Convex Combinations and TSP on Cubic Graphs
Lecture Notes

Alantha Newman

November 24, 2017
Revised July 2021

1 TSP on Cubic Graphs

Given an undirected, unweighted, simple graph $G = (V, E)$, our goal is to find a spanning, connected Eulerian submultigraph, F, of G such that $|F|$ is minimized. An edge in E may appear in F once, twice, or not at all. (In other words, $F \subseteq 2E$, where $2E$ represents the graph G in which every edge is doubled.) A *perfect matching* of G is a set of edges $M \subset E$ such that each vertex in V is adjacent to exactly one edge in M. A *cycle cover* of G is a subgraph $C \subset E$ that spans V and in which every vertex in V has degree exactly two. For example, a Hamilton cycle is a cycle cover. Note that a cycle cover C partitions the vertex set V into cycles.

Let us assume that G is a bridgeless, cubic graph. A graph is *bridgeless* iff it is 2-edge connected. A graph is *cubic* if each vertex has degree three. Here is a famous theorem in graph theory.

Theorem 1. [Pet91] *A bridgeless, cubic graph has a perfect matching.*

Let M denote a perfect matching in G. Observe that $C = E \setminus M$ is a cycle cover of G. We can use this cycle cover to construct a TSP tour as follows.

CYCLE-COVER-DOUBLE-SPANNING-TREE

Input: A bridgeless, cubic graph $G = (V, E)$ and cycle cover C of G.

1. Find a spanning tree, $S \subset E \setminus C$, on connected components of C.

2. Double edges in S. (Denote this by $2S$.)

3. Set $E' := C \cup 2S$.

Output: An Eulerian multigraph $G' = (V, E')$.

Claim 1. *If the cycle cover C consists of at most k cycles, then $|E'| < n + 2k$.*

Since G is not a multigraph (i.e. it contains no 2-cycles), each cycle in C contains at least three vertices. This implies that any cycle cover of G consists of at most $\frac{n}{3}$ cycles. By Claim 1, we conclude that G has a TSP tour of length at most $\frac{5n}{3}$. The main goal of this lecture is to prove that we can do much better.

2 Convex Combinations of Perfect Matchings

We will now show that we can find a cycle cover that does not contain any triangles. Such a cycle cover consists of at most $\frac{n}{4}$ cycles. A useful tool is the perfect matching polytope. Let $\delta(i) \subset E$ denote the set of edges with exactly one endpoint equal to i, and let $\delta(S) \subset E$ denote the set of edges with exactly one endpoint in S for $S \subset V$.

$$\sum_{ij \in \delta(i)} x_{ij} = 1 \text{ for all } i \in V,$$
$$\sum_{ij \in \delta(U)} x_{ij} \geq 1 \text{ for all cuts } U \subset V, \ |U| \text{ odd},$$
$$x_{ij} \geq 0 \text{ for all } ij \in E. \qquad (P_{PM}(G))$$

Definition 1. *For a graph $G = (V, E)$, the* everywhere $\frac{1}{3}$ vector *is the vector $\frac{1}{3}^E$.*

The next lemma is due to Naddef and Pulleyblank [NP81].

Lemma 1. *Let G be a bridgeless, cubic graph and let x^* be the everywhere $\frac{1}{3}$ vector for G. Then $x^* \in (P_{PM}(G))$.*

It is easy to check that x^* satisfies all the constraints in $(P_{PM}(G))$ with the exception of the constraints corresponding to $U \subset V$ such that $|\delta(U)| = 2$ and $|U|$ odd. This issue can be resolved via the following exercise.

Exercise 1. *Let G be a bridgeless, cubic graph and let $S \subset V$ be a subset of vertices. Show that $|\delta(S)|$ is even iff $|S|$ is even. (In particular, this shows that if $|\delta(S)| = 2$, then $|S|$ is even.)*

For a perfect matching M, let $\chi^M \in \mathbb{R}^E$ denote its indicator vector. Let $\lambda_i > 0$ for $i \in \{1, \ldots, \ell\}$. We say that a family of perfect matchings $\{M_1, M_2, \ldots, M_\ell\}$ and a set of convex multipliers $\{\lambda_1, \lambda_2, \ldots, \lambda_\ell\}$ is a *convex decomposition of x^** if the following holds:

$$x^* = \sum_{i=1}^{\ell} \lambda_i \chi^{M_i}, \quad \sum_{i=1}^{\ell} \lambda_i = 1, \text{ and } \lambda_i > 0 \text{ for all } i \in \{1, \ldots, \ell\}. \qquad (1)$$

A family of perfect matchings obtained via a convex decomposition of the everywhere $\frac{1}{3}$ vector exhibit some useful properties.

