability ($k = 1$, loose)

Theorem 3 (Gallai, Milgram [17]). *Let G be an arbitrary digraph. Then $\alpha \geq \pi$.*

Theorem 4 (Bessy, Thomassé [7], Gallai's conjecture [16]). *Let G be strong . Then $\alpha \geq \zeta$.*

Specializing these to acyclic transitive digraphs both imply the nontrivial part of Dilworth's theorem stating equality in the former theorem for acyclic transitive digraphs (posets), see for instance [33]. To deduce it from the latter theorem, we first have to make an acyclic transitive digraph strongly connected. This can be done for instance by adding a "supersource" and joining it to all the vertices of 0 indegree (sources), adding a "supersink" and joining all the vertices of 0 outdegree (sinks) to it, and adding an arc from the supersink to the supersource. (Just one vertex joined to and from all vertices is also a possible choice.)

2.3 Stability ($k = 1$, tight)

Any proof of the Gallai-Milgram theorem obviously provides the following:

Theorem 5. *Let G be an arbitrary digraph. Then there exists a path partition $\mathcal{P}$ of G and a stable set orthogonal to $\mathcal{P}$.*

Since Theorem 4 is the weakening of a min-max theorem, the condition of equality ("complementary slackness") easily implies:

Theorem 6. *Let G be a strong digraph. Then there exists a cycle cover $\mathcal{C}$ of G and a stable set S orthogonal to $\mathcal{C}$, where each element of S is covered by exactly one member of $\mathcal{C}$.*

Of course these statements can also be specialized to Dilworth's theorem, with the same reduction as before.

2.4 Coloring ($k = \lambda$, loose)

Theorem 7 (Gallai, Roy [18], [29]). *Let G be an arbitrary digraph. Then there exists a path of size at least χ.*

Theorem 8 (Bondy [8]). *Let G be a strong digraph. Then there exists a cycle of size at least χ.*

2.5 Coloring ($k = \lambda$, tight)

Any proof of the Gallai-Roy theorem obviously provides the following:

Theorem 9. *[8] Let G be an arbitrary digraph. Then for any longest path there exists a colouring whose color classes are orthogonal to the path.*

Bondy's theorem does correspond to an LP duality theorem, but the vertices are not stable sets and cycles ; the corresponding polyhedron does have fractional vertices [31], and it is not evident how it would imply an analogous theorem for the longest cycle in a strongly connected graph. Nevertheless, this tight version of Bondy's theorem is true, and this is actually what Bondy proved:

Theorem 10. *[8] Let G be a strong digraph. Then there exists a cycle and a colouring so that the color classes are orthogonal to the cycle.*

Here is a somewhat different structural sharpening of the Gallai-Roy theorem:

Conjecture 1 (Laborde, Payan, Xuong [25]). *Let G be an arbitrary digraph. Then there exists a stable set in G that meets every longest path.*

Could this be true replacing "longest path" by "longest cycle" in strongly connected graphs ?

2.6 General (Loose)

Conjecture 2 (Linial [26]). *Let G be a digraph. Then $\alpha_k \geq \min_{X \subseteq V}\{|X| + k\pi(V \setminus X)\}$.*

Theorem 11 (Sebő [31]). *Let G be a strong digraph. Then $\alpha_k \geq \min_{X \subseteq V}\{|X| + k\zeta(V \setminus X)\}$.*

Corollary 1 *(*Greene-Kleitman [21]). Let G be a transitive acyclic digraph. Then $\alpha_k = \min_{X \subseteq V}\{|X| + k\zeta(V \setminus X)\}$.

Indeed, for transitive acyclic digraphs " $\leq$ " is easy, and to prove the nontrivial inequality of the Greene-Kleitman theorem, the reduction of Subsection 2.2 to strongly connected graphs works again. So the corollary indeed follows from the preceding theorem. Note that the right hand side of the Greene-Kleitman theorem or of Linial's conjecture is usually written as $\min\{\sum_{P \in \mathcal{P}} \min\{|P|, k\}$: $\mathcal{P}$ *is a path partition*$\}$, and this sum is called the *k-norm* of $\mathcal{P}$.

2.7 General (Tight)

The following is a simple already unknown version of Berge's conjecture. It implies Linial's conjecture. (The two conjectures have been stated independently.)

Conjecture 3 (Berge [4]). *Let G be a digraph, and $k \in \mathbb{N}$, $k \geq 1$. Then there exists $X \subseteq V$, a path partition $\mathcal{P}$ of $V \setminus X$, and k disjoint stable sets orthogonal to $\mathcal{P}$ whose union contains X.*

We prove now the cycle cover version of this conjecture. Note that there are several options here for replacing "there exists" by "for all". Berge originally stated *for all $X \subseteq V$, and path partition $\mathcal{P}$ of $V \setminus X$ minimizing $|X| + k|\mathcal{P}|$*. We will call this *strongest conjecture* since it implies the above weaker assertion which implies in turn Linial's conjecture.

