Basic Packing of Arborescences

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Let $G = (V, E)$ be an undirected graph and $k$ be an integer. $G$ is called $k$-partition connected if, for every partition $\mathcal{P}$ of $V$, $e(\mathcal{P}) \geq k(|P| - 1)$.

**Theorem Tutte Nash-Williams (1961)**

There exists a packing of $k$ spanning trees in $G$ if and only if $G$ is $k$-partition connected.

**Theorem Edmonds (1973)**

There exists a packing of $k$-arborescences in $D$ if and only if $D$ is $k$-rooted connected at $r$.

**Theorem Frank (1978)**

There exists a $k$-rooted connected orientation of $G$ if and only if $G$ is $k$-partition connected.
Let $G = (V, E)$ be an undirected graph and $k$ be an integer. $G$ is called $k$-partition connected if, for every partition $\mathcal{P}$ of $V$, $e(\mathcal{P}) \geq k(|P| - 1)$.

**Theorem Tutte Nash-Williams (1961)**

There exists a packing of $k$ spanning trees in $G$\
$\iff$ $G$ is $k$-partition connected.

Let $D = (V, A)$ be a directed graph, $r \in V$ and $k$ be an integer. $D$ is called $k$-rooted connected at $r$ if, for every non-trivial subset $X$ of $V - r$, $\rho_D(X) \geq k$.

**Theorem Edmonds (1973)**

There exists a packing of $k r$-aborescences in $D$\
$\iff$ $D$ is $k$-rooted connected at $r$. 
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A matroid is a pair $\mathcal{M} = (S, r_M)$ where $S$ is a set and $r_M$ is a non-negative integer valued set-function defined on $S$ such that for all $U, T \subseteq S$

- $r_M(U) \leq r_M(T)$ if $U \subseteq T$,
- $r_M(U) \leq |U|$,
- $r_M(U \cup T) + r_M(U \cap T) \leq r_M(U) + r_M(T)$ (submodularity).

A set $U \subseteq S$ is called independent if $r_M(U) = |U|$.

A base if it is independent and of maximal rank.

For $U \subseteq S$ we define $\text{Span}(U) = \{ s \in S; r_M(U \cup \{s\}) = r_M(U) \}$. 

Example
A matroid is a pair $\mathcal{M} = (S, r_\mathcal{M})$ where $S$ is set and $r_\mathcal{M}$ is a non-negative integer valued set-function defined on $S$ such that for all $U, T \subseteq S$

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**Example**

$S$ is the set of columns of a matrix

$r_\mathcal{M}(U)$ is the rank of the subspace spanned by the vectors in $U$

$U$ is independent $\iff$ the vectors of $U$ are linearly independent.
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---

**Example**

S is the edge set of a graph

$r_\mathcal{M}(U)$ is the size of a maximal forest contained in $U$

U is independent $\Leftrightarrow$ U contains no cycle

If the graph is connected a base is spanning tree.
A matroid-based rooted-graph is a quadruple \((G, S, \pi, \mathcal{M})\) where
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- \(\mathcal{M}\) is a matroid on \(S\).
A packing of trees $(T_s)_{s \in S}$ in $G$ is called $\mathcal{M}$-basic if

$$\pi(s) \in V(T_s) \text{ for all } s \in S,$$

$$\{s \in S; v \in V(T_s)\} \text{ is a base of } \mathcal{M} \text{ for all } v \in V.$$

Notation: $S_X = \pi^{-1}(X)$ for $X \subseteq V$.

A placement $\pi$ is called $\mathcal{M}$-indepedent if $S_v$ is independent for all $v \in V$.

The graph $G$ is called $\mathcal{M}$-partition connected if, for all partition $P$ of $V$,

$$e(P) \geq \sum X \in P \left( r_{\mathcal{M}}(S) - r_{\mathcal{M}}(S_X) \right).$$

Theorem Katoh, Tanigawa 2012

Let $(G, S, \pi, \mathcal{M})$ be a matroid-based rooted-graph. $(G, S, \pi, \mathcal{M})$ contains a $\mathcal{M}$-basic packing of trees $\iff$ $\pi$ is $\mathcal{M}$-independent and $G$ is $\mathcal{M}$-partition connected.
A packing of trees \((T_s)_{s \in S}\) in \(G\) is called \(\mathcal{M}\)-basic if

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Let \((G, S, \pi, \mathcal{M})\) be a matroid-based rooted-graph.

\((G, S, \pi, \mathcal{M})\) contains a \(\mathcal{M}\)-basic packing of trees
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\[ \pi(s) \text{ is the root of } V(T_s) \text{ for all } s \in S, \]

\[ \{ s \in S; v \in V(T_s) \} \text{ is a base of } M \text{ for all } v \in V. \]

The directed graph \(D\) is called \(M\)-connected if, for all subset \(X\) of \(V\),

\[ \rho_D(X) \geq r_M(S) - r_M(S \setminus X). \]

**Theorem DdG, Nguyen, Szigeti 2012**

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\(\iff\) \(\pi\) is \(M\)-independent and \(D\) is \(M\)-connected.
Theorem Frank 1980

Let $G = (V, E)$ be a graph and $h$ an intersecting supermodular non-increasing set-function defined on $V$.

