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The chromatic gap and its extremes

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ABSTRACT

The *chromatic gap* is the difference between the chromatic number and the clique number of a graph. Here we investigate gap(n), the maximum chromatic gap over graphs on n vertices. Can the extremal graphs be explored? While computational problems related to the chromatic gap are hopeless, an interplay between Ramsey-theory and matching theory leads to a simple and (almost) exact formula for gap(n) in terms of Ramsey-numbers. For our purposes it is more convenient to work with the *covering gap*, the difference between the clique cover number and stability number of a graph and this is what we call the *gap* of a graph. Notice that the well-studied family of perfect graphs are the graphs whose induced subgraphs have gap zero. The maximum of the (covering) gap and the chromatic gap running on all induced subgraphs will be called *perfectness gap*.

Using $\alpha(G)$ for the cardinality of a largest stable (independent) set of a graph G, we define $\alpha(n) = \min \alpha(G)$ where the minimum is taken over triangle-free graphs on n vertices. It is easy to observe that $\alpha(n)$ is essentially an inverse Ramsey function, defined by the relation $R(3, \alpha(n)) \leqslant n < R(3, \alpha(n) + 1)$. Our main result is that $\mathrm{gap}(n) = \lceil n/2 \rceil - \alpha(n)$, possibly with the exception of small intervals (of length at most 15) around the Ramsey-numbers $R(3, m)$, where the error is at most 3.

The central notions in our investigations are the *gap-critical* and the *gap-extremal* graphs. A graph G is *gap-critical* if for every proper induced subgraph $H \subset G$, $\mathrm{gap}(H) < \mathrm{gap}(G)$ and *gap-extremal* if it is gap-critical with as few vertices as possible (among gap-critical graphs of the same gap). The strong perfect graph

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theorem, solving a long standing conjecture of Berge that stimulated a broad area of research, states that gap-critical graphs with gap 1 are the holes (chordless odd cycles of length at least five) and antiholes (complements of holes). The next step, the complete description of gap-critical graphs with gap 2 would probably be a very difficult task. As a very first step, we prove that there is a unique 2-extremal graph, $2C_5$, the union of two disjoint (chordless) cycles of length five.

In general, for $t \geqslant 0$, we denote by $s(t)$ the smallest order of a graph with gap t and we call a graph is *t-extremal* if it has gap t and order $s(t)$. Equivalently, $s(t)$ is the smallest order of a graph with perfectness gap equal to t. It is tempting to conjecture that $s(t) = 5t$ with equality for the graph tC_5. However, for $t \geqslant 3$ the graph tC_5 has gap t but it is not gap-extremal (although gap-critical). We shall prove that $s(3) = 13$, $s(4) = 17$ and $s(5) \in \{20, 21\}$. Somewhat surprisingly, after the uncertain values $s(6) \in \{23, 24, 25\}$, $s(7) \in \{26, 27, 28\}$, $s(8) \in \{29, 30, 31\}$, $s(9) \in \{32, 33\}$ we can show that $s(10) = 35$. On the other hand we can easily show that $s(t)$ is asymptotically equal to $2t$, that is, gap(n) is asymptotic to $n/2$. According to our main result the gap is actually equal to $\lceil n/2 \rceil - \alpha(n)$, unless n is in an interval $[R, R + 14]$ where R is a Ramsey-number, and if this exception occurs the gap may be larger than this value by only a small constant (at most 3).

Our study provides some new properties of Ramsey graphs them selves: it shows that triangle-free Ramsey graphs have high matchability and connectivity properties, leading possibly to new bounds on Ramsey-numbers.

1. Introduction

After the proof of the strong perfect graph conjecture [5], the problems concerning graph families that are close to perfectness become more interesting. Here we focus our attention on a parameter that we call the *chromatic gap* of a graph, gap(G), equal to the "duality gap" of a most natural integer linear programming formulation of the graph coloring problem.

Graphs in this paper are undirected, their vertex set is denoted by $V(G)$. A *cycle* is a connected subgraph with all degrees equal to 2. A *clique* is a subset of the vertices inducing a complete subgraph, and a *stable set* does not induce any edge. The notations C_i and K_i will refer to cycles, respectively cliques of order i $(i = 1, 2, \ldots)$.

The size of a largest clique (resp. stable set) in a graph G is denoted, by $\omega(G)$ (resp. $\alpha(G)$). We also speak about k-cliques or k-stable sets meaning that their cardinality is k. A 3-clique is also called a *triangle*. The *chromatic number*, $\chi(G)$, and *clique cover number*, $\theta(G)$, denote the minimum number of partition classes of $V(G)$ into stable sets and into complete subgraphs, respectively. Using $\overline{G}$ for the complement of G, we have obviously

$$\omega(G) = \alpha(\overline{G}), \qquad \chi(G) = \theta(\overline{G}) \tag{1}$$

and

$$\chi(G) \geqslant \omega(G) \geqslant \frac{|V(G)|}{\theta(G)}, \qquad \theta(G) \geqslant \alpha(G) \geqslant \frac{|V(G)|}{\chi(G)}. \tag{2}$$

Let us define the *chromatic gap* of a graph G as $\chi(G) - \omega(G)$, and the *covering gap* as $\theta(G) - \alpha(G)$. Although these parameters are equivalent (through (1)), for our purposes it is more convenient to work with the latter, so we define the *gap, or covering gap* of a graph G as gap$(G) = \theta(G) - \alpha(G)$.

Notice that *perfect graphs* are the graphs whose induced subgraphs have gap zero. The *perfectness gap* of a graph is the maximum of the (covering) gap and the chromatic gap running on all induced subgraphs.

A graph G is *gap-critical* if for every proper induced subgraph $H \subset G$, $\mathrm{gap}(H) < \mathrm{gap}(G)$. The perfect graph theorem [5] states that gap-critical graphs with gap 1 are the holes (chordless odd cycles of length at least five) and antiholes (complements of holes). The complete description of gap-critical graphs with gap 2 would probably be a very difficult task – it seems there is not even a plausible guess available. Trivial members can be obtained as a disjoint union of holes and/or antiholes. A non-trivial member (15 vertices, $\alpha = 6$, $\theta = 8$) is shown in [10, p. 427]. Deleting any pair of vertices of the Ramsey graph R_{13}, the unique graph with $\omega(G) = 2$, $\alpha(G) = 4$ on 13 vertices, gives another example of order 11 with $\alpha = 4$, $\theta = 6$. However, as we shall prove, the smallest order of a gap-critical graph with gap 2 is 10, the unique example is the trivial member, the union of two disjoint C_5. The graph R_{13} itself is also gap-critical with gap 3, in fact the smallest one (see Section 5).

Note that the definition of gap-critical graphs cannot be simplified by requiring only $\mathrm{gap}(G - v) < \mathrm{gap}(G)$ for every vertex v: indeed, for instance the gap of the circular graph $C(3,3)$ (on 10 cyclically ordered vertices, where any three cyclically consecutive ones form a clique) is 1, deleting any vertex the gap is 0 although C_5 subgraphs are still present. Here the smallest example. Consider a hole on 5 vertices $c_1 \dots c_5 c_1$ and replicate c_1 and c_3 (replicating a vertex v means adding a vertex adjacent to v and all neighbors of v). For the obtained graph G we have $\omega(G) = 3$, $\chi(G) = 4$, but for any $v \in V(G)$ $\omega(G - v) = \chi(G - v) = 3$, while G contains a C_5. So, the complementary graph $\bar{G}$ is not gap-critical, although $\mathrm{gap}(G - v) < \mathrm{gap}(G)$ for all $v \in V(G)$.

The central topic of our work is to determine the maximum gap of graphs of order n, denoted by $\mathrm{gap}(n)$ which leads to a study of gap-extremal graphs. For $t \geqslant 0$, we denote by $s(t)$ the smallest order of a graph with gap t. A graph is *t-extremal* if it has gap t and order $s(t)$; it is *gap-extremal*, if it is t-extremal for some t. Note that the empty graph has gap 0, so $s(0) = 0$, and – since C_5 is the unique smallest non-perfect graph – $s(1) = 5$, and C_5 is the only 1-extremal graph. It will be much more difficult to prove that $s(2) = 10$ (Theorem 5.2). It is tempting to conjecture that the pattern continues and $s(t) = 5t$ with equality for the graph tC_5, this is how we started ... However, classical Ramsey graphs provide better bounds. We shall prove that $s(3) = 13$, $s(4) = 17$ and $s(5) = 21$ or 20. From a general conjecture we think that the true value is 21. Somewhat surprisingly, after the uncertain values $s(6) \in \{23, 24, 25\}$, $s(7) \in \{26, 27, 28\}$, $s(8) \in \{29, 30, 31\}$, $s(9) \in \{32, 33\}$ we can show that $s(10) = 35$.

Gap-extremal graphs are obviously gap-critical. Holes and antiholes are gap-critical but if they have more than five vertices they are not gap-extremal; if they have more than eleven vertices their gap is also not maximal among graphs of the same order, since the gap of two disjoint C_5 is 2.

A large $\theta(G)$ might be the consequence of a small $\omega(G)$. But small clique number may mean not too many edges, so a large $\alpha(G)$ too! What happens with the gap in this competition? The trade between the size of cliques and stable sets is described by Ramsey-theory, itself having a lot of open questions. We will convert the relations provided by Ramsey-numbers into a balance between θ and α. Using Ramsey-numbers as a black box we will be able to (almost) determine our functions.

It will turn out to be essentially true that the graphs with a large gap are triangle-free. In other words, decreasing the clique-size, makes θ increase more than it does increase α. To work out this precisely will need a refined analysis based on details concerning Ramsey-numbers $R(3,.)$ and matchings. In Section 3 we prove simple statements about the gap, about Ramsey-numbers and about matchings that will provide the right tools for this work. In Section 4 we determine the gap-function with only a small constant error, and this relies mainly on a study of triangle-free graphs.

In view of this role, we will need to use variants of the notions and terms for triangle-free graphs separately. We will speak about *triangle-free t-extremal graphs* which means that their cardinality is minimum among triangle-free graphs of gap t. Note that a triangle-free gap-extremal graph is not necessarily a gap-extremal graph, since there might be a graph containing a triangle with smaller cardinality and the same gap. By analogy, the corresponding notations for triangle-free graphs will be $\mathrm{gap}_2(n)$, $s_2(t)$. Thus $\mathrm{gap}_2(n)$ is the maximum gap among triangle-free graphs on n vertices, $s_2(t)$

is the smallest order of a triangle-free graph with gap t. Clearly, $\mathrm{gap}(n) \geqslant \mathrm{gap}_2(n)$ for all $n \in \mathbb{N}$, and $s(t) \leqslant s_2(t)$ for all $t \in \mathbb{N}$. ($\mathbb{N}$ is the set of natural numbers $\{1, 2, \ldots\}$).

For any $n \in \mathbb{N}$, $t = \mathrm{gap}(n)$, adding $n - s(t)$ isolated points to a t-extremal graph we get a graph of maximum gap among graphs of order n. However, both α and θ increase by the addition of isolated vertices. When G is triangle-free, graphs of maximum gap, at the same time with minimum stability number among triangle-free graphs on n vertices will be particularly appreciated. Let $\alpha(n)$ denote the minimum of $\alpha(G)$ over triangle-free graphs G with n vertices. So, $\alpha(n)$ is defined by the relation $R(3, \alpha(n)) \leqslant n < R(3, \alpha(n) + 1)$. A graph G on n vertices will be called *stable gap-optimal*, if G is triangle-free, $\mathrm{gap}(G) = \mathrm{gap}_2(n)$, and $\alpha(G) = \alpha(n)$. It will turn out that there exist stable gap-optimal graphs for every n. Therefore it is unavoidable to know something about the function $\alpha(n)$, in fact it is just the inverse of the well studied Ramsey function $R(3, x)$.

We say that a graph is an (ω, α)-*Ramsey graph* ($\omega, \alpha \in \mathbb{N}$) if it is of maximum order among the graphs G without an ω-clique (a clique of size ω) and without an α-stable set (stable set of size α). By Ramsey's theorem [15], this maximum is finite. The smallest n such that for any graph G of order n either $\omega(G) \geqslant \omega$ or $\alpha(G) \geqslant \alpha$, is called the Ramsey-number $R(\omega, \alpha)$. We will use mainly Ramsey-numbers for $\omega = 3$. Clearly, the order of (ω, α)-Ramsey graphs is $R(\alpha, \omega) - 1$, and their maximum clique and stable set have size $\omega - 1$, $\alpha - 1$.

Clearly, the above introduced number $\alpha(n)$ ($n \in \mathbb{N}$) is actually defined by the relation $R(3, \alpha) \leqslant n < R(3, \alpha + 1)$. It is equal to *the number of Ramsey-numbers smaller than or equal to n.* Indeed, among the Ramsey-numbers $R(3, x)$ those with $1, 2, \ldots, x$ are smaller than or equal to n, and all the others are larger. It will turn out that $s(t + 1) - s(t)$ is usually 2, and the exceptions are at the Ramsey-numbers where this difference is equal to 4 with rare exceptions 5 of 3 (but these latter might actually all be for $t \leqslant 3$).