Theorem 12 (Sebő [31]). *Let G be a strong digraph, and $k \in \mathbb{N}$, $k \geq 1$. Then there exists $X \subseteq V$, a cycle cover $\mathcal{C}$ of $V \setminus X$, and k disjoint stable sets covering X, all orthogonal to $\mathcal{C}$. Furthermore, each element of the stable sets is covered by at most one member of $\mathcal{C}$.*

This last theorem easily implies all that has been previously stated about covers in strongly connected graphs:

It shows k stable sets whose union U satisfies $|U| = |X| + k|\mathcal{C}|$ for some $X \subseteq V$ and cycle cover $\mathcal{C}$ of $V \setminus X$. Theorem 11 follows since $\alpha_k \geq |U|$. Theorem 12 and Theorem 11 are central in our presentation. We show here how they can be proved through the integer decomposition property of coflow polyhedra, and how they can be useful.

Theorem 8 follows from Theorem 12 because choosing k to be the size of the longest cycle, $|X| + k|\mathcal{C}| \geq |X| + |\cup\mathcal{C}| \geq n$, so the union of the k disjoint stable sets provided by Theorem 12 is at least n. So G can be colored with k colors.

The $k = 1$ special case of Theorem 12 is Theorem 6, itself implying Theorem 4. Indeed, in this case the elements in X can be replaced by 1-element cycles.

Theorem 12 will, in turn, be proved in Section 3.3.

3 Cycle Covers

The ultimate goal of this section is to prove the second theorem of each subsection of Section 2. They have already been proved from Theorem 12, and here we will prove this latter. There are some interesting tools on the way, and they will lead us further: the integrality and integer decomposition property of *coflow* polyhedra, and the *coherent orders* of Knuth, Bessy and Thomassé. For the notations and basic notions from polyhedral combinatorics (including TDI, integer decomposition, etc.) we refer in this extended abstract to [32], [33]. The talk will be self-contained.

3.1 Coflows and Integer Decomposition

We wish to introduce here a ready to use helpful treatment of node-capacitated circulation problems. The idea is well-known: node-capacities can be reduced to edge-capacities by splitting each vertex v into two copies, an in-copy v_{in} and an out-copy v_{out} and adding the arc $v_{\text{in}}v_{\text{out}}$ with the given vertex-capacities (possibly lower and upper), see [32], [11]. It is less well-known that relevant cycle-cover or cycle packing problems arise in this way and have useful properties, such as box total dual integrality, primal integrality if the parameters are integers [11], furthermore integer decomposition (below). This elegant tool defined by Cameron and Edmonds is defined as follows:

The *coflow system of inequalities* $\mathcal{Q}(G,a,b,c)$, where $G=(V,E)$ is a digraph, $a,b : V(G) \to \mathbb{Z}$, $c : E \to \mathbb{Z}$ is the following system in $n := |V|$ variables x_v $(v \in V)$:

$$x(V_C) \leq c(E_C) \text{ for every cycle } C \text{ with vertex-set } V_C \text{ and edge-set } E_C\ ,$$

$$a \leq x \leq b.$$

The set of points $x \in \mathbb{R}^V$ satisfying the coflow inequalities $\mathcal{Q}(G,a,b,c)$ is called the *coflow polyhedron*, and is denoted by $Q(G,a,b,c)$. The *coflow (primal) problem* $P(G,a,b,c,w)$, where G,a,b,c are as before, and $w : V(G) \to \mathbb{Z}$, is the following:

$$\max\{w^\top x : x \in Q(G,a,b,c)\}$$

$D(G,a,b,c,w)$ will denote the dual linear program, and $\text{opt}(G,a,b,c,w)$ the common optimum of the primal and the dual (which can also be infinite).

In [31] the coflow approach is followed by applying the splitting of vertices case by case, without stating any general theorem. (Coflows have been so far absent from books and surveys - we hope to contribute to their inclusion.) Both in [11] and [31], primal integrality is deduced from the TDI property (the above Lemma) through Edmonds and Giles' theorem. However, [11] observes that primal integrality can be directly deduced proving that the coflow polyhedron is the projection of the dual of a circulation problem, and in [31] the dual of the stated flow problem provides an integer primal solution to coflows.

We express all this in a slightly simpler way using Charbit's matrix from [13], which is smaller and simpler than the network matrices from [11] (or [31]) :

Given a digraph $G=(V,E)$, $n := |V|$, $m := |E|$, let A denote the $2n \times m$ matrix whose first n rows consist of the usual incidence matrix of G (one $+1$ and one -1 per column, the rest is 0) and the second m rows the same, except that the $+1$ are replaced by 0. It is easy to see that this matrix A is totally unimodular (as a submatrix of a network matrix, or see [13] end of Section 3.2).

Lemma 1. *The coflow polyhedron $Q(G,a,b,c)$ can be written as*

$$Q(G,a,b,c) = \{(y_{n+1},\dots,y_{2n}) : y \in \mathbb{R}^{2n}, yA \leq c\}. \tag{1}$$

Proof. (see [13] proof of Theorem 3.3 Claim 2, but our context here is simpler.) Indeed, $(y_1, \dots, y_n)$ is a potential for the weight function $c_e - \sum\{y_{n+i}:$ vertex i is the tail of edge e$\}$. However, a potential exists if and only if there is no negative cycle, that is, along every negative cycle the sum of the y_{n+i} does not exceed the sum of the c_e on the same cycle. □

Lemma 2 *(Coflow Theorem [9],[11]). Any system of coflow inequalities $\mathcal{Q}(G, a, b, c)$ is TDI.*

Proof. First proof: For every $w : V(G) \to \mathbb{Z}$ the dual problem $D(G, a, b, c, w)$ is a problem of covering vertices by cycles which is a flow problem with w as lower capacities on the vertices. Second proof: The dual solutions are the same as the primal solutions of the LP with TU coefficient matrix in the preceding lemma. □

Surprisingly, these two three-line proofs are sufficient for getting our general graph theory results in a straightforward way.