Lemma 2. *Let G be a bridgeless, cubic graph and let x^* be the everywhere $\frac{1}{3}$ vector for G. Let $\mathcal{M} = \{M_1, M_2, \ldots, M_\ell\}$ denote a set of perfect matchings obtained via a convex decomposition of x^*. Then for all $i \in \{1, \ldots, \ell\}$, the perfect matching M_i contains exactly one edge from each 3-edge cut of G.*

Proof. Consider a 3-edge cut of G consisting of three edges, $\{a, b, c\}$. By Exercise 1, note that any perfect matching intersects these three edges in exactly one edge or in exactly three edges.

Suppose the matching M_i intersects this 3-edge cut in three edges, i.e. $|M_i \cap \{a, b, c\}| = 3$. By the definition of x^*, we have:

$$x_a^* + x_b^* + x_c^* = 1.$$

Thus, there must exist a matching, say M_j, such that $|M_j \cap \{a, b, c\}| < 1$. However, this cannot happen since, as stated above, every perfect matching intersects this 3-edge cut in at least one edge. $\qquad\square$

Corollary 2. *Let G be a bridgeless, cubic graph and let x^* be the everywhere $\frac{1}{3}$ vector for G. Let $\mathcal{M} = \{M_1, M_2, \ldots, M_\ell\}$ denote a set of perfect matchings obtained via a convex decomposition of x^*. Then for all $i \in \{1, \ldots, \ell\}$, the cycle cover $C_i = E \setminus M_i$ does not contain a triangle.*

Proof. For the sake of contradiction, suppose that $C_i = E \setminus M_i$ does contain a triangle. Denote the three vertices of this triangle by $T \subset V$. Then $|\delta(T)| = 3$ and $|\delta(T) \cap M_i| = 3$. This contradicts Lemma 2. $\qquad\square$

Thus, Corollary 2 and Claim 1 imply that a bridgeless, cubic graph has a TSP tour consisting of at most $\frac{3n}{2}$ edges, which matches the approximation guarantee of Christofides' algorithm.

3 Uniformly Covering Edges by Perfect Matchings

Suppose that the following statement were true.

Hypothesis 1. *For a bridgeless, cubic graph $G = (V, E)$, there is a family of perfect matchings, $\mathcal{M}$ such that each edge $e \in E$ appears in exactly one third of the matchings in $\mathcal{M}$.*

For example, suppose that in addition to being bridgeless and cubic, G is bipartite. Then we can find a set of three perfect matchings, such that each edge belongs to exactly one of these matchings. (This equivalent to saying that G is 3-edge colorable.)

Lemma 3. *Let $G = (V, E)$ be a bridgeless, cubic, bipartite graph and let x^* be the everywhere $\frac{1}{3}$ vector for G. Then there is a convex decomposition of x^* into three perfect matchings.*

Proof. Let M_1 be any perfect matching in G, e.g. use Petersen's Theorem. The cycle cover $C = E \setminus M_1$ will consist only of even cycles. Thus, we can decompose C into two perfect matchings, M_2 and M_3. $\qquad\square$

Moreover, Hypothesis 1 is implied by the infamous Berge-Fulkerson Conjecture.

Conjecture 1. [Ful71] *Let $G = (V, E)$ be a bridgeless, cubic graph and let x^* be the everywhere $\frac{1}{3}$ vector for G. Then there is a convex decomposition of x^* into six perfect matchings.*

In other words, the conjecture states that G contains six perfect matchings such that each edge $e \in E$ belongs to exactly two of them. The Berge-Fulkerson Conjecture is still unresolved. However, we can show that Hypothesis 1 is true in a probabilistic sense: For a bridgeless, cubic graph $G = (V, E)$, there is a polynomial-sized family of perfect matchings, $\mathcal{M} = \{M_1, M_2, \ldots, M_\ell\}$ and a probability distribution on $\mathcal{M}$ such that if a matching, say M_i, is chosen according to this probability distribution, then each edge belongs to M_i with probability $\frac{1}{3}$.