Although $s(t)$ will be determined with a constant error (modulo Ramsey-numbers), we also include a transparent easy proof in Section 2 that shows that $2t + c_1\sqrt{t \log t} \leqslant s(t) \leqslant 2t + c_2\sqrt{t \log t}$ (Corollary 2.3).

The main result of the paper is finding $\mathrm{gap}(n)$ and $s(t)$ with constant error in terms of Ramsey-numbers. First we shall prove that $\mathrm{gap}(n) = \mathrm{gap}_2(n) = \lceil n/2 \rceil - \alpha(n)$ except when n is even and there exist odd numbers n_1, n_2 such that $n = n_1 + n_2$ and $\alpha(n) = \alpha(n_1) + \alpha(n_2)$, in which case 1 must be added. The exceptional case can occur in an obvious way, when n is a Ramsey-number and n_1 or n_2 is equal to 1, or in a rather mysterious way (only if $n_1 = n_2 = 5$?), when we call n Ramsey-perfect.

A number n is *Ramsey-perfect* if n is not an even Ramsey-number and $n = n_1 + n_2$, where $n_1, n_2 \geqslant 5$ are odd and $\alpha(n) = \alpha(n_1) + \alpha(n_2)$. We know only one Ramsey-perfect number, 10 ($\alpha(10) = 2\alpha(5)$), and we believe that there are no others. One way this might still happen is $\alpha(n) = \alpha(n - 5) + \alpha(5)$, in that case $n - 1, n - 4$ must be both Ramsey-numbers – we call them (*Ramsey*) *twins*. Probably there are no Ramsey twins beyond 6, 9 but this is not proved, although Erdős and Sós [7] (see also in [4]) conjectured $R(3, m + 1) - R(3, m)$ tends to infinity with m. Our main results are summarized as follows.

- $\mathrm{gap}_2(n) = \lceil n/2 \rceil - \alpha(n) + \varepsilon(n)$, where $\varepsilon(n) = 1$ if n is an even Ramsey-number or a Ramsey-perfect number and 0 otherwise (Theorem 4.1).
- The functions $\mathrm{gap}(n), s(t)$ are determined with a small error by their restricted counterparts: for all $n, t \in \mathbb{N}$: $0 \leqslant \mathrm{gap}(n) - \mathrm{gap}_2(n) \leqslant 2$, $0 \leqslant s_2(t) - s(t) \leqslant 10$ (Theorem 4.11).
- A synthesis of this work: for all $n \in \mathbb{N} \setminus \bigcup_{\alpha \in \mathbb{N}} [R(3, \alpha), R(3, \alpha) + 14]$: $\mathrm{gap}(n) = \mathrm{gap}_2(n) = \lceil n/2 \rceil - \alpha(n)$, and always $\lceil n/2 \rceil - \alpha(n) \leqslant \mathrm{gap}(n) \leqslant \lceil n/2 \rceil - \alpha(n) + 3$ (Theorem 4.12).

It is worth noting that for Ramsey-numbers R that are at least 5 bigger than the preceding Ramsey-number (so maybe for all Ramsey-numbers larger than 28), only one $s(t)$ value is uncertain and equal to either $R + 1$ or $R + 2$. Also, our study reveals high matchability and connectivity properties of Ramsey graphs. For example, $(3, \alpha + 1)$-Ramsey graphs are $(R(3, \alpha + 1) - R(3, \alpha) - 3)$-connected, moreover, deleting at most $R(3, \alpha + 1) - R(3, \alpha) - 3$ vertices, the remaining $n \geqslant R(3, \alpha) + 2$ vertices, if n is even, induce a graph with a perfect matching (Corollary 4.5).

Finally we mention some related work. Bíró [1] raised the problem of finding the minimum of α while fixing n and θ, more precisely finding

$$\beta(n,\theta)=\min\{\alpha(G)\colon G \text{ graph},\ |V(G)|=n,\ \theta(G)=\theta\}$$

and gave the first bounds and a conjecture. Jahanbekam and West [12] stated another conjecture for constrained values of n and θ. If $\theta \geqslant \frac{n+1}{2}$ Theorem 4.1 easily provides the following formula for θ, implying these conjectures: $\beta(n,\theta)=n+\alpha(W)-W-\varepsilon$, where $W=2(n-\theta)+1$ and ε is 0 or 1, the latter if W is Ramsey-perfect or another (even more exceptional, possibly non-existing) case. A recent communication of Bíró, Füredi and Jahanbekam [2] proves a formula for $\beta(n,\theta)$ in the range $\theta \geqslant \frac{n+3}{2}$ with similar methods.[3] The equality between the two formulas can be proved easily. As far as we know, finding the exact values of gap(n) (without restricting ourselves to triangle-free graphs) and the solution of Bíró's problem *for arbitrary* θ both remain open.

2. Asymptotic of $s(t)$

Before giving the exact values of the function gap and gap_2 (up to a small constant) we show how to get easily the asymptotic of $s(t)$.

Proposition 2.1. $s(t) \leqslant s_2(t) \leqslant 2t + c_1\sqrt{t\log t}$.

Proof. The celebrated result of Kim [13] states that for every sufficiently large n there is a graph G_n with n vertices such that $\omega(G_n)=2$ and $\alpha(G_n) \leqslant 9\sqrt{n\log n}$. Define $f(t)$ as the smallest n for which there exists G_n such that

$$\left\lceil \frac{n}{2} \right\rceil - 9\sqrt{n\log n} \geqslant t. \tag{3}$$

Clearly $f(t)$ is an upper bound for $s_2(t)$ because by the definition of G_n and by (3)

$$\text{gap}(G_n)=\theta(G_n)-\alpha(G_n) \geqslant \left\lceil \frac{n}{2} \right\rceil - 9\sqrt{n\log n} \geqslant t. \tag{4}$$

One can easily check that the last inequality in (4) can be satisfied with $n=2t+\lfloor c_1\sqrt{t\log t}\rfloor$ where c_1 is a constant. This gives the required upper bound. □

Proposition 2.2. $s(t) \geqslant 2t+\alpha(2t) \geqslant 2t + c_2\sqrt{t\log t}$.

Proof. Let G be a graph with $\text{gap}(G)=t$ and with n vertices. Consider a clique cover of G obtained by greedily selecting a largest clique in the subgraph induced by the vertex set uncovered in previous steps. Suppose that in the first k steps cliques of size at least three were selected, covering $2k$ vertices plus a set $A \subseteq V(G)$, followed by l steps of selecting edges and covering Y, finally a set Z of independent vertices covers the rest of the vertices of G. Set $B=Y\cup Z$.

Then clearly,

$$\theta(G) \leqslant \frac{n-|Z|-|A|}{2} + |Z| = \frac{n-|A|}{2} + \frac{|Z|}{2} \leqslant \frac{n-|A|}{2} + \frac{\alpha(B)}{2}$$

therefore

$$\theta(G)-\alpha(G) \leqslant \frac{n-|A|}{2} + \frac{\alpha(B)}{2} - \alpha(G) \leqslant \frac{n-|A|}{2} + \frac{\alpha(B)}{2} - \alpha(B) = \frac{n-|A|-\alpha(B)}{2}$$

[3] Ref. [2] mentions the relation of $\beta(n,\theta)$ to the present work (to [11] or to an earlier version from November 2009) but misses the close tie to Theorem 4.1.

thus $2t + |A| + \alpha(B) \leqslant n = s(t)$. We gained $|A| + \alpha(B)$ over the $2t$ lower bound. However, we know that $3|A| + |B| \geqslant n \geqslant 2t$. It is easy to see that the gain is smallest for $|A| = 0$ thus we gain at least $\alpha(2t)$ as desired. □

Corollary 2.3. $s(t) = 2t + \theta(\sqrt{t \log t})$.

3. Matchings and Ramsey-numbers

In this section we explore the main properties of the gap of a graph, of gap-critical graphs, of the relation of these to matchings and the Ramsey-numbers.

3.1. Easy facts

Proposition 3.1. *If a graph G has k connected components $C_1, \dots, C_k$ then* $\mathrm{gap}(G) = \mathrm{gap}(C_1) + \cdots + \mathrm{gap}(C_k)$. *Every connected component of a gap-critical graph is gap-critical. Every connected component of a gap-extremal graph is gap-extremal.*

Proof. Both θ and α are sums of the θ and α of the components. □

Proposition 3.2. *The* $\mathbb{N} \to \mathbb{N}$ *functions* gap *and* gap_2 *are monotone increasing.*

Proof. Indeed, if $n_1 \leqslant n_2$, then adding $n_2 - n_1$ isolated vertices to a graph G of order n_1 of maximum gap, we get a graph of order n_2 of the same gap. □

Proposition 3.3. *For any $n_1, n_2 \in \mathbb{N}$ we have* $\mathrm{gap}(n_1 + n_2) \geqslant \mathrm{gap}(n_1) + \mathrm{gap}(n_2)$. *For any $t_1, t_2 \in \mathbb{N}$ we have* $s(t_1 + t_2) \leqslant s(t_1) + s(t_2)$.

Proof. Let G be a graph that consists of two components, G_1 on n_1 vertices, and G_2 on n_2 vertices, $\mathrm{gap}(G_1) = \mathrm{gap}(n_1)$ and $\mathrm{gap}(G_2) = \mathrm{gap}(n_2)$. Then G has $n_1 + n_2$ vertices, and $\mathrm{gap}(n_1 + n_2) \geqslant \mathrm{gap}(G) = \mathrm{gap}(n_1) + \mathrm{gap}(n_2)$. For the second part let G be a graph that consists of two components, a t_1-extremal graph G_1 on $s(t_1)$ vertices, and a t_2-extremal graph G_2 on $s(t_2)$ vertices. Then G has $s(t_1) + s(t_2)$ vertices, and $\mathrm{gap}(G) = t_1 + t_2$ thus $s(t_1 + t_2) \leqslant |V(G)| = s(t_1) + s(t_2)$. □

The equality is easily satisfied, for instance $\mathrm{gap}(5) = 1$, $\mathrm{gap}(17) = 4$, and $\mathrm{gap}(22) = 5$ as we will see in Section 5. We have a third, similar inequality where the condition of equality is less trivial (Theorem 3.19), that turns out to be very restrictive and the related notion of Ramsey-perfect numbers are crucial for the main results (Section 4.1).

Proposition 3.4. *For any $n_1, n_2 \in \mathbb{N}$ we have*

$$\alpha(n_1 + n_2) \leqslant \alpha(n_1) + \alpha(n_2).$$

Proof. Indeed, a graph G that consists of two components, G_1 on n_1 vertices, and G_2 on n_2 vertices, $\alpha(G_1) = \alpha(n_1)$ and $\alpha(G_2) = \alpha(n_2)$, has $n_1 + n_2$ vertices, and $\alpha(n_1 + n_2) \leqslant \alpha(G) = \alpha(n_1) + \alpha(n_2)$. □

Proposition 3.5. *Let G be a graph and Q a clique of G. Then*

$$\theta(G) \geqslant \theta(G - Q) \geqslant \theta(G) - 1,$$

$$\alpha(G) \geqslant \alpha(G - Q) \geqslant \alpha(G) - 1,$$

$$\mathrm{gap}(G) + 1 \geqslant \mathrm{gap}(G - Q) \geqslant \mathrm{gap}(G) - 1,$$

and there exists a chain of induced subgraphs of G with gaps equal to $\text{gap}(G), \text{gap}(G)-1, \dots, 0$. *Furthermore, if G is gap-critical,*

$$\theta(G-Q)=\theta(G)-1, \qquad \alpha(G-Q)=\alpha(G), \qquad \text{gap}(G-Q)=\text{gap}(G)-1.$$

Notice that the equality $\text{gap}(G-Q)=\text{gap}(G)-1$ may hold also for graphs that are not gap-critical (see the example in the Introduction: a hole on 5 vertices with two non-adjacent vertices replicated).

Proof of Proposition 3.5. $\theta(G) \leqslant \theta(G-Q)+1$ is true because adding Q to any clique cover of $G-Q$ we get a clique cover of G. $\alpha(G) \leqslant \alpha(G-Q)+1$ holds because any stable set meets Q in at most one vertex. The third inequality follows from these first two and the obvious bounds $\alpha(G-Q) \leqslant \alpha(G)$, $\theta(G-Q) \leqslant \theta(G)$. The statement about the chain of induced subgraphs follows by noting that the deletion of a vertex changes the gap by at most 1, in the beginning it is $\text{gap}(G)$, and at the end it is 0.

If G is gap-critical, $\text{gap}(G-Q)=\text{gap}(G)+1$, $\text{gap}(G-Q)=\text{gap}(G)$ cannot occur in the proven inequalities, so the only option is $\text{gap}(G-Q)=\text{gap}(G)-1$, and then $\theta(G-Q)=\theta(G)-1$ and $\alpha(G-Q)=\alpha(G)$. □

A vertex of a graph is *simplicial* if its neighbors induce a complete graph.

Proposition 3.6. *If G is gap-critical, then it has no simplicial vertex.*

Proof. If $v \in V(G)$ is a simplicial vertex, $\alpha(G-N[v])=\alpha(G)-1$, since $S \cup \{v\}$ is a stable set for any stable set S of $G-N[v]$, contradicting Proposition 3.5 for $Q=N[v]$. □

The following generalizes the condition on $N(v)$ if $\alpha \leqslant 2$:

Proposition 3.7. *Let G be a graph such that* $\alpha(G) \leqslant 2$ *and there exists* $v \in V(G)$ *where* $G(N(v))$ *is perfect. Then* $\text{gap}(G) \leqslant 1$.