Lemma 3 *(Baum, Trotter [3]). If A is totally unimodular, $\{y : yA \le c\}$ has the integer decomposition property.*

Proof. Indeed, suppose $\bar{y}A \le kc$. We have to show an integer vector y_k, $y_kA \le c$ for which

$$(\bar{y} - y_k)A \le (k-1)c.$$

Then the statement follows by induction on k. We have to find a linear solution to

$$\bar{y}A - (k-1)c \le y_kA \le c,$$

where $\bar{y}$, A, k, c are fixed and the entries of y_k are the variables. This system of linear inequalities has a solution, since $y_k := (1/k)\bar{y}$ is a solution. Since A is unimodular, then it also has an integer solution, and the claim is proved. □

We mimic now the same proof once more for handling projections:

Lemma 4. *If $Q \subseteq \mathbb{R}^{m'}$, $Q = \{y : yA \le c\}$ where A is totally unimodular, and $m < m'$, then*

$$P := \{(y_1, \dots, y_m) : y \in Q\}$$

has the integer decomposition property.

Proof. Let $x \in kP \cap \mathbb{Z}^m$, that is, $x/k \in P$. By definition there exists $(x_{m+1}, \dots,$ $x_{m'}$, so that $(x/k, x_{m+1}, \dots, x_{m'}) \in Q$. So So $(x, kx_{m+1}, ...kx_{m'}) \in kQ$. It is sufficient to show that $x' := (kx_{m+1}, ...kx_{m'})$ can be chosen to be integer, because then by the integer decomposition property of Q:

$$(x, x') = y^1 + \ldots + y^k,\ y^i \in Q, \text{ and } y^i \text{ is an integer vector},$$

and letting x^i be the vector formed by the first m entries of y^i, we get

$$x = x^1 + \ldots + x^k,\ x^i \in P \cap \mathbb{Z}^m \quad (i = 1, \dots k).$$

Now x' is a feasible solution of the equation $xB + x'A' \leq c$, where B is the matrix formed by the first n rows of A, and where A' is the rest of A. Since A is totally unimodular, A' is also totally unimodular, so the equation

$$x'A' \leq kc - xB,$$

– where k, c, x, B are fixed, all integer, and x' is variable –, also has an integer solution x' (it does have a feasible solution x', A' is totally unimodular, and the right hand side is integer). □

In [31] flows are applied in each special case separatly: c takes there only two different values in all of these– 0 and $k \in \mathbb{N}$, which makes the proofs and algorithms simpler. *The integer decomposition property of $Q(G,a,b,c)$ is also proved for a special case, and turns out to be crucial for proving Theorem 11 and 12.*

Integer decomposition makes possible the inclusion of *unions of vertices of* $0-1$ *coflow polyhedra, establishing that these also form coflow polyhedra, analogously with a similar matroid property.*

It is unfortunate that these special cases were proved one by one in [31], [13], without knowing about coflows. Several colleagues advised a similar unified treatment – Attila Bernáth made a very concrete suggestion. Then I learned about coflows from Irith Hartman, but the nontrivial graph theoretic proof of the integer decomposition property in particular cases – where the tree is hiding the forest – persisted. I have realized Lemma 4 only recently, and that proving this property for coflows in general provides a much simpler proof of Theorem 15 than the proof we followed in [31]:

Theorem 13 *(Coflow ID)***.** *Coflow polyhedra have the integer decomposition property.*

Proof. By Lemma 3, coflow polyhedra are of the form of the condition of Lemma 4, therefore, by this latter Lemma, they have the integer decomposition property. □

3.2 Coherence

Graph theory courses characterize strongly connected graphs with the existence of an "ear decomposition" [33, Theorem 6.9]. Knuth's characterization is then at hand, and provides considerably more information:

Theorem 14 *(Knuth)***.** *Let $G = (V,E)$ be a strong digraph. Then for every $v \in V$ there exists an order $v_1, \ldots, v_n$ on V such that $v_1 = v$, and*

(i) Every $e \in E$ is contained in a cycle C with at most one backward arc.
(ii) Every $v \in V$ can be reached from v_1 using only forward arcs.

A *backward arc* (with respect to a given order of the vertices) is an arc $v_iv_j \in E$, $i > j$, the other arcs are *forward arcs.*

Bessy and Thomassé [7] found the relevant part (i) independently, and developed it as a key to their proof of Gallai's conjecture Theorem 4. (A second ingredient was Dilworth's theorem that has been traded for circulations in [31].) Following them we call an order satisfying (i) *coherent*. They proved, equivalently to (i), that every strong digraph has a coherent order. The equivalence of this fact with Knuth's theorem has been realized by Iwata and Matsuda [23].

Four simple proofs of the existence of coherent orders in strongly connected graphs, each providing its own insight, can be found in [7], [23], [24], [31].