Lemma 4. *Let $G = (V, E)$ be a bridgeless, cubic graph and let x^* be the everywhere $\frac{1}{3}$ vector for G. Then there is a convex decomposition of x^* into $|E| + 1$ perfect matchings.*

Proof. By Claim 1, we have that $x^* \in (P_{PM}(G))$. Applying Carathéodory's Theorem, we can write x^* as a convex combination of at most $|E| + 1$ extreme points of the polyhedron $(P_{PM}(G))$, and by a fundamental theorem of Edmonds, each of these extreme points is integral and corresponds to a perfect matching [Edm65, Sch03].

For a matching M, let $\chi^M \in \mathbb{R}^E$ denote its indicator vector. Let $\lambda_i > 0$ for $i \in \{1, \ldots, \ell\}$, where $\ell \leq |E| + 1$. Then we have the following convex decomposition of x^* into perfect matchings $\{M_1, M_2, \ldots, M_\ell\}$.

$$\sum_{i=1}^{\ell} \lambda_i = 1, \quad x^* = \sum_{i=1}^{\ell} \lambda_i \chi^{M_i}.$$

For each edge $e \in E$, we have:

$$\sum_{i:e \in M_i} \lambda_i = \frac{1}{3}.$$

Therefore, if we choose a matching M_i with probability λ_i, this probability distribution has the required properties. $\qquad\square$

4   A Cycle Cover with at Most $\frac{n}{6} + 1$ Cycles

By Corollary 2, we can find a cycle cover C that contains no triangles. This implies that the number of cycles in C is at most $\frac{n}{4}$. In fact, we can find a cycle cover with even fewer cycles.

Theorem 3. *A bridgeless, cubic graph has a cycle cover containing at most $\frac{n}{6} + 1$ cycles.*

Our goal is to prove Theorem 3. One way of lower bounding the number of components or cycles in a cycle cover is to lower bound the length of all of its cycles. In other words, to prove Theorem 3, we could show that there exists a cycle cover consisting only of cycles on at least six vertices. But such a cycle cover might not exist. Moreover, it is not necessary to require that all cycles in the cycle cover are long, or, for example, even to require that there are no 4-cycles. We simply want to show that there are *few* cycles. We now show how to do this. Theorem 4 is due to Mömke and Svensson [MS16], although it is rephrased here.

Theorem 4. *Let $G = (V, E)$ be a bridgeless, cubic graph and let M be a perfect matching randomly chosen such that each edge in E is included in M with probability $\frac{1}{3}$. Then $C = E \setminus M$ is a cycle cover for which the expected number of components (i.e. cycles) is at most $\frac{n}{6} + 1$.*

4.1   Proof of Theorem 4

Consider an arbitrary depth-first-search tree of G with a root r. Partition the edges E into tree edges, T, and back edges, B, so that $E = T \cup B$. Each edge in T is directed away from r and each edge in B is directed towards r. Then each edge in B forms a unique directed cycle with edges in T. Pair each edge in B with the first edge in such a cycle. For example, pair edge $(i, j) \in B$ with edge $(j, h) \in T$ that belongs to this cycle. Denote these paired edges from T by T_1, and denote all the remaining edges in T by T_2. (Note that one edge in B is not paired since r has two incoming back edges.)

Thus, we have partitioned the edges in E into three disjoint sets, B, T_1 and T_2. We observe that $|B| = \frac{n}{2} + 1, |T_1| = |B| - 1$, and $|T_2| = \frac{n}{2} - 1$. Now, (only) for the sake of the analysis, we remove the edges in M from E in three phases. Recall that M is chosen such that each edge in E belongs to M with probability $\frac{1}{3}$.

1. Remove $B \cap M$. This does not disconnect the graph, i.e. it does not create any new components, because we have not touched the (spanning) tree T.

2. Remove $T_1 \cap M$. This does not disconnect the graph either. We can prove this by induction. The inductive hypothesis is that for every $i, j \in V$, there is a path between i and j in the current (sub)graph. This is true initially in the graph $E \setminus (B \cap M)$, since no edges in T have yet been removed.

   Now consider a path from r to a leaf in T. Choose the first edge on the path (closest to the root) that belongs to $T_1 \cap M$ and remove it. Suppose this edge is (i, j). Since (i, j) belongs to T_1 and to M, there is a back edge, say (h, i) that does not belong to M. Note that there is a path in T from j to h. Thus, after removing edge (i, j), there is still a path from j to i through vertex h.