Proof. Consider $G_1 := G(N[v])$ which is now perfect, and $Q := G-N[v]$ which is a clique because of $\alpha(G) \leqslant 2$. By Proposition 3.5, $0=\text{gap}(G_1)=\text{gap}(G-Q) \geqslant \text{gap}(G)-1$. □

At last we state easy but crucial lower bounds for $s(t)$ and $s_2(t)$, and an interesting relation between these bounds and the equality $s(t)=s_2(t)$.

Proposition 3.8. *If there exists a* $(t+1)$*-extremal graph G with* $\omega(G) \geqslant k$ $(k \in \mathbb{N})$, *then* $s(t+1) \geqslant s(t)+k$, *in particular, for any* $t \in \mathbb{N}$: $s(t+1) \geqslant s(t)+2$.

Proof. Let K be a k-clique in G. By Proposition 3.5, $\text{gap}(G \setminus K)=\text{gap}(G)-1$. So

$$s(t) \leqslant \left|V(G \setminus K)\right| = \left|V(G)\right| - k = s(t+1)-k. \qquad \square$$

We prove three simple but important statements on the relation of s and s_2:

Proposition 3.9. *If* $s(t+1)=s(t)+2$, *then* $s(t)=s_2(t)$, $s(t+1)=s_2(t+1)$.
If $s(t) \neq s_2(t)$ *or* $s(t+1) \neq s_2(t+1)$, *then* $s(t+1) \geqslant s(t)+3$.
If $s(t)=s_2(t)$ *and* $s(t+1) \neq s_2(t+1)$, *then* $s_2(t+1) \geqslant s_2(t)+4$.

Proof. Let G be $t+1$-extremal, and suppose $s(t+1)=s(t)+2$. If G is not triangle-free, by Proposition 3.5, $s(t+1) \geqslant s(t)+3$, so G is triangle-free, and deleting the two endpoints of an edge, the gap decreases by 1, so what we get is t-extremal, and the first statement follows. The second statement is just the indirect reformulation of the first. The third follows by $s_2(t+1) > s(t+1) \geqslant s(t)+3 = s_2(t)+3$, using the preceding inequality. □

3.2. Gaps and matchings

As usual, $\nu(G)$ denotes the size of a maximum matching of G, the maximum number of pairwise disjoint edges; let $\zeta(G)$ denote the minimum number of edges that cover the vertices of G. If G is a triangle-free graph, $\theta(G) = \zeta(G)$. The reader can find in any textbook or check that for connected graphs $\nu(G) + \zeta(G) = n$.

A graph is *factor-critical* if the removal of any vertex yields a graph with a perfect matching. (It is convenient to include graphs of order 1 under this term.) A graph is *bicritical* if deleting any two vertices there is a perfect matching. Clearly, *factor-critical and bicritical graphs are connected*. The following is a simple but ingenious and important result of Gallai [8] (in English in [16] or [15, Exercise 26, p. 58]).

Theorem 3.10. *(See Gallai [8].) If G is connected and $\nu(G \setminus v) = \nu(G)$ for all $v \in V(G)$, then G is factor-critical, and in particular it has an odd number of vertices.*

Proposition 3.11. *If G is a triangle-free and gap-critical graph then every component of G is factor-critical of (odd) order at least* 5.

Proof. Let H be a component of a triangle-free, gap-critical graph. By Proposition 3.1 H is gap-critical. Since H is triangle-free, $\theta(H) = \zeta(H)$ and by Proposition 3.5, for all $v \in V(H)$ we have $\zeta(H \setminus v) = \theta(H \setminus v) = \theta(H) - 1 = \zeta(H) - 1$. So

$$\nu(H \setminus v) = \left|V(H \setminus v)\right| - \zeta(H \setminus v) = \left|V(H)\right| - \zeta(H) = \nu(H),$$

whence H is factor-critical by Theorem 3.10.

If some component is a vertex, deleting that isolated vertex the gap does not decrease. It cannot be a triangle either. □

The following proposition gives a lower bound on the gap and this bound will turn out to be *very sharp*, in fact an equality. The intuition behind it: *in a triangle-free graph G $\theta(G) = \theta(G - v) + 1$ for every vertex $v \in V(G)$* implies $\theta(G) = \lceil \frac{V(G)}{2} \rceil$, which is the smallest possible value in a triangle-free graph. That is, if we want $\theta(G)$ to be largest possible comparing to $\theta(G - v)$, then θ takes its smallest possible value.

Proposition 3.12. *For any triangle-free graph G, $\operatorname{gap}(G) \geqslant \lceil \frac{|V(G)|}{2} \rceil - \alpha(G)$, and for connected triangle-free gap-critical graphs the equality holds. If there exists a triangle-free gap-extremal graph of order n with k components of order $n_1, \ldots, n_k$,*

$$\operatorname{gap}_2(n) = \left\lceil \frac{n_1}{2} \right\rceil - \alpha(n_1) + \cdots + \left\lceil \frac{n_k}{2} \right\rceil - \alpha(n_k).$$

Proof. Since G is triangle-free, $\theta(G) \geqslant \lceil \frac{|V(G)|}{2} \rceil$ so $\operatorname{gap}(G) = \theta(G) - \alpha(G) \geqslant \lceil \frac{|V(G)|}{2} \rceil - \alpha(G)$. If G is gap-critical and connected, by Proposition 3.11 it is factor-critical, so $\theta(G) = \lceil \frac{|V(G)|}{2} \rceil$, settling the first claim. Now if G is triangle-free gap-extremal, then by Proposition 3.1 all of its components are connected gap-critical graphs, and by the already proven assertion, $\operatorname{gap}(G_i) = \lceil \frac{n_i}{2} \rceil - \alpha(G_i)$.

If $\alpha(G_i) > \alpha(n_i)$ then replacing G_i by H_i of the same order n_i, triangle-free, $(\theta(H_i) \geqslant \lceil n_i \rceil)$, and $\alpha(H_i) = \alpha(n_i) < \alpha(G_i)$, the gap increases, contradicting that G_i is gap-extremal. So $\theta(G_i) = \lceil \frac{n_i}{2} \rceil$, $\alpha(G_i) = \alpha(n_i)$, finishing the proof with an application of Proposition 3.1. □

Is the triangle-free condition essential in these statements? For some of the claims it can be dropped! Gallai himself proved in [9]: *If the complement of a k-color-critical graph is connected, it has at least $2k - 1$ vertices.* By Proposition 3.5 the complements of gap-critical graphs are color-critical, so we immediately get:

Table 1
Inequalities for Proposition 3.15.

$40 \leqslant R(3,10) \leqslant 43$
$46 \leqslant R(3,11) \leqslant 51$
$52 \leqslant R(3,12) \leqslant 59$
$59 \leqslant R(3,13) \leqslant 69$
$66 \leqslant R(3,14) \leqslant 78$
$73 \leqslant R(3,15) \leqslant 88$
$79 \leqslant R(3,16)$
$92 \leqslant R(3,17)$
$99 \leqslant R(3,18)$
$106 \leqslant R(3,19)$
$111 \leqslant R(3,20)$
$122 \leqslant R(3,21)$
$125 \leqslant R(3,22)$
$136 \leqslant R(3,23)$
$143 \leqslant R(3,24)$
$153 \leqslant R(3,25)$
$159 \leqslant R(3,26)$
$167 \leqslant R(3,27)$
$172 \leqslant R(3,28)$
$182 \leqslant R(3,29)$

Proposition 3.13. *If G is a connected gap-critical graph, $\theta(G) \leqslant \lceil \frac{|V(G)|}{2} \rceil$.*

Stehlík [19] proved the sharpening of Gallai's general theorem stating that *there exists a coloration where all color classes are of size at least two*, extending Gallai's proof [15,16] of Theorem 3.10 [8]. Despite these promising generalizations, we were not able to make essential use of Proposition 3.13 or prove in any other way that gap-extremal graphs cannot contain a triangle. However, Proposition 3.9, the main results of the paper and further verifications for small t (see Section 5) suggest that it is true:

Conjecture 3.14. *Every gap-extremal graph is triangle-free.*

3.3. Gaps and Ramsey-numbers

Let W_8 be the Wagner' graph [18], a cycle on 8 vertices with its four long chords. Deleting one of these chords we get W_{81} and deleting two neighboring chords we get W_{82}. Let R_{13} be the graph on $\{r_1, \dots, r_{13}\}$ with the following edges: $r_i r_{i+1}$ and $r_i r_{i+5}$, $i = 1, \dots, 13$, where the addition is taken modulo 13. It is well known [18] that R_{13} is the largest graph such that $\omega = 2$ and $\alpha = 4$. Note that $\mathrm{gap}(R_{13}) = 3$.

The following is mostly an extract of [18], except for the lower bounds on $R(3,24), \dots, R(3,29)$ that are from [22]:

Proposition 3.15. *The Ramsey-numbers $R(3,l)$ for values $l = 2,3,4,5,6,7,8,9$ are $3,6,9,14,18,23,28,36$, and the corresponding Ramsey graphs are unique for $l = 2$, $l = 3$ and $l = 5$: K_2, C_5 and R_{13} respectively. For $l = 4$ there are three Ramsey graphs, W_8, W_{81}, W_{82}. Moreover all the inequalities from Table 1 hold.*

$R(4,4) = 18$, and the unique $(4,4)$-Ramsey graph on 17 vertices is a cycle of length 17 with all chords between vertices at distance 2, 4, 8.

The following is a result of Xiaodong, Zheng and Radziszowski [21, Theorem 3] see also [18, 2.3 (g)].

Proposition 3.16. *(See [21].) If $p, q \geqslant 2$, $R(3, p+q-1) \geqslant R(3,p) + R(3,q) + \min\{p,q\} - 2$.*

Proposition 3.17.

(1) $\alpha+1 \geqslant R(3,\alpha+1)-R(3,\alpha) \geqslant 3$ (*provided* $\alpha \geqslant 2$ *for the second inequality*) *and both inequalities are strict if both* $R(3,\alpha)$ *and* $R(3,\alpha+1)$ *are even.*
(2) $R(3,\alpha+2)-R(3,\alpha) \geqslant 7$ *provided* $\alpha \geqslant 3$.
(3) $R(3,\alpha+3)-R(3,\alpha) \geqslant 11$ *provided* $\alpha \geqslant 2$.
(4) $R(3,\alpha+4)-R(3,\alpha) \geqslant 17$ *provided* $\alpha \geqslant 3$.
(5) $R(3,\alpha+k)-R(3,\alpha) \geqslant R(3,k+1)+k-1$, *if* $\alpha \geqslant k+1 \geqslant 3$.
(6) *The right hand side of* (5) *for* $\alpha \geqslant 3$ *and* $k=5,6,7$ *are*: 22, 28, 34.
(7) *The right hand side of* (5) *for* $\alpha \geqslant 4$, $k=8,9,10,11,12,13$ *are*: 43, 48, 55, 62, 70, 78.
(8) $R(3,\alpha+14)-R(3,\alpha) \geqslant 86$, *if* $\alpha \geqslant 3$.

Proof. First, we prove (1): The upper bound is the easy and most well-known upper bound $R(3,\alpha+1) \leqslant R(3,\alpha)+R(2,\alpha+1)$ [15], where the equality does not hold if both terms on the right hand side are even, and where of course $R(2,\alpha+1)=\alpha+1$ (to see this, start the usual induction with a vertex of even degree). Since equality would imply that $R(2,\alpha+1)=\alpha+1$ is even too (that is, α is odd), we have the assertion concerning the upper bound. The lower bound of (1) is a result in [3] and also a special case of Proposition 3.16 by substituting $q=2$ and $R(3,2)=3$.

Second, we check (2) by substituting $p=\alpha \geqslant 3$, $q=3$ and $R(3,3)=6$ into Proposition 3.16. Third, substituting $p=\alpha \geqslant 4$, $q=4$ and $R(3,4)=9$ into Proposition 3.16 provides (3) for $\alpha \geqslant 4$, and for $\alpha=2,3$ it can be checked in Proposition 3.15. (4) for $\alpha \geqslant 5$ is a specialization, and can be checked directly in Table I for $\alpha=3,4$, (5) is just a rewriting of Proposition 3.16.

Finally, if we specialize (5) to $k=5,\dots,14$, we get (6), (7), (8) for $\alpha \geqslant 6,\dots,\alpha \geqslant 15$, respectively. For $\alpha=3,\dots,9$ we still get the inequalities from [18, Tables II and I], for the lower bounds are provided until $l=23$, and the upper bounds until $l=15$: for instance, $R(3,23) \geqslant 136$, $R,(3,9)=36$, so $R(3,23)-R,(3,9) \geqslant 100$. For the lower bounds concerning the highest arguments we have to rely on upper bounds [22] copied into Proposition 3.17. The inequalities with the largest values that we have to check are $R(3,\alpha+14)-R(3,\alpha) \geqslant 86$, for $\alpha=4,\dots,14$. (For $\alpha \geqslant 15$ we have from (5) and substituting $R(3,15) \geqslant 73$ from Table I [18] $R(3,\alpha+14)-R(3,\alpha) \geqslant R(3,15)+13 \geqslant 86$.) We make the last checking, for $\alpha=14$: $R(3,28)-R(3,14) \geqslant 86$. Indeed, from Proposition 3.15 $R(3,28) \geqslant 172$ (copied from [22]) and $R(3,14) \leqslant 78$ (from Proposition 3.15), so in fact $R(3,28)-R(3,14) \geqslant 94 \geqslant 86$. □

If $R(3,\alpha+1)-R(3,\alpha)=3$, we will say that $R(3,\alpha)$, $R(3,\alpha+1)$ are *twins*.