Knuth proved this theorem as an application of his "Wheels within Wheels" theorem [24]. Iwata and Matsuda found Knuth's theorem in the archives, and proved it shortly and constructively using the ear decomposition of strongly connected graphs providing a measurable computational progress as well: it takes $O(nm)$ time to construct the order in Theorem 14, whereas a construction in [31] based on different ideas takes $O(n^2m^2)$ time. To prove the statement by induction (ii) is useful.

Fixing an order, the *index* (or winding) $\mathrm{ind}(C)$ of a cycle C is the number of its backward arcs, except for cycles $\{v\}$ $(v \in V)$ for which we define $\mathrm{ind}(\{v\}) = 1$ (like if it had one "backward loop"). If $\mathcal{C}$ is a set of cycles, we denote

$$\mathrm{ind}(\mathcal{C}) := \sum_{C \in \mathcal{C}} \mathrm{ind}(C).$$

For any cycle C in any graph with any order, $\mathrm{ind}(C) \geq 1$, so for any set of cycles $\mathcal{C}$, we have $\mathrm{ind}(\mathcal{C}) \geq |\mathcal{C}|$.

3.3 Topping

At the end of Section 2 we deduced the Greene-Kleitman theorem, and well-known results on cycle-covers, from Theorem 12. It is now the turn of Theorem 12 itself, completed by the following topping:

Given a graph $G = (V, E)$ with an order on the vertex set, let us call a set S satisfying

$$\text{(COMB)} \qquad |S \cap C| \leq \mathrm{ind}(C), (i = 1, \dots k).$$

a *cyclic stable set*. (This notion is equivalent to a geometric notion of Bessy and Thomassé [7]. The equivalence is proved in [31, (5)].)

The only thing we need here about cyclic stable sets though is that they *are indeed stable sets provided $G = (V, E)$ with the given order is coherent.* This is true, because by coherence every arc $e = ab \in E$ $(a, b \in V)$ is contained in a cycle C with $\mathrm{ind}(C) = 1$, so we get for the sets S satisfying (COMB): $|S \cap \{a, b\}| \leq |S \cap C| = 1$. So for every arc $ab \in E$, S can contain at most one of a and b.

Theorem 15 *([31] Theorem 3.1). Let G be a strong digraph given with a coherent order.*

$$\max\{|S_1 \cup \ldots \cup S_k| : \ S_i \ (i=1,\ldots k) \textit{ is a cyclic stable set } \} =$$
$$= \min\{|R(\mathcal{C})| + k\,\mathrm{ind}(\mathcal{C}) : \ \mathcal{C} \textit{ is a set of cycles}\}.$$

Proof. Let $G = (V, E)$ be strong and $k \in \mathbb{Z}$. Apply Theorem 14 to G and fix the coherent order it provides. Let B be the set of backward arcs, and define $c_{B,k}(e) := k$ if $e \in B$, and 0 otherwise $a := 0 \in \mathbb{Z}^n$, $b := w := 1 \in \mathbb{Z}^n$ (constant 0 and constant 1 vectors, that we will simply denote by 0 and 1). Then $\mathcal{Q}(G, 0, 1, c_{B,k})$ is the following system:

(kBT) $\quad x(V_C) \le k\,\mathrm{ind}(C)$ for every cycle C with vertex-set V_C, $0 \le x \le 1$.

Claim 1: The optimum of $P(G, 0, 1, c_{B,k}, 1)$ is $\max|S_1 \cup \ldots \cup S_k|$, $S_i \subseteq V$ $(i = 1, \ldots k)$ satisfies (COMB).

Such a union defines a primal solution, so the optimum is at least this quantity. To prove the equality, we show that the optimum x_{opt} of $P(G, 0, 1, c_{B,k}, 1)$ can be written in this form. Note $Q(G, 0, \infty, c_{B,k}) = kQ(G, 0, \infty, c_{B,1})$, so $P(G, 0, 1, c_{B,k}, 1)$ is the problem

$$\max\, 1^\top x, \text{ subject to } x = (x_1, \ldots, x_n) \in kQ(G, 0, \infty, c_{B,1}), \quad x \le 1.$$

Because of Theorem 13 applied to $Q(G, 0, \infty, c_{B,1})$:

$$x_{\mathrm{opt}} = x^1 + \ldots + x^k, \ x^i \in Q(G, 0, \infty, c_{B,1}) \text{ for all } i = 1, \ldots k,$$

and because of $x_{\mathrm{opt}} \in \{0, 1\}^n$ and $x^i \ge 0\,(i = 1, \ldots k)$ *we have* $x^i \in \{0, 1\}^n$, that is, x_i is the incidence vector of a set S_i satisfying (COMB).

Claim 2: The optimum of $D(G, 0, 1, c_{B,k}, 1)$ is

$$\min\{|R(\mathcal{C})| + k\,\mathrm{ind}(\mathcal{C}) : \ \mathcal{C} \text{ is a set of cycles}\}.$$

Indeed, (kBT) is a TDI system (Lemma 2), and therefore the optimum of $D(G, 0, 1, c_{B,k}, 1)$ is a $0-1$ vector. Since for a given dual solution $R(\mathcal{C}) := \{v \in V :$ the dual variable for v is 1 $\}$, the dual optimum of (kBT) is as claimed.

