Discrete Mathematics 64 (1987) 101–103
North-Holland

NOTE

A QUICK PROOF OF SEYMOUR'S THEOREM ON t-JOINS

András SEBÖ*

Eötvös Loránd University and Computer and Automation Institute, Hungarian Academy of Sciences, Budapest, Hungary

Received 10 December 1985
Revised 13 May 1986

A very short proof of Seymour's theorem, stating that in bipartite graphs the minimum cardinality of a t-join is equal to the maximum cardinality of an edge-disjoint packing of t-cuts, is given.

Let G be a graph and $t: V(G) \to \{0, 1\}$, where $t(V(G))$ is even. (If $X \subseteq V(G)$, then $t(X) := \sum \{t(x): x \in X\}$.) A *t-join* is a set $F \subseteq E(G)$ with $d_F(x) \equiv t(x) \pmod 2$, $\forall x \in V(G)$. ($d_F(x)$ denotes the number of edges of F incident with x, where loops count twice.) t-joins contain Chinese postman tours, matchings and minimum weight paths as a special case. (cf. [1, 7]).

If $X \subseteq V(G)$, let $\delta(X) = \{xy \in E(G): y \notin X, x \in X\}$. If $t(X) \equiv 1 \pmod 2$, then $\delta(X)$ is called a *t-cut*. t-cuts contain plane multicommodity flows as a special case [8]. For basic definitions concerning graphs we refer to [4].

Let $\tau(G, t) = \min\{|F: F \subseteq E(G), F$ is a t-join$\}$, and $\nu(G, t) = \max\{|C|: C$ is a family of disjoint t-cuts$\}$. It is easy to see that $\tau(G, t) \geq \nu(G, t)$.

Theorem (Seymour [8]). *If G is bipartite, then $\tau(G, t) = \nu(G, t)$.*

If G is an arbitrary graph, then replacing every edge by a path of length two, we get a bipartite graph for which Seymour's theorem can be applied. The resulting minimax theorem for G was proved earlier by Lovász [3]. Both Lovász' and Seymour's proofs use rather sophisticated linear programming techniques and are quite involved. In [2] Frank, Sebö and Tardos presented a short proof for a sharper theorem, using a new technique. The extension of this technique has led to a Gallai–Edmonds type structure theorem for t-joins [6]. The present note is

* Supported by the joint project "Algorithmic Aspects of Combinatorial Optimization" of the Hungarian Academy of Sciences (Magyar Tudományos Akadémia) and the German Research Association (Deutsche Forschungsgemeinschaft, SFB 21)

based on the recent observation that the method used in [6] to prove this structure theorem, gives rise to very short proofs for some of its corollaries.

Let us introduce some notations and terminology:

$$\text{For } a \neq b \in V(G), \quad t^{a,b}(x) \equiv \begin{cases} t(x) & \text{if } x \in V(G)\setminus\{a, b\} \\ t(x)+1 & \text{if } x \in \{a, b\} \end{cases} \pmod 2.$$

The *contraction* of an edge $e = xy \in E(G)$ in (G, t) means deleting e and identifying x and y and defining $t(v_{xy}) \equiv t(x) + t(y) \pmod 2$ where v_{xy} is the new vertex that arises; $\Gamma(x)$ is the set of neighbours of x; an (a, b)-*path* $(a, b \in V(G))$ means a simple path in G between a and b. If P is a path $P(x, y)$ $(x, y \in V(P))$ denotes its subpath between x and y.

The following simple observations will be used without reference in the sequel: A t-join F is minimum if and only if for every circuit C, $|F \cap C| \leqslant |F \setminus C|$, [5].

If F_1 is a minimum t_1-join and F_2 is a minimum t_2-join, then for each circuit C in $F_1 \triangle F_2$, $|C \cap F_1| = |C \cap F_2|$.

If F is a minimum t-join then for every $a \neq b \in V(G)$ there exists a minimum $t^{a,b}$-join F' and an (a, b)-path P such that $F = F' \triangle P$. (This follows by observing that for any minimum $t^{a,b}$-join F'', $F \triangle F''$ is the union of an (a, b)-path P and circuits $C_1, \ldots, C_k$ which are pairwise edge-disjoint. Since both F and F'' are minimum, the circuits have the same number of edges in the two joins. Thus, $F' = F'' \triangle (C_1 \cup \cdots \cup C_k)$ is also a minimum $t^{a,b}$-join and $F = F' \triangle P$ holds.)

Proof of Seymour's theorem. Let the function t differ from the 0-function and $a \neq b \in V(G)$ be such that $\tau(G, t^{a,b})$ is minimum. (If $t \equiv 0$ the theorem is trivial.)

Claim. If F is a minimum t-join, then $d_F(a) = d_F(b) = 1$.

Let the minimum $t^{a,b}$-join F' and the (a, b)-path P be such that $F = F' \triangle P$. Then $d_{F'}(a) = d_{F'}(b) = 0$, since if $bb' \in F'$ say, then $F' \setminus bb'$ is a $t^{a,b'}$-join, a contradiction with the choice of a and b. Since $d_P(a) = d_p(b) = 1$ the claim is proved.

Contract every edge of $\delta(b)$ to get (G^*, t^*). It is enough to prove that $F^* := F \setminus \delta(b)$ is a minimum t^*-join of G^* since then the claim implies $\tau(G^*, t^*) = F \setminus \delta(b)| = |F| - 1 = \tau(G, t) - 1$ and Seymour's theorem follows by induction. ($\delta(b)$ is a t-cut disjoint from $E(G^*)$.)

Suppose indirectly, that $K \subset E(G^*)$ is a circuit in G^* with: $|K \cap F^*| > |K \setminus F^*|$. Then $|K \cap F^*| \geqslant |K \setminus F^*| + 2$ follows, because G^* is bipartite. K corresponds in G to an x_1, x_2-path $(x_1, x_2 \in \Gamma(b))$ and since F is a minimum t-join, $|(K \cup \{bx_1, bx_2\}) \cap F| \leqslant |(K \cup \{bx_1, bx_2\}) \setminus F|$. As a consequence we have equality in the last two inequalities, and $bx_1, bx_2 \notin F$. The latter equality implies that $T = F \triangle (K \cup \{bx_1, bx_2\})$ is also a minimum t-join. However, $d_T(b) = 3$ contradicting the claim. $\square$

Note that the sharper theorem of Frank–Sebö–Tardos [2] can be proved in the same way.

Acknowledgments

I would like to express my thanks to Bill Cook, András Frank and Éva Tardos for their suggestions.

References

[1] J. Edmonds and E.L. Johnson, Matching, Euler tours and the Chinese postman, Math. Programming 5 (1973) 88–124.
[2] A. Frank, A. Sebö and E. Tardos, Covering directed and odd cuts, Math. Programming Study 22 (1984) 99–112.
[3] L. Lovász, On two minimax theorems in graph theory, J. Combin. Theory 21 (1976) 96–103.
[4] L. Lovász, Combinatorial Problems and Exercises (Akadémiai Kiadó, 1979).
[5] Mei Gu Guan, Graphic programming using odd or even points, Chinese Math 1 (1962) 273–277.
[6] A. Sebö, On the structure of odd joins, J. Combin. Theory, to appear.
[7] A. Sebö, On the Chinese postman problem: algorithms, structure and applications. Discrete Appl. Math., to appear.
[8] P. Seymour, On odd cuts and plane multicommodity flows, Proc. London Math. Soc. 3 42 (1981) 178–192.