Proposition 3.18. $\mathrm{gap}_2(n) \geqslant \lceil \frac{n}{2} \rceil - \alpha(n)$.

Proof. Indeed, by Proposition 3.12 for any triangle-free graph G on n vertices $\mathrm{gap}_2(n) \geqslant \mathrm{gap}(G) \geqslant \lceil \frac{n}{2} \rceil - \alpha(G)$, and if we apply this to a triangle-free graph G with $\alpha(G)=\alpha(n)$ we get the claim. □

We will now need to deduce conditions on the equality in Proposition 3.4. These computations will enable us to conclude that there exist stable gap-optimal graphs of any order $n \in \mathbb{N}$, and this will be crucial for our formulas describing the gap. A combination of the inequalities of Proposition 3.16 and the upper bound of Proposition 3.17 (1) yield the following characterization of the equality in Proposition 3.4 that will be crucial for describing the gap-function, through Ramsey-perfect numbers.

Theorem 3.19. *Let* $n, n_1, n_2, n_3 \in \mathbb{N}$. *Equality in* $\alpha(n_1+n_2) \leqslant \alpha(n_1)+\alpha(n_2)$ *implies that there exist* $\varepsilon, \varepsilon_1, \varepsilon_2$ *such that* $n_1+n_2-\varepsilon$, $n_1+1+\varepsilon_1$, $n_2+1+\varepsilon_2$ *are all Ramsey-numbers, and* $\varepsilon, \varepsilon_1, \varepsilon_2 \in \{0,1\}$, $\varepsilon+\varepsilon_1+\varepsilon_2 \leqslant 1$.
Furthermore if $n_i \geqslant 3$ *for* $i=1,2,3$ *then* $\alpha(n_1+n_2+n_3) < \alpha(n_1)+\alpha(n_2)+\alpha(n_3)$.

In the last, strict inequality the condition is necessary: $\alpha(6)=3=3\alpha(2)$; if say $n_3=2$, then $n := n_1+n_2+n_3$ may be a Ramsey-number, $n-3$ its twin, and $n-2$ could be Ramsey-perfect. However, luckily, we are interested in these equalities only if the numbers n_1, n_2, n_3 are odd, and then a stronger inequality holds.

Note that even in the first part of the theorem, $\alpha(n_1+n_2) = \alpha(n_1) + \alpha(n_2)$ with $n_2 = 1$ can be useful. This holds if and only if $n_1 + 1$ is a Ramsey-number. If in addition $n_1 + 1$ is even, a Ramsey graph on n_1 vertices and an isolated vertex provides the maximum gap (Theorem 4.1).

Proof of Theorem 3.19. We reprove the easy inequality $\alpha(n_1+n_2) \leqslant \alpha(n_1) + \alpha(n_2)$ (see Proposition 3.4) in a complicated way, in order to deduce the conditions of equality. Set $\alpha_i = \alpha(n_i)$. Then $n_i \leqslant R(3, \alpha_i + 1) - 1$ $(i = 1, 2)$.

Lemma 3.20. *For arbitrary $\alpha_1, \alpha_2 \in \mathbb{N}$*

$$R(3, \alpha_1 + 1) - 1 + R(3, \alpha_2 + 1) - 1 \leqslant R(3, \alpha_1 + \alpha_2) + 1, \quad \textit{and} \tag{9}$$

equality implies that Proposition 3.17 (1) (first part) holds with equality for the smaller of α_1, α_2.

Proof. By symmetry we may suppose $\alpha_1 \geqslant \alpha_2$.

If $\alpha_2 = 1$ then (9) and Proposition 3.17 (1) (first part) are equalities. If $\alpha_2 \geqslant 2$ we can substitute $p = \alpha_1 + 1$, $q = \alpha_2$ into Proposition 3.16 and add 1 to both sides:

$$R(3, \alpha_1 + 1) - 1 + R(3, \alpha_2) - 1 + \alpha_2 + 1 \leqslant R(3, \alpha_1 + \alpha_2) + 1. \tag{10}$$

Applying Proposition 3.17 (1) to α_2,

$$R(3, \alpha_2) + \alpha_2 + 1 \geqslant R(3, \alpha_2 + 1), \tag{11}$$

and (10), (11) gives lemma (together with the remark on equality). □

From the definitions and from Lemma 3.20, $n_1 + n_2 \leqslant R(3, \alpha_1 + 1) - 1 + R(3, \alpha_2 + 1) - 1 \leqslant R(3, \alpha_1 + \alpha_2) + 1$, from where we indeed can read $\alpha(n_1 + n_2) \leqslant \alpha_1 + \alpha_2$, and the equality holds if and only if

$$R(3, \alpha_1 + \alpha_2) \leqslant n_1 + n_2 \leqslant R(3, \alpha_1 + \alpha_2) + 1.$$

These inequalities allow at most one of n_1 or n_2 be one less than $R(3, \alpha_1 + 1) - 1$ or $R(3, \alpha_2 + 1) - 1$ respectively, that is, $\varepsilon_1 + \varepsilon_2 \leqslant 1$, and in case of equality, $n_1 + n_2 = R(3, \alpha_1 + \alpha_2)$, that is, $\varepsilon = 0$.

Next we prove the second part of Theorem 3.19, the strict inequality when n is decomposed into three numbers. We could apply Lemma 3.20 twice and each time the conditions for the equality in it, but then the result we get would be too weak. We repeat the proof, applying Proposition 3.16 directly, twice, choosing its arguments carefully:

Lemma 3.21. *For arbitrary natural numbers $\alpha_1 \geqslant \alpha_2 \geqslant \alpha_3 \geqslant 2$,*

$$R(3, \alpha_1 + 1) - 1 + R(3, \alpha_2 + 1) - 1 + R(3, \alpha_3 + 1) - 1 \leqslant R(3, \alpha_1 + \alpha_2 + \alpha_3 - 1) + 2. \tag{12}$$

Lemma 3.21 concludes the proof of Theorem 3.19 since $n_1 + n_2 + n_3$ is less than or equal to the left hand side of (12). Since $n_i \geqslant 3$ implies $\alpha_i \geqslant 2$, Lemma 3.21 shows that $n_1 + n_2 + n_3$ is also bounded from above by the right hand side of (12). Then, because of Proposition 3.17 (1) (second inequality providing the lower bound 3), the right hand side can be upper bounded by $R(3, \alpha_1 + \alpha_2 + \alpha_3) - 1$, proving that $\alpha(n_1 + n_2 + n_3) \leqslant \alpha_1 + \alpha_2 + \alpha_3 - 1$, showing the claimed strict inequality of Theorem 3.19. □

Proof of Lemma 3.21. Apply the upper bound of Proposition 3.17 (1) to get that the left hand side is less than or equal to

$$R(3,\alpha_1+1)-1+\bigl(R(3,\alpha_2)+\alpha_2+R(3,\alpha_3)+\alpha_3\bigr), \tag{13}$$

where the sum in the parentheses can in turn be bounded according to Proposition 3.16:

$$R(3,\alpha_2)+R(3,\alpha_3)+\alpha_2+\alpha_3 \leqslant R(3,\alpha_2+\alpha_3-1)-(\alpha_3-2)+\alpha_2+\alpha_3. \tag{14}$$

Substituting this to (13) and applying Proposition 3.16 again to the result,

$$\begin{aligned}(13) &\leqslant R(3,\alpha_1+1)+R(3,\alpha_2+\alpha_3-1)+\alpha_2+1\\ &\leqslant R(3,\alpha_1+\alpha_2+\alpha_3-1)-(\alpha_2+1-2)+\alpha_2+1,\end{aligned}$$

after noting that $\alpha_2+1\leqslant \min\{\alpha_1+1,\alpha_2+\alpha_3-1\}$. □

Corollary 3.22. *A number $n\in\mathbb{N}$ is Ramsey-perfect if and only if there exist $\alpha_1\geqslant\alpha_2\geqslant 2$ that satisfy $n=R(3,\alpha_1+\alpha_2)+1=R(3,\alpha_1+1)-1+R(3,\alpha_2+1)-1$, where $R(3,\alpha_i+1)$ is even $(i=1,2)$. Moreover, then the equality holds in* (10), (11).

Proof. Indeed, if n is Ramsey-perfect, let $n_1,n_2\geqslant 5$ be odd numbers such that $n=n_1+n_2$, $\alpha(n)=\alpha(n_1)+\alpha(n_2)$. Since n_1 and n_2 satisfy the condition of Theorem 3.19, the theorem can be applied. Denote $\alpha:=\alpha(n)$, $\alpha_1:=\alpha(n_1)\geqslant 2$, $\alpha_2:=\alpha(n_2)\geqslant 2$. Since n is not a Ramsey-number, $\varepsilon=1$, and then $\varepsilon_1=\varepsilon_2=0$. In other words $n=R(3,\alpha)+1$, $n_i=R(3,\alpha_i+1)-1$ are odd, $(i=1,2)$, $n=n_1+n_2$, $\alpha=\alpha_1+\alpha_2$, showing the assertion. Moreover, Lemma 3.20 is satisfied with equality, whence (10), (11) as well. Conversely, if the equality and the parity condition are satisfied with $\alpha_1,\alpha_2\geqslant 2$, then defining, $n_1:=R(3,\alpha_1+1)-1$, $n_2:=R(3,\alpha_1+1)-1$ we see that $n=R(3,\alpha_1+\alpha_2)$ is Ramsey-perfect. □

The lack of other examples of twins or other Ramsey-perfect numbers is not really surprising: only the first nine Ramsey values are known. Yet we believe that all the applied inequalities cannot be tight for arbitrary large Ramsey-numbers, so we state two conjectures:

Conjecture 3.23. *The natural number n is Ramsey-perfect if and only if n is even and $n-1$ is the bigger of Ramsey twins.*

Conjecture 3.24. *The only Ramsey twins are* $\{3,6\}$ *and* $\{6,9\}$.

Corollary 3.25. *Let G be triangle-free-extremal with a minimum number of components. Then G has at most two components, and two if and only if $n:=|V(G)|$ is Ramsey-perfect, when*

$$\operatorname{gap}(G)=\lceil n/2\rceil-\alpha(n)+1,$$

otherwise n is odd, G is connected, and

$$\operatorname{gap}(G)=\lceil n/2\rceil-\alpha(n).$$

In both cases the triangle-free-extremal graphs are stable gap-optimal, and in the second case any triangle-free graph on n vertices and stability number $\alpha(n)$ is stable gap-optimal.

Proof. Let G be a triangle-free-t-extremal graph with a minimum number of components, $t\in\mathbb{N}$, and let $G_1,\dots,G_k$ be its components, of order $n_1,\dots,n_k$, $n:=|V(G)|=n_1+\cdots+n_k$. According to Proposition 3.11 all the components are factor-critical, in particular all the n_i are odd, $\theta(G_i)=\lceil n_i/2\rceil$ $(i=1,\dots,k)$, and by Proposition 3.12,

$$\operatorname{gap}(G)=\lceil n_1/2\rceil-\alpha(n_1)+\cdots+\lceil n_k/2\rceil-\alpha(n_k). \tag{15}$$

It follows now from Theorem 3.19 that $k \leqslant 2$, because otherwise three components can be replaced by one, contradicting the choice of G:

$$\left\lceil \frac{n_1+n_2+n_3}{2} \right\rceil \geqslant \lceil n_1/2 \rceil + \lceil n_2/2 \rceil + \lceil n_3/2 \rceil - 1,$$
$$\alpha(n_1+n_2+n_3) \leqslant \alpha(n_1)+\alpha(n_2)+\alpha(n_3)-1.$$

Two components can also be replaced by just one, unless the equality is satisfied in both of the following inequalities:

$$\left\lceil \frac{n_1+n_2}{2} \right\rceil \geqslant \lceil n_1/2 \rceil + \lceil n_2/2 \rceil - 1, \qquad \alpha(n_1+n_2) \leqslant \alpha(n_1)+\alpha(n_2).$$

So $k=1$, or $k=2$, and then (15) specializes to the claimed formula, since for $k=2$

$$\operatorname{gap}(G) = \lceil n_1/2 \rceil - \alpha(n_1) + \lceil n_2/2 \rceil - \alpha(n_2) = \left\lceil \frac{n_1+n_2}{2} \right\rceil + 1 - \alpha(n_1+n_2),$$

and this happens if and only if n is Ramsey-perfect.