We have arrived at the end of the proof now: by the duality theorem of linear programming $\mathrm{OPT}(G, 0, 1, c_{B,k}, 1)$ is equal to both the quantities in Claim 1 and Claim 2, and by (COMB) the sets S_i $(i = 1, \ldots, k)$ are all cyclic stable sets. □

Theorem 11 is an immediate corollary since $\mathrm{ind}(\mathcal{C}) \ge |\mathcal{C}|$.

The stable sets S_i $(i = 1, \ldots k)$ and the set of cycles $\mathcal{C}$ provided by the theorem satisfy by complementary slackness (get it directly from the equalities of the theorem or in its proof) : $|S_i \cap C| = \mathrm{ind}(C) \ge 1$ $(i = 1, \ldots k)$, $C \in \mathcal{C}$, so we can delete from each S_i all but one of the elements of $S_i \cap C$, *finishing the proof of Theorem 12 as well.*

The original proof of theorems 15, 11, 12 was quite tedious – the integer decomposition property was proved through a complicated graph theory argument using potentials (arriving at a geometric surplus though). Theorem 13 provides a shorter way. (Which can also be converted into an algorithm.) Theorem 15 is actually the most general result we can prove for the union of k stable sets. It is similar to sums of matroids.

Let us finally deduce from Theorem 15 the two fundamental results of Bessy and Thomassé [7] originally proved with two entirely different methods. They are both minmax theorems, so "structural versions" follow by complementary slackness.

Corollary 2 *([7] Theorem 1). Let G be a strong digraph given with a coherent order.*

$$\max\{|S| : \ S \textit{ is a cyclic stable set } \} = \min\{\mathrm{ind}(\mathcal{C}) : \ \mathcal{C} \textit{ covers } V\}.$$

Proof. Apply Theorem 15 to $k = 1$, noting that the one-element subsets of X can be replaced by a cycle of index 1. □

For $k = 1$ the integer decomposition actually becomes simply flow integrality and we get back the simple proof of Theorem 6 in the introduction of [31] (Subsection 0.3).

Corollary 3 *([7], combination of Lemma 3 and Theorem 3). The minimum of k such that G can be colored with k cyclic stable sets is equal to the maximum of $\lceil |C|/\,\mathrm{ind}(C)\rceil$ over all cycles of C.*

Proof. Apply Theorem 15 to $k := \max\{\lceil |C|/\,\mathrm{ind}(C)\rceil \ : \ C$ is a cycle of G $\}$. Then $k\,\mathrm{ind}(C) \geq |C|$ for every cycle, and therefore the right hand side in Theorem 15 is n. Less cyclic stable sets are not enough, since $k-1$ cyclic stable sets meet each cycle C in at most $(k-1)\,\mathrm{ind}(C)$ elements, which is less than $|C|$ for the cycles for which the above maximum is reached. □

These two corollaries showed the way: they are the two ice-cream balls, the theorem is the topping.

4 Path Partitions

We prove Berge's conjecture in the following cases:

4.1 Long Paths

Theorem 16. *Let $G = (V, E)$ be a digraph, $k, m \in I\!N$ and $\mathcal{P} = \{P_1, \ldots, P_m\}$ a path partition. Then there exists*

(i) either k disjoint stable sets orthogonal to $\mathcal{P}$,
(ii) or a subpartition $\mathcal{Q} = \{Q_1, \ldots, Q_{m-1}\}$ of paths s.t. $\mathrm{ini}(\mathcal{Q}) \subseteq \mathrm{ini}(\mathcal{P})$*,* $\mathrm{ter}(\mathcal{Q}) \subseteq \mathrm{ter}(\mathcal{P})$*, and*

$$|R(\mathcal{Q})| \leq k - 1.$$

Proof. We prove the statement by induction on $n := |V|$. Suppose it holds for all $n' < n$ with all values of m and k, and prove it for G.

We can suppose that $|P| \geq k$ for all $P \in \mathcal{P}$, because if say $|P_m| < k$, define $Q_i := P_i$ for all $i = 1, \dots, m-1$. We see that (ii) holds: $|V \setminus (Q_1 \cup \dots \cup Q_{m-1})| = |P_m| \leq k-1$.

Let $a_i := \mathrm{ini}(P_i)$, and let a_i' be the second vertex of P_i, and $P_i' := P_i \setminus \{a_i\}$, that is, $\mathrm{ini}(P_i') = a_i'$ $(i = 1, \dots, m)$.

We distinguish now two cases:

Case 1: $\mathrm{ini}(\mathcal{P})$ is not a stable set, that is, say $a_1 a_2 \in E$.

If $|P_1| = k$, we can replace $\mathcal{P}$ by $\mathcal{P} \setminus \{P_1\}$, and add a_1 to P_2 as first vertex: we see that (ii) holds then.