There exists an orientation $D$ of $G$ s. t. $\rho_D(X) \geq h(X)$ $\forall \emptyset \neq X \subset V$

$\iff e(\mathcal{P}) \geq \sum_{X \in \mathcal{P}} h(X)$ for all partition $\mathcal{P}$ of $V$. 

Corollary

Let $(G, S, \pi, M)$ be a matroid-based rooted-graph.

There exists an orientation $D$ of $G$ s. t. $(D, S, \pi, M)$ is $M$-connected

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There exists a $k$-rooted connected orientation of $G$ 

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Choosing $h(X) = r_M(S) - r_M(S_X)$ for all $\emptyset \neq X \subset V$.

**Corollary**

Let $(G, S, \pi, M)$ be a matroid-based rooted-graph.

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Corollary of Frank Theorem
Let \((G, S, \pi, \mathcal{M})\) be a matroid-based rooted-graph.

There exists an orientation \(D\) of \(G\) s.t. \((G, S, \pi, \mathcal{M})\) is \(\mathcal{M}\)-connected
\(\iff\) \((G, S, \pi, \mathcal{M})\) is \(\mathcal{M}\)-partition connected .
Let $uv$ be an arc and choose $s \in S_u$.

Define the matroid-based rooted digraph $(D', S', \pi', \mathcal{M}')$ where:

- $D' = D - uv$,
- $S' = S + s'$,
- $s'$ is placed at $v$,
- $s'$ is parallel to $s$ in $\mathcal{M}'$. 

If $\pi'$ is independent and $D'$ is $\mathcal{M}'$-connected then there exists (by induction) an $\mathcal{M}'$-basic packing in $D'$.
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![Diagram](image-url)
Let \( uv \) be an arc and choose \( s \in S_u \).
Define the matroid-based rooted digraph \( (D', S', \pi', \mathcal{M}') \) where

- \( D' = D - uv \),
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- \( s' \) is placed at \( v \),
- \( s' \) is parallel to \( s \) in \( \mathcal{M}' \).
Let $uv$ be an arc and choose $s \in S_u$. Define the matroid-based rooted digraph $(D', S', \pi', \mathcal{M}')$ where:

- $D' = D - uv$,
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- $s'$ is placed at $v$,
- $s'$ is parallel to $s$ in $\mathcal{M}'$.

If $\pi'$ is independent and $D'$ is $\mathcal{M}'$-connected then there exists (by induction) an $\mathcal{M}'$-basic packing in $D'$. 

![Diagram of the rooted digraph](image-url)
Let $uv$ be an arc and choose $s \in S_u$.
Define the matroid-based rooted digraph $(D', S', \pi', M')$ where

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If $\pi'$ is independent and $D'$ is $M'$-connected then there exists (by induction) an $M'$-basic packing in $D'$
Let $uv$ be an arc and choose $s \in S_u$.
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- $D' = D - uv$,
- $S' = S + s'$,
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- $s'$ is parallel to $s$ in $M'$.

If $\pi'$ is independent and $D'$ is $M'$-connected then there exists (by induction) an $M'$-basic packing in $D'$ and an $M$-basic packing in $D$. 
\( \pi' \) is independent \( \iff \) \( s \notin \text{Span}(S_v) \).

\( D' \) is NOT \( M' \)-connected \( \iff \) \( uv \) enters a vertex set \( X \) such that
\[
\rho_D(X) = r_M(S) - r_M(S_X) \text{ and } s \in \text{Span}(S_X).
\]
\[ \pi' \text{ is independent } \iff s \notin \text{Span}(S_v). \]
\[ D' \text{ is NOT } M'-\text{connected } \iff uv \text{ enters a vertex set } X \text{ such that } \]
\[ \rho_D(X) = r_M(S) - r_M(S_X) \text{ and } s \in \text{Span}(S_X). \]

A vertex set \( X \) is **tight** is \( \rho_D(X) = r_M(S) - r_M(S_X) \).
A vertex set \( Y \) **dominates** a vertex set \( X \) if \( S_X \subseteq \text{Span}(S_Y) \).
An arc \( uv \) is **good** if \( v \) does NOT dominate \( u \).
\( \pi' \) is independent \( \iff \) \( s \notin \text{Span}(S_v) \).

\( D' \) is NOT \( \mathcal{M}' \)-connected \( \iff \) \( uv \) enters a vertex set \( X \) such that \( \rho_D(X) = r_{\mathcal{M}}(S) - r_{\mathcal{M}}(S_X) \) and \( s \in \text{Span}(S_X) \).