In both cases G is stable gap-optimal, and conversely, if $n = s_2(t)$ is neither an even Ramsey-number nor Ramsey-perfect, then according to Proposition 3.12 every graph H on n vertices and stability number $\alpha(n)$ satisfies: $\operatorname{gap}(G) \geqslant \lceil n/2 \rceil - \alpha(n) = \operatorname{gap}_2(n)$, so there is equality throughout, and G is stable gap-optimal. □

4. Finding the gap with constant error

In this section we first determine the functions $\operatorname{gap}_2(n)$ and $s_2(t)$ exactly, and then the functions $\operatorname{gap}(n)$ and $s(t)$ with small errors (2 and 10 respectively), moreover we prove that the error may occur only after Ramsey-numbers on an interval of length 13.

4.1. Finding the triangle-free gap

Recall that $\operatorname{gap}_2(n)$ is the maximum of the gap of a triangle-free graph of order n, and $s_2(t)$ denotes the minimum order of a triangle-free graph of gap t. The main result of this section is a simple formula for these functions if the inverse Ramsey-numbers $\alpha(n)$ are used as black boxes.

Theorem 4.1. $\operatorname{gap}_2(n) = \lceil n/2 \rceil - \alpha(n) + \varepsilon(n)$, *where* $\varepsilon(n) = 1$ *if* n *is an even Ramsey-number, or if it is Ramsey-perfect, and* 0 *otherwise.*

Proof. Let $f(n) := \lceil n/2 \rceil - \alpha(n) + \varepsilon(n)$.

Claim 1. $\operatorname{gap}_2(n) \geqslant f(n)$ *for all* $n \in \mathbb{N}$.

Indeed, if n is neither an even Ramsey-number nor Ramsey-perfect, this is just Proposition 3.18. If n is an even Ramsey-number, then $\alpha(n-1) = \alpha(n) - 1$ and $\lceil \frac{n-1}{2} \rceil = \lceil \frac{n}{2} \rceil$, so by the monotonicity of gap_2 (see Proposition 3.2):

$$\operatorname{gap}_2(n) \geqslant \operatorname{gap}_2(n-1) \geqslant f(n-1) = \lceil n/2 \rceil - \alpha(n) + 1.$$

More generally, if $n = n_1 + n_2$ where n_1, n_2 are odd numbers and $\alpha(n) = \alpha(n_1) + \alpha(n_2)$, then $\lceil \frac{n}{2} \rceil + 1 = \lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2}{2} \rceil$, and applying Proposition 3.3 and then Proposition 3.18:

$$\operatorname{gap}_2(n) \geqslant \operatorname{gap}_2(n_1) + \operatorname{gap}_2(n_2) \geqslant \left\lceil \frac{n_1}{2} \right\rceil - \alpha(n_1) + \left\lceil \frac{n_2}{2} \right\rceil - \alpha(n_2) = \lceil n/2 \rceil - \alpha(n) + 1.$$

Corollary 3.25 establishes the theorem for the values $n = s_2(t)$ $(t = 1, 2, \ldots)$, thus we get

Claim 2. *If $n = s_2(t)$ for some $t \in \mathbb{N}$, then $\mathrm{gap}_2(n) = f(n)$.*

Claim 3. *The function $f(n)$ is monotone increasing.*

Indeed, since $\lceil n/2 \rceil$ is a monotone increasing function, we have $f(n+1) \geqslant f(n)$ unless $\alpha(n)$ is increasing, or unless $\varepsilon(n)$ is decreasing when n grows to $n+1$. We prove that in both of these less trivial events actually $f(n+1) = f(n)$:

Assume first that $\alpha(n+1) > \alpha(n)$. Then $\alpha(n+1) = \alpha(n) + 1$, that is, $n+1$ is the Ramsey-number $R(3, \alpha(n) + 1)$. If in addition n is even, $\lceil \frac{n+1}{2} \rceil = \lceil \frac{n}{2} \rceil + 1$, and $\varepsilon(n+1) = 0 = \varepsilon(n)$ since $n+1$ is an odd Ramsey-number, so neither n nor $n+1$ is an even Ramsey-number or Ramsey-perfect by Theorem 3.19. So

$$f(n+1) = \left\lceil \frac{n+1}{2} \right\rceil - \alpha(n+1) + \varepsilon(n+1) = \left\lceil \frac{n}{2} \right\rceil + 1 - \bigl(\alpha(n) + 1\bigr) + \varepsilon(n) + 0 = f(n).$$

If n is odd – and still $\alpha(n+1) > \alpha(n)$ –, then $\lceil \frac{n+1}{2} \rceil = \lceil \frac{n}{2} \rceil$, but then $n+1$ is an even Ramsey-number, so

$$f(n+1) = \left\lceil \frac{n+1}{2} \right\rceil - \alpha(n+1) + \varepsilon(n+1) = \left\lceil \frac{n}{2} \right\rceil - \bigl(\alpha(n) + 1\bigr) + \varepsilon(n) + 1 = f(n).$$

Second, assume that $\alpha(n+1) = \alpha(n)$, but $\varepsilon(n+1) = \varepsilon(n) - 1$. Then $\varepsilon(n) = 1$, so n is even, and therefore $\lceil \frac{n+1}{2} \rceil = \lceil \frac{n}{2} \rceil + 1$, so again $f(n+1) = f(n)$ proving the claim.

To finish the proof of the theorem, suppose for a contradiction that $\mathrm{gap}_2 \neq f$. Let x be the smallest integer x for which $t := \mathrm{gap}_2(x) \neq f(x)$. By Claim 1, $\mathrm{gap}_2(x) > f(x)$. Then, by Claim 3, we have for all $y \leqslant x$: $t = \mathrm{gap}_2(x) > f(x) \geqslant f(y) = \mathrm{gap}_2(y)$ by the minimality of x. So $s_2(t) = x$, and then, by Claim 2, $\mathrm{gap}_2(x) = f(x)$, a contradiction that proves the theorem. □

Corollary 4.2. *For all $\alpha \in \mathbb{N}$, $\mathrm{gap}_2(R(3, \alpha)) = \lceil \frac{R(3,\alpha)+1}{2} \rceil - \alpha = \mathrm{gap}_2(R(3, \alpha) - 1)$, in particular, Ramsey-numbers are not in the image of the function s_2.*

Proof. If n is even, $\varepsilon(n) = 1$, so $\lceil n/2 \rceil - \alpha + \varepsilon(n) = \lceil \frac{n+1}{2} \rceil - \alpha$. If n is odd, $\varepsilon(n) = 0$ and $\lceil \frac{n}{2} \rceil = \lceil \frac{n+1}{2} \rceil$, so again $\lceil n/2 \rceil - \alpha + \varepsilon(n) = \lceil \frac{n+1}{2} \rceil - \alpha$. In both cases $\mathrm{gap}_2(n) = \mathrm{gap}_2(n-1)$, so $n \neq s_2(t)$ for any t. □

Corollary 4.3. *For every $\alpha \in \mathbb{N}$ for which $R(3, \alpha + 1) - R(3, \alpha) \geqslant 4$, exactly the odd numbers of the interval $[R(3, \alpha) + 3, R(3, \alpha + 1) - 1]$ are the values of the function $s_2(t)$, for $t = \mathrm{gap}_2(R(3, \alpha)) + 1, \ldots, \mathrm{gap}_2(R(3, \alpha + 1)) - 1$.*

Proof. This is an immediate consequence of Theorem 4.1, since for the integers n of the given interval both $\alpha(n)$ and $\varepsilon(n)$ are constant, and $\lceil n/2 \rceil$ increases exactly on odd numbers. □

Corollary 4.4. *For every n there exists a stable gap-optimal graph G, defined from an arbitrary $(3, \alpha + 1)$-Ramsey graph G_α $(\alpha = 1, 2, \ldots)$:*

- *If $n \in [R(3, \alpha) + 2, R(3, \alpha + 1) - 1]$ or if $n = R(3, \alpha) + 1$ is not Ramsey-perfect or if $n = R(3, \alpha)$ is odd, let G be an arbitrary, order n induced subgraph of G_α.*
- *If n is Ramsey-perfect, $n = R(3, \alpha) + 1 = n_1 + n_2$, $n_i := R(3, \alpha_i + 1) - 1$ is odd $(i = 1, 2)$, $\alpha = \alpha_1 + \alpha_2$, then let G consist of two components: G_{α_1} and G_{α_2}.*
- *If $n = R(3, \alpha)$ is even, let G consist of $G_{\alpha - 1}$ and an isolated vertex.*

If n or $n - 1$ is equal to $R(3, \alpha)$ then G is not necessarily connected, but otherwise every stable gap-optimal graph is connected.

For $n = 6$ the only stable gap-optimal graph is C_5 and an isolated vertex. For $n = 7$ and any number $R(3,\alpha) + 1$ which is not Ramsey-perfect, a graph having two components, a Ramsey graph and a K_2 is stable gap-optimal, and may actually coincide with G_α.

Proof of Corollary 4.4. In the first case $\mathrm{gap}(G) = \lceil n/2 \rceil - \alpha(n) = \mathrm{gap}_2(n)$ according to Theorem 4.1 G is indeed stable gap-optimal.

In the second and third cases, if $n = R(3,\alpha) + 1$ or $n = R(3,\alpha)$, the defined graphs are readily stable gap optimal, and so are the graphs of the remark before the proof if $n = R(3,\alpha) + 1$ but n is not Ramsey-perfect. If n is neither of these two numbers, it cannot be written as the sum of two nonzero numbers whose inverse Ramsey-numbers sum up to $\alpha(n)$ (see Theorem 3.19), so the defined stable gap-optimal G is connected. $\square$

Corollary 4.5. *$(3,\alpha+1)$-Ramsey graphs are $(R(3,\alpha+1) - R(3,\alpha) - 3)$-connected, moreover, deleting at most $R(3,\alpha+1) - R(3,\alpha) - 3$ vertices, the remaining $n \geqslant R(3,\alpha) + 2$ vertices, if n is even, induce a graph with a perfect matching.*

Proof. Apply Corollary 4.4 to odd $n \in [R(3,\alpha)+3, R(3,\alpha+1)-1]$: any induced subgraph G of G_α on n vertices has optimal gap. Fix this G and have a look at Theorem 3.19: $\varepsilon(n) = \varepsilon(n-1) = 0$, and we see that the jump-points of the function $\mathrm{gap}_2(n) = \lceil n/2 \rceil - \alpha$, that is the values of the function $s_2(t)$ on the considered interval are exactly the odd numbers. So G is a triangle-free-extremal graph, and either by Corollary 4.4 or by Corollary 3.25 it is connected, and by Proposition 3.11 it is factor-critical, and the graphs in the assertion arise by deleting a vertex in such a graph. $\square$

Corollary 4.6. *Order n induced subgraphs of $(3,\alpha+1)$-Ramsey graphs induce a factor-critical graph if $n \geqslant R(3,\alpha) + 3$ is odd, and a bicritical graph if $n \geqslant R(3,\alpha) + 4$ is even.*

Indeed, this corollary is an immediate consequence of Corollary 4.5.

We now determine the recurrence relations for the function s_2. Why? Doesn't Theorem 4.1 tell us all we need? Indeed, it does already tell the most important information, the following theorem and its proof are secondary, the reader can skip it at first reading. However, besides an automatic conversion of Theorem 4.1 from the gap_2 function to s_2, it also has a new content: it shows that for a Ramsey-perfect number n, the interval $[n, n+3]$ cannot contain a Ramsey-number again. Besides making the formulas simpler (at the price of a slightly more difficult proof), it reveals some interesting relations between the distance of consecutive Ramsey-numbers and Ramsey-perfectness.

Corollary 4.7. *For all t, $s_2(t)$ is odd or Ramsey-perfect. Moreover, the function s_2 is determined by the following recursive relations*:

1 *If neither $s_2(t) + 1$, nor $s_2(t) + 2$ are Ramsey, then*:
  - 1.1 *$s_2(t+1) = s_2(t) + 2$ if $s_2(t)$ is not Ramsey-perfect.*
  - 1.2 *$s_2(t+1) = s_2(t) + 3$ if $s_2(t)$ is Ramsey-perfect, and $s_2(t) + 3$ is not Ramsey.*
  - 1.3 *$s_2(t+1) = s_2(t) + 4$ if $s_2(t)$ is Ramsey-perfect, $s_2(t) + 3$ is Ramsey, moreover $s_2(t) + 4$ is Ramsey-perfect.*
  - 1.4 *$s_2(t+1) = s_2(t) + 5$ if $s_2(t)$ is Ramsey-perfect, $s_2(t) + 3$ is Ramsey, but $s_2(t) + 4$ is not Ramsey-perfect.*

2 *If either $s_2(t) + 1$ or $s_2(t) + 2$ are Ramsey, then*:
  - 2.1 *$s_2(t+1) = s_2(t) + 3$, if $s_2(t) + 3$ is Ramsey-perfect.*
  - 2.2 *$s_2(t+1) = s_2(t) + 4$ otherwise, except if $s_2(t) + 4$ is Ramsey.*
  - 2.3 *$s_2(t+1) = s_2(t) + 5$, if $s_2(t) + 4$ is Ramsey.*

Proof. Let $n = s_2(t)$ then by definition, $\mathrm{gap}_2(n) > \mathrm{gap}_2(n-1)$, and let $\alpha := \alpha(n)$, $\varepsilon = \varepsilon(n)$. Suppose that n is odd, or Ramsey-perfect. We will show that the recursive relations 1.1–2.3 hold, and $s_2(t+1)$ is also odd or Ramsey-perfect.