So suppose $|P_1| \geq k+1$, and apply the induction hypothesis to $G - a_1$ and the same path partition restricted to $V \setminus \{a_1\}$, that is, with the only change of replacing P_1 by $P_1 \setminus \{a_1\}$. If now (i) holds, then, using also that $|P_1 \setminus \{a_1\}| \geq k$, (i) also holds for G. So suppose (ii) holds for $G - a_1$, and let $\mathcal{Q}' = \{Q_1', \dots, Q_{m-1}'\}$ be the path partition satisfying (ii). Since $\mathrm{ini}(\mathcal{Q}')$ is an $m-1$ element subset of $\{a_1', a_2, \dots, a_m\}$, it contains a path Q_1' with $\mathrm{ini}(Q_1') = a_1'$ or $\mathrm{ini}(Q_1') = a_2$. Adding a_1 as a first vertex to Q_1' we get a path partition of G that satisfies (ii).

Case 2: $\mathrm{ini}(\mathcal{P})$ is a stable set.

Apply the statement to $G' := G - \mathrm{ini}(\mathcal{P})$, $k' := k-1$.

If then (i) holds, then adding the stable set $\mathrm{ini}(\mathcal{Q})$ to the provided $k-1$ stable sets, we get that (i) holds for G and $\mathcal{P}$ with parameter k.

Otherwise alternative (ii) holds for $G' = (V', E')$ with $\mathcal{P}'$ and k', that is, we have a subpartition of paths $\mathcal{Q}' = \{Q_1', \dots, Q_{m-1}'\}$, in G' such that $\mathrm{ini}(\mathcal{Q}') \subseteq \mathrm{ini}(\mathcal{P}')$ (and $\mathrm{ter}(\mathcal{Q}') \subseteq \mathrm{ter}(\mathcal{P}') = \mathrm{ter}(\mathcal{P})$), that is, with an appropriate choice of the notation $\mathrm{ini}(\mathcal{Q}') = \{a_1', \dots, a_{m-1}'\}$, furthermore $\mathrm{ini}(Q_i') = \{a_i'\}$ for all $i = 1, \dots, m-1$, and

$$|R(\mathcal{Q}')| = |V' \setminus (Q_1' \cup \dots \cup Q_{m-1}')| \leq k-2.$$

Define now $Q_i := Q_i' \cup a_i$ $(i = 1, \dots, m-1)$. Clearly, $\mathcal{Q} := \{Q_1, \dots, Q_{m-1}\}$ is a subpartition of paths in G, and

$$|R(\mathcal{Q})| = |V \setminus (Q_1 \cup \dots \cup Q_{m-1})| = |V' \setminus (Q_1' \cup \dots \cup Q_{m-1}')| + 1 \leq k-1,$$

since $V \setminus (Q_1 \cup \dots \cup Q_{m-1}) = (V' \setminus (Q_1' \cup \dots \cup Q_{m-1}')) \cup \{a_m\}$, proving that alternative (ii) holds for G with $\mathcal{P}$ and k. □

This implies Berge's conjecture when there exists an optimal path partition with only paths of length at least k, which turns out to be equivalent to a result of Aharoni, Hartman and Hoffman [2]. Their proof is based on improving paths, and probably implies all the claims of the Theorem, in a more involved way though.

Corollary 4. *[2] If G is a digraph and $\mathcal{P}$ is a path partition where $k|\mathcal{P}| = \min\{|X| + k\pi(V \setminus X) : X \subset V\}$, then there exist k disjoint stable sets orthogonal to $\mathcal{P}$.*

4.2 Acyclic Digraphs

For acyclic digraphs Berge's and Linial's conjectures are consequences of Theorem 13 on the lines of, and more simply than the proof of Theorem 15, but I do not see how to deduce them directly from the statement of Theorem 15. The results have been proved in [1], [2], [10], [14], [26], [30]. Let us show how Theorem 13 replaces all the difficulties:

Let G be acyclic, and $1, \ldots, n$ an order of the vertices with only forward arcs. Add all backward arcs, that is $\hat{G} := G \cup B$, $B := \{ij,\, i > j\}$. Note that this order is coherent for $\hat{G}$, and a cycle with β backward arcs is the disjoint union of vertex-sets of β cycles each having 1 backward arc.

Consider now the polyhedron $Q(\hat{G}, -\infty, 1, c_{B,k}) \subseteq kQ(\hat{G}, -\infty, \infty, c_{B,1})$. It can have negative vertices! (We cannot avoid this, since since we want equality constraints for the dual problem.) By Theorem 13 $Q(\hat{G}, -\infty, \infty, c_{B,1})$ – which is now simply $\{x \in \mathbb{R}^n : x(P) \leq 1$ for every path $P\}$ – has the integer decomposition property again, and by Theorem **??** $\mathcal{Q}(G, -\infty, 1, c_{B,k})$ is TDI. Now we can finish using Theorem 13 exactly like in the proof of Theorem 15. (Negative variables do not disturb, since by complementary slackness a primal optimal solution $x \in Q(\hat{G}, -\infty, 1, c_{B,k})$ satisfies $x(P) = k$ for all paths P of an optimal path partition; because of $x \leq 1$, x has at least k different 1 entries; because of Theorem 13 we have $x = x^1 + \ldots + x^k$, $x^i \in Q(\hat{G}, -\infty, \infty, c_{B,1})$, and then the positive coordinates of the x_i meet every path, and in different vertices for $i \neq j = 1, \ldots, n$.)