   Now we repeat this argument for the next edge in $T_1 \cap M$ that occurs on the path from r to the leaf. Since removing these edges never disconnects the graph, at the end of this phase, we still only have one component.

3. Remove $T_2 \cap M$. In the worst case, the removal of each edge in $T_2 \cap M$ could create a new component. In other words,

$$\mathbb{E}\left[\text{Number of cycles in } C\right] \quad \leq \quad \mathbb{E}\left[|M \cap T_2|\right].$$

   As observed above, $|T_2| = \frac{n}{2} - 1$. Moreover, we assumed that each edge in E—and therefore in T_2—belongs to M with probability $\frac{1}{3}$. Therefore,

$$\mathbb{E}[|M \cap T_2|] \quad \leq \quad \frac{n}{2} \cdot \frac{1}{3} \quad = \quad \frac{n}{6}.$$

Thus, in expectation, we are left with at most $n/6 + 1$ components after removing the (random) perfect matching M from G. This concludes the proof of Theorem 4.

4.2 Subcubic Graphs

Theorem 4 can be extended to bridgeless, subcubic graphs. In a subcubic graph, each vertex has degree two or three. Theorem 5 is due to Mömke and Svensson [MS16], although it is rephrased here. They replace each degree two vertex with a gadget to transform the graph into a cubic graph and then a random perfect matching is obtained for this bridgeless, cubic graph. See Figure 1. The resulting perfect matching corresponds to a set $M \subset E$ that includes each edge from E with probability $1/3$ and such that the set $E \setminus M$ is Eulerian. Moreover, a vertex with degree three in G has degree two in $E \setminus M$, and a vertex with degree two in G has degree two or zero in $E \setminus M$.

Figure 1: A vertex of degree two is replaced by a gadget resulting in a cubic graph.

Theorem 5. *Let $G = (V, E)$ be a bridgeless, subcubic graph and let $M \subset E$ be randomly chosen such that (i) each edge in E is included in M with probability $\frac{1}{3}$, (ii) $E \setminus M$ is Eulerian, and (iii) a vertex with degree three in G has degree two in $E \setminus M$. Then the expected number of components in $E \setminus M$ is at most $\frac{2n - |E|}{3}$.*

Using the double spanning tree algorithm on the components, Theorem 5 results in a tour of length at most:

$$|E \setminus M| + 2 \cdot \mathbb{E}\left[\text{Number of components in } E \setminus M\right] \;=\; \frac{2|E|}{3} + 2 \cdot \frac{2n - |E|}{3} \;=\; \frac{4n}{3}.$$

Proof of Theorem 5. Consider an arbitrary depth-first-search tree T with a root of degree two in G. The set of back edges B has size $|E| - n + 1$. Following the proof of Theorem 4, each edge in B is in a unique cycle with the edges of T. Let each edge in B be paired with the first edge in this cycle. This set of edges is T_1, and we note that $|T_1| = |B|$. The remaining edges are the set T_2, and we note that

$$|T_2| \;=\; |E| - 2(|E| - n + 1) \;<\; 2n - |E|.$$

Since removing the edges in $M \cap (B \cup T_1)$ from E does not disconnect the graph, the expected number of components is $|M \cap T_2| = |T_2|/3$. Thus, the expected number of components in $E \setminus M$ is at most $(2n - E)/3$. $\qquad\square$

References

[Edm65] Jack Edmonds. Maximum matching and a polyhedron with 0,l-vertices. *Journal of Research of the National Bureau of Standards*, 69B(1965):125–130, 1965.

[Ful71] Delbert Ray Fulkerson. Blocking and anti-blocking pairs of polyhedra. *Mathematical programming*, 1:168–194, 1971.

[MS16]   Tobias Mömke and Ola Svensson. Removing and adding edges for the traveling salesman problem. *Journal of the ACM*, 63(1):2, 2016.

[NP81]   Denis Naddef and William R. Pulleyblank. Matchings in regular graphs. *Discrete Mathematics*, 34(3):283–291, 1981.

[Pet91]   Julius Petersen. Die Theorie der regulären Graphs. *Acta Mathematica*, 15(1):193–220, 1891.

[Sch03]   Alexander Schrijver. Combinatorial optimization: Polyhedra and efficiency. Volume A: Paths, flows, matchings. *Algorithms and Combinatorics*, 24(1), 2003.