1.1: If neither $n+1$, nor $n+2$ are Ramsey-numbers, and n is not Ramsey-perfect, then by assumption n is odd and α, ε are constant in the interval $[n, n+2]$. Therefore by Theorem 4.1 $\lceil\frac{n}{2}\rceil = \lceil\frac{n+1}{2}\rceil < \lceil\frac{n+2}{2}\rceil$, so 1.1 holds.

1.2: If $n = s_2(t)$ is Ramsey-perfect then according to Corollary 3.22 there exist $\alpha, \alpha_1, \alpha_2 \in \mathbb{N}$ such that

$$(1) \qquad n = R(3,\alpha) + 1 = R(3,\alpha_1+1) - 1 + R(3,\alpha_2+1) - 1, \quad n \text{ is even.}$$

According to Theorem 4.1, $\text{gap}_2(n) = \text{gap}_2(n+1) = \text{gap}_2(n+2)$, since while the ceiling increases by 1, ε decreases by 1. Now $\text{gap}_2(n+3) = \text{gap}_2(n) + 1$ unless $n+3$ is a Ramsey-number again, and 1.2 is checked.

1.3: If $n+3$ is a Ramsey-number (and otherwise the same condition holds as in 1.2), then in addition to $\text{gap}_2(n) = \text{gap}_2(n+1) = \text{gap}_2(n+2)$ we have $\text{gap}_2(n+2) = \text{gap}_2(n+3)$, since both θ and α have increased. However, $n+4$ may or may not be Ramsey-perfect, and in the former case $\text{gap}_2(n+4) = \text{gap}_2(n) + 1$, that is, $s(t+1) = n+1$, as claimed.

1.4: In case $n+4$ is not Ramsey-perfect (and otherwise the same condition holds as in 1.2) $\text{gap}_2(n+3) = \text{gap}_2(n+4)$ and $n+4$ is even, so θ, α remain the same as for $n+3$. However, $n+5$ is odd, and cannot be Ramsey again since $n+3$ is Ramsey; θ increases, but α does not: $\text{gap}_2(n+5) > \text{gap}_2(n) = \text{gap}_2(n+4)$, so $s_2(t+1) = n+5$, as claimed.

2.1: If $n+3$ is Ramsey-perfect then $n+2$ is an odd Ramsey-number, and by Theorem 4.1 we have by parity, and because of $\varepsilon(n) = \varepsilon(n+1) = \varepsilon(n+2) = 0$, $\varepsilon(n+3) = 1$, $\alpha(n) = \alpha(n+1) = \alpha$, $\alpha(n+2) = \alpha(n+3) = \alpha + 1$: $\text{gap}_2(n) = \text{gap}_2(n+1) = \text{gap}_2(n+2) < \text{gap}_2(n+3)$ as claimed.

2.2: If the same hold but $n+3$ is not Ramsey-perfect, then all the relations of 2.1 hold except that we have now $\varepsilon(n+3) = 0$, and therefore $\text{gap}_2(n) = \text{gap}_2(n+1) = \text{gap}_2(n+2) = \text{gap}_2(n+3) < \text{gap}_2(n+4)$, where $n+4$ is indeed odd.

2.3: If $n+1$ or $n+2$ is a Ramsey-number, and $n+4$ is a Ramsey-number again, then n is odd, $n+1$ and $n+4$ are twins. So $n+1$ is an even Ramsey-number, and $\alpha(n+1) = \alpha(n) + 1$, $\varepsilon(n+1) = \varepsilon(n) + 1 = 1$ compensate one another, so Theorem 4.1 gives this time $\text{gap}_2(n) = \text{gap}_2(n+1) = \text{gap}_2(n+2) = \text{gap}_2(n+3) = \text{gap}_2(n+4) < \text{gap}_2(n+5)$. Note that $s_2(t+1) = n+5$ is even in this case, in accordance with the fact that $n+5$ is Ramsey-perfect because of $\alpha(n+5) = \alpha(n) + \alpha(5) = \alpha + 2$. $\square$

Corollary 4.7 gives concrete values of $s_2(i)$ for $i < 12$, because we do not know whether 40 or 41 is a Ramsey-number.

Corollary 4.8. *The values of* $s_2(i)$, $i = 1, \ldots, 11$ *are* $5, 10, 13, 17, 21, 25, 29, 31, 33, 35, 39$.

In fact, we will prove $s_2(i) = s(i)$ almost everywhere, and we conjecture it is true everywhere. This is a slightly weaker conjecture than Conjecture 3.14.

Conjecture 4.9. $\text{gap}(n) = \text{gap}_2(n)$ *for all* $n \in \mathbb{N}$, *and* $s(t) = s_2(t)$ *for all* $t \in \mathbb{N}$.

In the next section we show that the possible exceptions to this conjecture are at constant distance from Ramsey-numbers, and at any such place the difference of the function value from the "usual" $\lceil n/2\rceil - \alpha(n)$ is also a small constant.

4.2. Bounding the gap-function

The first assertion of the following lemma states that once the relation $s(t) = s_2(t)$ holds, it surely holds again and again (together with the equivalent equality $\text{gap}(t) = \text{gap}_2(t)$) until the next Ramsey-number; the second assertion ensures that the relation $s(t) = s_2(t)$ holds again after exceptions restricted to a small interval (of size at most 29) after each Ramsey-number.

Lemma 4.10. *Assume $R(3,\alpha) \leqslant s(t) = s_2(t) < R(3,\alpha+1)$. Then:*

- *For all $t' \in \mathbb{N}$ such that $s(t) \leqslant s(t') \leqslant R(3,\alpha+1)$: $s(t') = s_2(t')$.*
- *There exists $t' \in \mathbb{N}$, $t < t' \leqslant t+29$ such that*

$$s(t) = n < R(3,\alpha+1) < s(t') = s_2(t') \leqslant R(3,\alpha+1) + 85 \leqslant R(3,\alpha+15) - 1.$$

Proof. Let us first prove the first assertion. Suppose that $s(t') \neq s_2(t')$ for some t, t' such that

$$R(3,\alpha) \leqslant s(t) = s_2(t) < s_2(t') \leqslant R(3,\alpha+1), \tag{2}$$

and t' is smallest possible under (2). Clearly, $t' = t+1$. Since $s(t') \neq s_2(t')$ but $s(t) = s_2(t)$, by the third part of Proposition 3.9, $s_2(t) + 4 \leqslant s_2(t')$. This implies that neither $s_2(t)+1$, nor $s_2(t)+2$ is a Ramsey-number, thus $s_2(t')$ is defined from $s_2(t)$ in Case 1 (1.1, 1.2, 1.3 or 1.4) of Corollary 4.7. This cannot happen in 1.3 or in 1.4 because $s_2(t) + 3 < s_2(t) \leqslant R(3,\alpha+1)$ so $s_2(t) + 3$ cannot be a Ramsey-number. But it cannot happen in 1.1 or in 1.2 either because there $s_2(t') \leqslant s_2(t) + 3$, contradicting $s_2(t) + 4 \leqslant s_2(t')$ and finishing the proof.

Now to prove the second assertion, let $T := \max\{t' \colon\ s_2(t') < R(3,\alpha+1)\}$. By the condition of the theorem, and the proven first part $n = s(T) = s_2(T)$. Because of Corollary 4.7 part 1.1,

$$s(T) \geqslant R(3,\alpha+1) - 2, \quad \text{and} \quad T = \operatorname{gap}(n) = \operatorname{gap}\bigl(R(3,\alpha+1)\bigr).$$

Suppose for a contradiction that $s(T+i) \neq s_2(T+i)$ $(i = 1,\dots,k)$.

By the second part of Proposition 3.9 $s(T+i) \geqslant s(T+i-1) + 3$, so $s(T+i) \geqslant s(T) + 3i \geqslant R(3,\alpha+1) - 2 + 3i$ $(i = 1,\dots,k)$.

Claim. $k \leqslant 29$.

Indeed, otherwise $s(t+29) \geqslant s(t) + 3 \times 29 \geqslant R(3,\alpha+1) - 2 + 87 = R(3,\alpha+1) + 85$. On the other hand, by Proposition 3.17 (8) $R(3,\alpha+1) + 85 \leqslant R(3,\alpha+15) - 1$, so by Proposition 3.18, and then applying Corollary 4.2:

$$\begin{aligned}
\operatorname{gap}_2\bigl(R(3,\alpha+1)+85\bigr) &\geqslant \left\lceil \frac{R(3,\alpha+1)+85}{2} \right\rceil - (\alpha+14) \\
&\geqslant \left\lceil \frac{R(3,\alpha+1)+1}{2} \right\rceil + 42 - (\alpha+14) \\
&= \left\lceil \frac{R(3,\alpha+1)+1}{2} \right\rceil + 42 - (\alpha+1) - 13 \\
&= \operatorname{gap}_2\bigl(R(3,\alpha+1)\bigr) + 29.
\end{aligned}$$

So $s_2(t+29) \leqslant R(3,\alpha+1) + 85 \leqslant s(t+29)$ and therefore there is equality throughout, proving the claim, and the theorem. $\square$

Theorem 4.11. *For all $n, t \in \mathbb{N}$: $0 \leqslant \operatorname{gap}(n) - \operatorname{gap}_2(n) \leqslant 2$, $0 \leqslant s_2(t) - s(t) \leqslant 10$.*

Proof. Let $p < r$ be two integers so that $s(p) = s_2(p)$, $s(r) = s_2(r)$, and $s(t) \neq s_2(t)$ for all $t \in \mathbb{N}$ such that $p < t < r$. According to Lemma 4.10 with $\alpha := \alpha(s(p))$, we have

$$s(p) = R(3,\alpha+1) - 1, \quad \text{or} \quad s(p) = R(3,\alpha+1) - 2, \tag{3}$$

where we can suppose $\alpha + 1 \geqslant 6$ since $s(p) < R(3,5) = 14$ is not possible because then $p = 3$, and the choice $t = p+1 = 4$ would contradict Theorem 5.5. By Lemma 4.10

$$s(r) - s(p) \leqslant 87, \qquad r - p \leqslant 29, \qquad \alpha\bigl(s(r)\bigr) - \alpha\bigl(s(p)\bigr) \leqslant 14.$$

Moreover, for $t \in [p, r]$: $s(t) \geqslant s(p) + 3(t - p)$ that is,

$$\text{gap}(n) \leqslant p + \left\lfloor \frac{n - s(p)}{3} \right\rfloor \tag{4}$$

for any integer n in the interval $[s(p), s(r)]$, and in this same interval we are checking

$$\text{gap}_2(n) \geqslant p + \left\lfloor \frac{n - s(p)}{2} \right\rfloor - \beta\big(n - s(p)\big), \tag{5}$$

where $\beta : [0, 86] \to [0, 14]$ is the following function:

- $\beta(x) = 0$ if $x = 0$, $\beta(x) = 1$ in the interval $[1, 3]$,
- $\beta(x) = 2$ in the interval $[4, 7]$, 3 in the interval $[8, 11]$, 4 in the interval $[12, 17]$,
- 5 in $[18, 22]$, 6 in $[23, 28]$, 7 in $[29, 34]$, 8 in $[35, 43]$, 9 in $[44, 48]$, 10 in $[49, 55]$,
- 11 in $[56, 62]$, 12 in $[63, 70]$, 13 in $[71, 78]$, 14 in $[79, 86]$.

We first prove (5), and then the following:

Claim. $0 \leqslant \text{gap}(n) - \text{gap}_2(n) \leqslant 2$ *for all* $n \in [s(p), s(r)]$.

We will be done then, since every $n \in \mathbb{N}$ belongs to such an interval by Lemma 4.10. The second assertion of the theorem also follows then: let $t \in \mathbb{N}$, and apply the first assertion to $n := s(t)$. Then by the proven assertion, if we are not done, $\text{gap}_2(n) < t = \text{gap}(n) \leqslant \text{gap}_2(n) + 2 \leqslant \text{gap}_2(n + 10)$, where in the last inequality we have used the immediate consequence of Corollary 4.7 that $\text{gap}_2(n + 5) \geqslant \text{gap}_2(n) + 1$. This means $n < s_2(t) \leqslant n + 10 = s(t) + 10$.

Proof of (5). Indeed, according to Proposition 3.18, for $n \in [s(p), s(r)]$:

$$\begin{aligned}\text{gap}_2(n) &\geqslant \left\lceil \frac{n - s(p) + s(p)}{2} \right\rceil - \alpha(n) + \alpha - \alpha \geqslant \left\lceil \frac{s(p)}{2} \right\rceil - \alpha + \left\lfloor \frac{n - s(p)}{2} \right\rfloor - \big(\alpha(n) - \alpha\big) \\ &\geqslant p + \left\lfloor \frac{n - s(p)}{2} \right\rfloor + \beta\big(n - s(p)\big),\end{aligned}$$

where at last we applied that $\lceil \frac{s(p)}{2} \rceil - \alpha = \text{gap}_2(s_2(p)) = p$ by (the first part of) Lemma 4.10; instead of the obvious estimate $\alpha(n) - \alpha(s(p)) \leqslant \alpha(n - s(p))$ (Proposition 3.4) we used the particular situation of the number $s(p)$ close to the Ramsey-number $R(3, \alpha + 1)$, see (3): the function β provides a universal upper bound for $\alpha(s(p) + x) - \alpha(s(p))$, *independently of* $s(p)$: this difference is the number of Ramsey-numbers in the interval $[s(p), s(p) + x]$. We have to check

$$\alpha\big(s(p) + x\big) - \alpha\big(s(p)\big) \leqslant \beta(x) \quad \text{for all } 0 \leqslant x \leqslant 86.$$

Since $\alpha \geqslant 4$, all the inequalities of Proposition 3.17 concerning $1 \leqslant k \leqslant 14$ are valid. For $x = 0$ the upper bound is obvious, for $x = 1$ it follows from Proposition 3.17 (1), since $R(3, \alpha + 2) \geqslant R(3, \alpha + 1) + 3 \geqslant s(p) + 4$, for $x = 2$ from Proposition 3.17 (2), since $R(3, \alpha + 2) \geqslant R(3, \alpha + 1) + 7 \geqslant s(p) + 8$, etc., proving (5).