4.3 Corollaries for Path Partitions

Berger and Hartman studied the two next-to-extreme cases of Berge's conjecture [5], [6]:"$k = 2$" and "$k = \lambda - 1$" – the $k = 1$ and $k = \lambda$ cases being completely settled, see the subsections 2.2, 2.3, 2.4, 2.5. It is somewhat discouraging for the continuation that the path partition and cycle cover versions are so far completely unrelated even in the Gallai-Milgram case $k = 1$.

However, the following theorems show some connections at the other extreme, and for strongly connected graphs a larger interval can be allowed for k. These are the starting steps of a research with Irith Hartman in the frame of the French-Israeli collaboration project, intending to prove Berge's conjecture.

The following result is a direct corollary of Theorem 15, and it provides a common statement and proof of the Gallai-Roy theorem, and a theorem of Berger and Hartman [6] according to which Berge's strongest conjecture (and therefore Linial's conjecture as well) is true if $k = \lambda - 1$. Their original proof is quite involved. Note that we prove only the version Conjecture 3 ignoring short ($< k$) paths that are not singletons, still implying Linial's conjecture.

Theorem 17. *Let $G = (V, E)$ be a digraph, and $k \in \mathbb{N}$, $k \geq \lambda - 1$. For any subpartition $\mathcal{P}$ of paths minimizing $|R(\mathcal{P})| + k|\mathcal{P}|$, there exists k disjoint stable sets orthogonal to $\mathcal{P}$ whose union contains $R(\mathcal{P})$.*

Proof. Let $\mathcal{P}$ satisfy the condition. The number $|R(\mathcal{P})| + k|\mathcal{P}|$ is called the *k-norm* of $\mathcal{P}$ [22].

Add a new vertex v_0 to the graph, a cycle C_0 through v_0 with $k+1$ new vertices besides v_0, and add all the edges $v_0 v, v v_0$ $(v \in V)$. Denote $\hat{V} := V \cup V(C_0)$. Order V starting with v_0, continuing on C_0 until the vertex before v_0, then continuing with the other members of $\mathcal{P}$ in some order, from the sources to the sinks, and finally adding the vertices of $R(\mathcal{P})$ in arbitrary order. Let $\hat{G} = (\hat{V}, \hat{E})$ be the constructed graph.
Note: C_0 serves the goal of covering v_0 in a predictable way, and the presence of C_0 will also have the useful consequence that we will never color v_0. Adding v_0 without adding C_0 may slightly change the problem and cause technical difficulties. Let $\mathcal{C} := \{C_0\} \cup \{v_0 \cup P : P \in \mathcal{P}\}$.

The defined order is coherent, since for all $uv \in E : v_0, u, v$ is a cycle with one backward arc, and C_0 is also a cycle that has one backward arc.

All cycles in $\mathcal{C}$ have one backward arc, so $\operatorname{ind}(\mathcal{C}) = |\mathcal{C}| = |\mathcal{P}|+1$. We show *that $\mathcal{C}$ minimizes the right hand side of Theorem 15* (see after Claim 2), and then this theorem will provide the statement. Let $\mathcal{Q} = \{Q_0, \ldots, Q_m\}$ $(Q_0, \ldots, Q_m$ are cycles) minimize the right hand side of this theorem, and among the possible choices $R(\mathcal{Q})$ be maximum. In fact we will show

$$|R(\mathcal{C})| + k \operatorname{ind}(\mathcal{C}) \leq |R(\mathcal{Q})| + k \operatorname{ind}(\mathcal{Q}), \tag{1}$$

by showing through claims 1, 2 a path partition $\mathcal{P}'$ in G, $R(\mathcal{P}') = R(\mathcal{Q})$, $|\mathcal{P}'| = |\mathcal{Q}| - 1$, and then by the minimality of the k-norm of $\mathcal{P}$ we have

$$|R(\mathcal{P})| + k|\mathcal{P}| \leq |R(\mathcal{P}')| + k|\mathcal{P}'|, \tag{2}$$

implying (1): indeed, the left hand side of (1) is k plus the left hand side of (2), and according to the following, the right hand side of (1) is *at least k plus the right hand side of (2)*:

$$|R(\mathcal{P}')| + k|\mathcal{P}'| + k = |R(\mathcal{Q})| + k|\mathcal{Q}| \leq |R(\mathcal{Q})| + k|\operatorname{ind}(\mathcal{Q})|.$$

Claim 1: There exists $i \in \{1, \ldots, m\}$ so that $Q_i = C_0$, and therefore $Q_0 = C_0$ can be supposed.

Indeed, if C_0 does not occur, then the vertices of C_0 different of v_0 cannot occur in $\mathcal{Q}$ at all. Since their number is $k+1$, by adding C_0 to $\mathcal{Q}$ we decrease $|R(\mathcal{Q})|$ by $k+1$ and $k|\mathcal{Q}|$ increases only by k ($|\mathcal{Q}|$ increases by 1), contradicting the optimal choice of $\mathcal{Q}$.

Claim 2: $P_i' := Q_i \setminus v_0$ $(i = 0, 1, \ldots, m)$ are pairwise disjoint.

Indeed, $Q_i \setminus v_0 \leq \lambda \leq k+1$, so if it meets $Q_j \setminus v_0$ $(j \neq i)$, then $\mathcal{Q} \setminus \{Q_i\}$ contradicts the choice of $\mathcal{Q}$, because $|R(\mathcal{Q} \setminus \{Q_i\})| \leq |R(\mathcal{Q})| + k$: $v_0 \notin R(\mathcal{Q})$ by Claim 1, and the possible other common point is not in $R(\mathcal{Q})$ either. On the other hand $k \operatorname{ind}(\mathcal{Q} \setminus \{Q_i\}) = k \operatorname{ind}(\mathcal{Q}) - k \operatorname{ind}(Q_i) \leq k \operatorname{ind}(\mathcal{Q}) - k$, so the right hand side of the formula of Theorem 15 does not increase, $R(\mathcal{Q})$ increases, again contradicting the choice of $\mathcal{Q}$.

Now by Claim 2, $\mathcal{P}' := \{P'_1, \ldots, P'_m\}$ is a set of disjoint paths satisfying the promised relations $R(\mathcal{P}') = R(\mathcal{Q})$, $|\mathcal{P}'| = |\mathcal{Q}| - 1$, so (1) is satisfied and $\mathcal{C}$ is an optimal set of cycles in Theorem 15.

Theorem 15 provides now exactly what we want, unless v_0 is a colored vertex. However, one can suppose that v_0 is contained in at least 2 members of $\mathcal{C}$, since in case of $\mathcal{C} = \{C_0\}$ the set $\mathcal{C}' = \{C_0, \{v_0\} \cup P\}$, where P is a longest path is a set of cycles which also minimizes the right hand side. Then by complementary slackness in Theorem 15, v_0 is not contained in any of the sets S_i provided by the theorem. $\square$

For $k = \lambda - 1$ the theorem can be restated as follows:

Corollary 5. *Let $G = (V, E)$ be a directed graph. If $\mathcal{P}$ is a maximum number of disjoint maximum paths of G, there exists a set $U \subseteq V$ consisting of exactly one vertex of each $P \in \mathcal{P}$, and a (complete) coloring of $G - U$ where each color class is orthogonal to $\mathcal{P}$.*

Indeed, $|R(\mathcal{P})| + (\lambda - 1)|\mathcal{P}|$ is minimum provided $\mathcal{P}$ is a maximum set of disjoint maximum paths.

Applying the theorem to $k = \lambda$ we can reformulate it into the following very similar form where U can in addition be chosen to be a cyclic stable set (it is one of the colors) the paths are not necessarily part of a maximum packing, however all of the colors may have to meet $R(\mathcal{P})$ (while U did not). Both corollaries extend the Gallai-Roy theorem.

Corollary 6. *Let $G = (V, E)$ be a directed graph. If $\mathcal{P}$ is any number of disjoint maximum paths of G, there exists a coloring of G where the color classes are orthogonal to $\mathcal{P}$.*

The following result exploits some simple properties of paths, but the application of these prevents to use the gadget reductions of the previous proof and we cannot avoid assuming strong connectivity.

Theorem 18. *Conjecture 3 is true provided G is strongly connected and $k \geq \lambda - \sqrt{\lambda}$.*

Proof. Let G be strongly connected, $k \geq \lambda - \sqrt{\lambda}$, and choose a coherent order. Apply Theorem 15 and let $S_1, \ldots, S_k$ the stable sets in the maximum, X and $\mathcal{C}$ the set and cycle cover in the minimum, moreover, suppose that among the possible choices, $|X|$ is biggest possible. As in the previous proof, $k \geq \lambda/2$ easily implies that $\text{ind}(C) = 1$ for all $C \in \mathcal{C}$. The problem is that the cycles in $\mathcal{C}$ are not necessarily disjoint.

If a cycle has at most $\lambda - \lfloor\sqrt{\lambda}\rfloor$ vertices not covered by any other cycle, then delete it from $\mathcal{C}$ and add to X the vertices that get now uncovered, contradicting the choice of X. So the difference of any two cycles has size larger than $\lambda - \lfloor\sqrt{\lambda}\rfloor$. *If for two intersecting cycles C_1, C_2 we have $|C_1 \setminus C_2|, |C_2 \setminus C_1| > \lambda - \lfloor\sqrt{\lambda}\rfloor$, then their union contains a path with more than λ vertices.* (This bound is essentially tight.)

Indeed, we have then $|C_1 \cap C_2| \le \lfloor\sqrt{\lambda}\rfloor$. But then $C_1 \cap C_2$ divides both cycles into paths, and one of these paths has at least $\lfloor\sqrt{\lambda}\rfloor$ vertices outside C_1, say. (If all these subpaths of C_2 have at most $\sqrt{\lambda} - 1$ vertices outside C_1, then $|C_2 \setminus C_1| \le \sqrt{\lambda}(\sqrt{\lambda} - 1) = \lambda - \sqrt{\lambda}$.) Take such a path P for instance in C_2. Then $|C_1 \cup P| > \lambda - \lfloor\sqrt{\lambda}\rfloor + \lfloor\sqrt{\lambda}\rfloor = \lambda$. It is easy to see that $|C_1 \cup P|$ contains a Hamiltonian path, that is, a path of length larger than λ, contradicting the definition of λ.

So the cycles in $\mathcal{C}$ are pairwise disjoint and then we are done again by complementary slackness. □

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