Proof of the claim. Of course $\text{gap}(n) \geqslant \text{gap}_2(n)$. Combining (5) and (4) we have

$$0 \leqslant \text{gap}(n) - \text{gap}_2(n) \leqslant \left\lfloor \frac{n - s(p)}{3} \right\rfloor - \left\lfloor \frac{n - s(p)}{2} \right\rfloor + \beta\big(n - s(p)\big),$$

which gives our estimate by taking the maximum of the 86 values $x = n - s(p)$, but actually only the 14 values

$$x = n - s(p) = 1, 4, 8, 12, 18, 23, 29, 35, 44, 49, 56, 63, 71, 79,$$

matters since while β is constant, the function $\operatorname{gap}_2(n)$ increases faster than $\operatorname{gap}(n)$, and the bound improves. For the given values the differences are

$$1, 1, 1, 2, 2, 2, 2, 2, 1, 2, 1, 2, 1, 1$$

in order, proving $0 \leqslant \operatorname{gap}(n) - \operatorname{gap}_2(n) \leqslant 2$ for the interval $[s(p), s(r)]$. $\square$

Remark. As can be expected, the somewhat modified computation of this proof provides the result of Lemma 4.10 as well. Indeed, $\operatorname{gap}_2(s(p)+86) \geqslant p+43-14 = p+29$, that is, $s_2(p+29) \leqslant s(p)+86$. On the other hand, $s(p+29) \geqslant s(p)+29\times 3 = s(p)+87$. However, $s_2(p+29) \geqslant s(p+29)$, a contradiction, proving actually $r-p \leqslant 28$.

Last, we summarize the results of the two preceding theorems, completed with the remark that the both the worst differences between gap and gap_2, $s(t)$ and $s_2(t)$ or the exception of Theorem 4.1 occur in a very small radius of Ramsey-numbers. This can be considered as a synthesis of this work.

Theorem 4.12. *For all* $n \in \mathbb{N} \setminus \bigcup_{\alpha\in\mathbb{N}}[R(3,\alpha), R(3,\alpha)+14]$: $\operatorname{gap}(n) = \operatorname{gap}_2(n) = \lceil n/2\rceil - \alpha(n)$, *and always* $\lceil n/2\rceil - \alpha(n) \leqslant \operatorname{gap}(n) \leqslant \lceil n/2\rceil - \alpha(n) + 3$.

Proof. The last inequality follows from the error of 2 in Theorem 4.11 added to the additive term 1 of Theorem 4.1. For the first part let $\alpha \in \mathbb{N}$, $t := \operatorname{gap}_2(R(3,\alpha))$, and assume $R(3,\alpha+1) \geqslant R(3,\alpha)+16$, otherwise there is nothing to prove. Then $s_2(t) < R(3,\alpha)$ (Corollary 4.2), and $s_2(t+1) \leqslant s_2(t)+4 \leqslant R(3,\alpha)+3$ (Corollary 4.7). Set

$$I := \big[s_2(t+1)+1, s_2(t+1)+12\big] \cap \mathbb{N} \subseteq \big[R(3,\alpha), R(3,\alpha)+15\big] \subseteq \big[R(3,\alpha), R(3,\alpha+1)\big).$$

Claim. *If I does not contain any Ramsey-number, then there exists $t' \in \mathbb{N}$:*

$$s\big(t'\big) = s_2\big(t'\big) \in I.$$

Indeed, by the condition α is constant on I, so by Theorem 4.1, $s_2(t+7) = s_2(t+1)+12 \leqslant R(3,\alpha)+15$. On the other hand, by Proposition 3.9 we have $s(t+7) \geqslant s(t+3)+12$. If the claim is not true, the equality does not hold here, whence $s_2(t+1) > s(t+3)$. This means that defining $n = s(t+3)$, we have $\operatorname{gap}_2(n) \leqslant t$ and $\operatorname{gap}(n) = t+3$, contradicting Theorem 4.11 and proving the claim.

Now by Lemma 4.10, for the t' provided by the claim and for any $n \in [s_2(t'), R(3,\alpha+1)]$ we have $\operatorname{gap}_2(n) = \operatorname{gap}(n)$. According to the claim, $s(t') \leqslant R(3,\alpha)+15$ finishing the proof. $\square$

5. Graphs with small gap

In this section we explore the smallest gap-extremal graphs and for small orders we show the graphs of maximum gap. Graphs on at most 4 vertices are perfect, so $s(1) \geqslant 5$, and the only 1-extremal graph is C_5.

We will need the following lemma of merely technical use. A graph G is *clique-Helly* if its inclusion-wise maximal cliques (viewed as set of vertices) have the Helly property: if a collection of maximal cliques of G pairwise intersect, then they have a common vertex. A triangular claw is a graph T_6 on 6 vertices, and 9 edges consisting of a triangle $\Delta \subseteq V(T_6)$ and a 3-stable set $S \subseteq V(T_6)$, $V(T_6) = \Delta \cup S$ so that every vertex of S is joined to a different pair of vertices of Δ. This graph is not clique-Helly, and as shown below, it is in a sense the basic example of a non-clique-Helly graph. We omit the simple proof of the following lemma:

Lemma 5.1. *(See [14].) If a graph G does not contain a triangular claw as an induced subgraph then it is clique-Helly.*

Theorem 5.2. *The graph $2C_5$ is gap-extremal, in particular, $s(2) = s_2(2) = 10$ and the only 2-extremal graph is $2C_5$. Therefore the graphs consisting of a C_5 and an arbitrary graph on $\{1\}$, $\{1,2\}$, $\{1,2,3\}$, $\{1,2,3,4\}$ have maximum gap for $n = 6,7,8,9$ respectively. In addition*:

(1) *For $n = 6$ this is the unique graph of maximum gap, and it is stable gap-optimal.*
(2) *For $n = 7$ the gap of C_7 and $\bar{C}_7$ is maximum, as well as that of $R - v$ where R is a $(3,4)$-Ramsey graph and $v \in V(R)$. The latter graphs are stable gap-optimal.*
(3) *For $n = 8$ the only stable gap-optimal graphs are the $(3,4)$-Ramsey graphs.*
(4) *For $n = 9$ a graph G on n vertices is stable gap-optimal if and only if it is triangle-free and $\alpha(G) = 4$.*

Proof. We first prove (1) and (2). By Proposition 3.8 $s(2) \geqslant s(1) + 2 = 7$, so $\text{gap}(6) = \text{gap}(7) = 1$, and (2) immediately follows. A graph G of maximum gap on 6 vertices is imperfect, so it contains C_5 as induced subgraph. The vertex v not contained in this C_5 is an isolated vertex, since otherwise the edge vu and the matching of $C_5 - u$ is a clique cover with 3 edges, whence $\text{gap}(G) = 0$, a contradiction which proves (1).

Suppose that G is a 2-extremal graph. Since $\text{gap}(2C_5) = 2$, we have $n := |V(G)| \leqslant 10$. The only thing we have to prove now is $G = 2C_5$, since then $\text{gap}(8) = \text{gap}(9) = 1$ follow and (3) and (4) can be readily checked: by Proposition 3.15 $\alpha(8) = 3$, $\alpha(9) = 4$, so for any triangle-free graph G on 8 vertices with $\alpha(G) = 3$ we have $\text{gap}(G) \geqslant \lceil n/2 \rceil - 3 = 1$, and for any triangle-free graph G on 9 vertices with $\alpha(8) = 4$ we have $\text{gap}(G) \geqslant \lceil n/2 \rceil - 3 = 1$. It follows that their gap is maximum, and on 8 vertices these are exactly the $(3,4)$-Ramsey graphs. Conversely, stable gap-optimal graphs are triangle-free and their stability number is as claimed by definition, so the assertion follows from the proven part.

Suppose now for a contradiction that $G \neq 2C_5$. Let $\alpha := \alpha(G)$, $\omega := \omega(G)$, $\theta := \theta(G)$.

Claim 1. *If K is a clique of G, then $G - K$ has at least 7 vertices.*

By Proposition 3.5 $\text{gap}(G - K) = 1$, so it has at least 5 vertices. If it has exactly 5 vertices, then it is a C_5. Then $\theta(G) \leqslant 4$, so $\alpha(G) \leqslant \theta(G) - 2 \leqslant 2$, and the equality holds everywhere. Pick a vertex v of this C_5. Then $N(v)$ is the union of a stable set and a clique, so it does not contain a C_5, C_7 or $\bar{C}_7$ (it is *split graph*), so $N(v)$ induces a perfect graph, and we conclude $\text{gap}(G) \leqslant 1$ by Proposition 3.7. If $G - K$ has 6 vertices, then by (1) $G - K$ has an isolated vertex v, whence $N(v)$ is simplicial in G, contradicting Proposition 3.6.

Claim 2. $\alpha = \omega = 3$, $\theta = 5$, $n = 10$.

Apply Claim 1 to an arbitrary clique K. Since $n \leqslant 10$, we get $|K| \leqslant 3$. If there exists a clique K for which equality holds, we have $n = 10$, $\omega = 3$.

If $\omega \leqslant 2$, then by Proposition 3.11 every component of G is factor-critical, that is odd, and at least two of them are imperfect: $G = 2C_5$. So $\omega = 3$ and $n = 10$.

Now by Proposition 3.15, $R(4,3) = R(3,4) = 9$, so since $\omega = 3$, $\alpha \geqslant 3$. But $\alpha \geqslant 4$ is not possible, because then by Proposition 3.5 $\alpha(G - K) \geqslant 4$, $\text{gap}(G - K) = 1$, $\theta(G - K) \geqslant 5$. Since $G - K$ has 7 vertices but is neither C_7 nor $\bar{C}_7$, it contains a C_5, and the two vertices that are not in this C_5 are isolated ones because of $\theta(G - K) \geqslant 5$. If v is one of them, then again, it is a simplicial vertex in G, contradicting Proposition 3.6, and finishing the proof of the claim.

Claim 3. *G contains two disjoint triangles.*

Because of $\theta(G - v) = 4$, we have $\omega(G - v) = 3$ for all $v \in V$. If G does not contain 2 disjoint triangles, then the triangles of G pairwise intersect, so either G is clique-Helly and they all intersect, a contradiction to $\omega(G - v) = 3$, or by Lemma 5.1, G contains a triangular claw $\Delta \cup S$ where $S = \{s_1, s_2, s_3\} \subseteq V(G)$ is a stable set, and $\Delta = \{t_1, t_2, t_3\} \subseteq V(G)$ is a triangle, and s_i is adjacent to $T \setminus \{t_i\}$

($i=1,2,3$). Note that $\Delta \cup S$ may be assumed to be induced because adding an edge to it yields either a K_4 or two disjoint triangles.

We may assume that $G-\{t_1,t_2\}$ is triangle-free because else, there are two disjoint triangles. Since $\alpha(G-\{t_1,t_2\})=3$, $G-\{t_1,t_2\}$ must be one of W_8, W_{81}, W_{82} (Proposition 3.15). So, $G-\{t_1,t_2\}$ has a cycle $w_1 \dots w_8 w_1$, and the only other edges are among $w_i w_{i+5}$, $i=1,\dots,4$. We suppose up to symmetry $t_3=w_1$. We consider now two cases.

Case 1. t_1 is not adjacent to w_2 and w_8. Because of the triangular claw, w_1 and t_1 have a common neighbor that must be w_5. Also t_2 and w_1 must have a common neighbor, that cannot be w_5 because $\omega=3$, so it is w_2 or w_8, say w_2 up to symmetry. Now, we may assume $t_2w_3, t_1w_4, t_1w_6 \notin E(G)$ because otherwise there are two disjoint triangles. So, the common neighbor s_3 of t_1t_2 must be w_7 and we may assume $t_2w_6, t_2w_8 \notin E(G)$ because otherwise there are two disjoint triangles. Hence, $\{t_2, w_3, w_6, w_8\}$ is a stable set, a contradiction.

Case 2. t_1 has at least one neighbor among w_2 and w_8. Symmetrically, we may assume that t_2 also has at least one neighbor among w_2 and w_8. Since $\omega=3$, we may assume $t_1w_8, t_2w_2 \in E(G)$ and $t_1w_2, t_2w_8 \notin E(G)$. Now, we may assume $t_1w_7, t_2w_3 \notin E(G)$ because otherwise there are two disjoint triangles. Hence, $\{t_1, w_2, w_7, w_4\}$ is a stable set unless $t_1w_4 \in E(G)$, so $t_1w_4 \in E(G)$ and symmetrically, $t_2w_6 \in E(G)$. Now, $t_2w_5 \notin E(G)$ because else there are two disjoint triangles. Hence, $\{t_2, w_3, w_5, w_8\}$ is a stable set, a contradiction. This proves the claim.

So, G contains two vertex-disjoint triangles, $T_1=\{a_1,a_2,a_3\}$, $T_2=\{b_1,b_2,b_3\}$. If the remaining four vertices contain a triangle or two independent edges, we have $\theta(G) \leqslant 4$, a contradiction. Therefore three of these vertices form an independent set $C=\{c_1,c_2,c_3\}$ and we have the following cases according to the adjacencies of the last vertex d (which has a neighbor among c_1, c_2, c_3 because $\alpha(G)=3$).

Case 1. $dc_i \in E(G)$ for $i=1,2,3$. Each vertex of T_1 must have a neighbor in C because $\alpha(G)=3$. If $a_1c_1, a_2c_1 \in E(G)$ then we must have $a_3c_2 \in E(G)$ or $a_3c_3 \in E(G)$ because there is no K_4. But then, we can cover G with two triangles and two edges. So we proved that no two vertices in T_1 can have a common neighbor in C. Hence, we may assume that the only edges between T_1 (and similarly T_2) and C are c_ia_i (and similarly c_ib_i), $i=1,2,3$. Using that $\alpha(G)=3$, it follows that $a_ib_i \in E(G)$ and now a_i, b_i, c_i for $i=1,2,3$ give three disjoint triangles showing that $\theta(G) \leqslant 4$, a contradiction.

Case 2. $dc_3 \in E(G)$, $dc_1, dc_2 \notin E(G)$. Suppose first that every vertex of T_1 has a neighbor in $\{c_1,c_2\}$. Since there is no K_4 we may assume $a_1c_1, a_2c_2, a_3c_2 \in E(G)$, so we can cover G with two triangles and two edges, a contradiction. So there must be a vertex in T_1 with no neighbor in $\{c_2,c_1\}$, say a_1, and by the same argument a similar vertex in T_2, say b_1. Using five times that $\alpha(G)=3$, we get that $a_1c_3, b_1c_3, a_1b_1, da_1, db_1 \in E(G)$, a contradiction because $\{a_1, b_1, c_3, d\}$ is a clique.

Case 3. $dc_2, dc_3 \in E(G)$, $dc_1 \notin E(G)$. We claim that c_1 is non-adjacent to at least two vertices of both T_1, T_2. If not, say c_1 is adjacent to a_2, a_3, then $c_2a_1, c_3a_1 \notin E(G)$ otherwise we have a cover with two triangles and two edges. Depending on $c_1a_1 \in E(G)$ or not, we have either a clique or an independent set of size four, a contradiction that proves the claim. Therefore, w.l.o.g. c_1 is non-adjacent to a_2, a_3, b_2, b_3. If $c_1a_1 \notin E(G)$ or $c_1b_1 \notin E(G)$ or $a_1b_1 \in E(G)$ then c_1 is a simplicial vertex, a contradiction. Thus $c_1a_1, c_1b_1 \in E(G), a_1b_1 \notin E(G)$.

Next we note that each of a_2, a_3 must have a neighbor in $\{c_2, c_3\}$, else there is an S_4. But a_2, a_3 may not have a common neighbor in $\{c_2, c_3\}$ because then there is a cover with two triangles and two edges. Hence w.l.o.g. the only edges between T_1 and C are c_1a_1, c_2a_2, c_3a_3. Similarly, the only edges between T_2 and C are c_1b_1, c_2b_2, c_3b_3.

Now $\alpha(G)=3$ implies $a_2b_2, a_3b_3 \in E(G)$. Moreover $da_2, da_3, db_2, db_3 \notin E(G)$ otherwise there is a clique cover with two triangles and two edges. Then $a_2b_3, a_3b_2 \in E(G)$ for otherwise a_2, b_3, c_1, d or

a_3, b_2, c_1, d would form an independent set. But now have the final contradiction since a_2, a_3, b_2, b_3 span a clique. $\square$

To slightly shorten the proof, one could use Chvátal's [6] theorem stating that the *Grötzsch graph* (the fourth in Mycielski's well-known construction [17], being the "Mycielskian" of C_5 which is the third) is *the only triangle-free graph on at most* 11 *vertices with chromatic number at least* 4. The complement of the Grötzsch graph is therefore the only graph on at most 11 vertices with $\alpha \leqslant 2$ and $\theta \geqslant 4$. Also the following lemma could be used. For a proof, see Lemma 1.16 in [20].

Lemma 5.3. *If G is a graph on at least* 10 *vertices then either G contains a clique or a stable set on four vertices, or G contains two disjoint triangles.*

Theorem 5.4. *The graph R_{13} is gap-extremal, in particular, $s(3) = s_2(3) = 13$, and the only* 3*-extremal graph is R_{13}. Any triangle-free graph G on* 11 *or* 12 *vertices and $\alpha(G) \leqslant 4$ is stable gap-optimal and connected.*

Proof. Suppose that G is a 3-extremal graph, $\alpha := \alpha(G)$, $\omega := \omega(G)$, $\theta := \theta(G)$. Since R_{13} is triangle-free, $\theta(R_{13}) = \zeta(R_{13}) = 7$, and $\alpha(R_{13}) = 4$ (it is a $(3,5)$-Ramsey graph). So $\text{gap}(R_{13}) = 3$, and therefore $n := |V(G)| \leqslant 13$. We have to prove $G = R_{13}$. If $\omega = 2$ this is true since then by Proposition 3.11 G is factor-critical, $\theta(G) = 7$, so $\alpha(G) = 4$. Therefore G is a $(3,5)$-Ramsey graph, and by Proposition 3.15 $G = R_{13}$. So suppose $\omega \geqslant 3$.

Claim 1. *$n = 13$, $\omega = 3$, $\alpha = 4$, $\theta = 7$, and for every triangle T, $G - T$ is a $2C_5$.*

If K is an arbitrary clique, $\text{gap}(G - K) = 2$, so by Theorem 5.2, $G - K$ is of order at least 10, whence $|K| \leqslant 3$, and therefore $\omega = 3$. If T is a triangle, $n \leqslant 13$ implies that $G - T$ is of order at most 10. So $G - T$ is of order 10 and gap 2, and $n = 13$. By Proposition 3.5, $\text{gap}(G - T) = 2$, and since $G - T$ has 10 vertices, the unicity in Theorem 5.2 states that it is $2C_5$.

Now by the equalities of Proposition 3.5 concerning gap-critical graphs, $\alpha(G) = \alpha(G - Q) = \alpha(2C_5) = 4 = \theta - 3$, finishing the proof of the claim.

Claim 2. *Let T be a triangle, and $C, D \subseteq V(G)$ be the two C_5 components of $G - T$. Then for every $t \in T$ either $\alpha(\{t\} \cup C) = 2$ or $\alpha(\{t\} \cup D) = 2$.*

Indeed, if there exists $t \in T$ so that both are 3, then there exist $c_1, c_2 \in C$, and $d_1, d_2 \in D$ so that t, c_1, c_2, d_1, d_2 form a stable set in G, contradicting Claim 2.

So suppose $t \in T$, $\alpha(\{t\} \cup C) = 2$. Then $C \setminus N(t)$ is the subset of an edge of C, and therefore t forms a triangle T_1 and T_2 with two different edges of C. But this is impossible, because by Claim 1 both $G - T_1$ and $G - T_2$ are $2C_5$ graphs, however, $(C - T_1) \cup T \setminus \{t\} \neq (C - T_2) \cup T \setminus \{t\}$, because $T_1 \neq T_2$.

The remaining additional claim follows now from Proposition 3.12: if G is a triangle-free graph on 11 or 12 vertices and $\alpha(G) = \alpha(n) = 4$, then $\text{gap}(G) \geqslant 6 - 4 = 2$, so the equality holds and $\text{gap}(G)$ is maximum. Moreover G is connected since $R(3,2) = 3$, $R(3,3) = 6$, $R(3,4) = 9$ imply that two vertex-disjoint graphs with stability numbers 2 and 2 or 1 and 3 have at most 12 vertices. $\square$

Theorem 5.5. *The* $(3,6)$*-Ramsey graphs are* 4*-extremal, in particular $s(4) = s_2(4) = 17$. A graph is* 4*-extremal and triangle-free if and only if it is a* $(3,6)$*-Ramsey graph*; *for all other* (*possibly non-existing*) 4*-extremal graphs G, $\alpha(G) = 4$, and $\theta(G) = 8$.*

Proof. Let G be 4-extremal. According to Proposition 3.12 the gap of $(3,6)$-Ramsey graphs on 17 vertices is at least $9 - 5 = 4$. So $n := |V(G)| \leqslant 17$.

Since $s_2(4) = 17$ from Corollary 4.8, we may assume $\omega := \omega(G) \geqslant 3$. Then by Proposition 3.8 $s(4) \geqslant s(3) + 3 = 16$ (see Theorem 5.4), and $s(4) \leqslant s_2(4) = 17$. The statement $s(4) = 17$ follows now from the next claim.

Claim. *For any clique K, $G-K$ is of order at least* 13, $\omega \leqslant 4$, *and* $|V(G)| = 17$.

Indeed, by Proposition 3.5, $\mathrm{gap}(G-K) = 3$. So $G-K$ is of order at least 13, so $|K| \leqslant 4$. Suppose $n = 16$. Then $\omega = 3$ and for any triangle K, $G-K$ is of order exactly 13 of gap 3, so it is a $(3,5)$-Ramsey graph, in particular it is triangle-free. Consequently there are no two disjoint triangles in G, and $\alpha(G) = \alpha(G-K) = \alpha(R_{13}) = 4$.

On the other hand $n-2 = 14 = R(3,5)$, so for all $u, v \in V(G)$, $\omega(G - \{u,v\}) = 3$.

So, by Lemma 5.1, G is clique-Helly or has a triangular claw. In the first case, since there are no two disjoint triangles, the triangles pairwise intersect, so they intersect, a contradiction to $\omega(G - \{u,v\}) = 3$. Hence, there is a triangular claw $\{t_1, t_2, t_3, s_1, s_2, s_3\}$ (our usual notation). Since $\omega(G - \{t_1, t_2\}) = 3$, $G - \{t_1, t_2\}$ contains a triangle, hence G contains two disjoint triangles. This contradiction finishes the proof of the claim.

Let K be an ω-clique of G. By Proposition 3.13, $\theta(G-K) \leqslant 7$, so by Proposition 3.5 $\theta(G) \leqslant 8$, and since $\mathrm{gap}_2(G) = 4$: $\alpha(G) \leqslant 4$. The strict inequality here, that is, $\alpha \leqslant 3$ would imply either $\omega \leqslant 3$ and then applying $R(4,4) = 18$ (Proposition 3.15) we get that G is a $(4,4)$-Ramsey graph; or by claim, $\omega = 4$, and $G-K$ is of gap 3 and order 13, so isomorphic to R_{13}. In the former case we see that $\theta = 6$, implying $\alpha = 2$, but $(4,4)$-Ramsey graphs have $\alpha = 3$, a contradiction; in the latter case $\alpha(G) = \alpha(G-K) = 4$ is proved, finishing the proof of the theorem. □

Surprisingly, the next case we can treat is $s(10)$:

Lemma 5.6. $s(t) \geqslant s_2(4) + 3(t-4)$ *for* $t = 5, \ldots, 10$.

Proof. Note that $s_2(i) - s_2(i-1)$ for the six values $i = 5, \ldots, 10$ is equal to $4, 4, 4, 2, 2, 2$, that is, 3 in average.

If the statement does not hold let t_0 be the smallest value for which this inequality is violated. Then

$$s(t_0) < s_2(4) + 3(t_0 - 4) \leqslant s_2(t_0).$$

Clearly, $s(t_0) - s(t_0 - 1) = 2$ since if not, according to Proposition 3.9 $s(t_0) - s(t_0 - 1) \geqslant 3$ so we could have chosen $t_0 - 1$ or a smaller value instead of t_0. Therefore any t_0-extremal graph is triangle-free, in contradiction with $s(t_0) < s_2(t_0)$. □

Using Lemma 5.6 for a lower bound and Corollary 4.8 as upper bound, $s(5) \in \{20, 21\}$, $s(6) \in \{23, 24, 25\}$, $s(7) \in \{26, 27, 28\}$, $s(8) \in \{29, 30, 31\}$, $s(9) \in \{32, 33\}$, and $s(10) = 35$.

Corollary 5.7. *We have* $s(10) = 35$, *the* $(3,9)$*-Ramsey graphs are all* 10*-extremal, and all other* 10*-extremal graphs contain a triangle.*

Proof. $s(10) \leqslant 35$, since by Proposition 3.18 $\mathrm{gap}_2(35) \geqslant \lceil 35/2 \rceil - \alpha(35) = 18 - 8 = 10$, so $s_2(10) \leqslant 35$. Substituting $s(4) = 17$ (Theorem 5.5) and $t = 10$ into Lemma 5.6 we get $s(10) \geqslant 35$. □

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