A vertex set \( X \) is **tight** is \( \rho_D(X) = r_{\mathcal{M}}(S) - r_{\mathcal{M}}(S_X) \).

A vertex set \( Y \) dominates a vertex set \( X \) if \( S_X \subseteq \text{Span}(S_Y) \).

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A vertex set \( X \) is **tight** if \( \rho_D(X) = r_{\mathcal{M}}(S) - r_{\mathcal{M}}(S_X) \).

A vertex set \( Y \) **dominates** a vertex set \( X \) if \( S_X \subseteq \text{Span}(S_Y) \).

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**Claim**

Every vertex \( v \) of a tight set \( X \) inducing no good arc dominates \( X \).
\[ \pi' \text{ is independent} \iff s \notin \text{Span}(S_v). \]
\[ D' \text{ is NOT } M' \text{-connected} \iff uv \text{ enters a vertex set } X \text{ such that} \]
\[ \rho_D(X) = r_M(S) - r_M(S_X) \text{ and } s \in \text{Span}(S_X). \]

A vertex set \( X \) is **tight** if \( \rho_D(X) = r_M(S) - r_M(S_X) \).

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**Claim**

Every vertex \( v \) of a tight set \( X \) inducing no good arc dominates \( X \).

If there exists no good arc, the packing in which each arborescence has no arc is \( M \)-basic.
\[ \pi' \text{ is independent } \iff s \notin \text{Span}(S_v). \]

\[ D' \text{ is NOT } \mathcal{M}'\text{-connected } \iff uv \text{ enters a vertex set } X \text{ such that} \]

\[ \rho_D(X) = r_M(S) - r_M(S_X) \text{ and } s \in \text{Span}(S_X). \]

A vertex set \( X \) is **tight** is \( \rho_D(X) = r_M(S) - r_M(S_X). \)

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**Claim**

Every vertex \( v \) of a tight set \( X \) inducing no good arc dominates \( X \).

If there exists no good arc, the packing in which each arborescence has no arc is \( \mathcal{M} \)-basic. So we may assume that there exists good arcs.
\( \pi' \) is independent \( \iff \) \( s \notin \text{Span}(S_v) \).

\( D' \) is NOT \( \mathcal{M}' \)-connected \( \iff \) \( uv \) enters a vertex set \( X \) such that
\[
\rho_{D}(X) = r_{\mathcal{M}}(S) - r_{\mathcal{M}}(S_X) \text{ and } s \in \text{Span}(S_X).
\]

A vertex set \( X \) is **tight** is \( \rho_{D}(X) = r_{\mathcal{M}}(S) - r_{\mathcal{M}}(S_X) \).

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**Claim**

Every vertex \( v \) of a tight set \( X \) inducing no good arc dominates \( X \).

If there exists no good arc, the packing in which each arborescence has no arc is \( \mathcal{M} \)-basic. So we may assume that there exists good arcs and that each good arc \( uv \) enters a tight set \( X \) that dominates \( u \).
\[ \pi' \text{ is independent } \iff s \notin \text{Span}(S_v). \]

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A vertex set \(X\) is **tight** is \(\rho_{D}(X) = r_{\mathcal{M}}(S) - r_{\mathcal{M}}(S_X)\).

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**Claim**

Every vertex \(v\) of a tight set \(X\) inducing no good arc dominates \(X\).

If there exists no good arc, the packing in which each arborescence has no arc is \(\mathcal{M}\)-basic. So we may assume that there exists good arcs and that each good arc \(uv\) enters a tight set \(X\) that dominates \(u\).

- Choose \((uv, X)\) with \(X\) minimal
\( \pi' \) is independent \iff \( s \notin \text{Span}(S_v) \).

\( D' \) is NOT \( \mathcal{M}' \)-connected \iff \( uv \) enters a vertex set \( X \) such that
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\rho_D(X) = r_M(S) - r_M(S_X) \text{ and } s \in \text{Span}(S_X).
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A vertex set \( X \) is tight if \( \rho_D(X) = r_M(S) - r_M(S_X) \).

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**Claim**

Every vertex \( v \) of a tight set \( X \) inducing no good arc dominates \( X \).

If there exists no good arc, the packing in which each arborescence has no arc is \( \mathcal{M} \)-basic. So we may assume that there exists good arcs and that each good arc \( uv \) enters a tight set \( X \) that dominates \( u \).

- Choose \((uv, X)\) with \( X \) minimal
- \( X \) induces a good arc \( u'v' \)
\[
\pi' \text{ is independent } \iff s \notin Span(S_v).
\]
\[
D' \text{ is NOT } M'-\text{connected } \iff uv \text{ enters a vertex set } X \text{ such that }
\rho_D(X) = r_M(S) - r_M(S_X) \text{ and } s \in Span(S_X).
\]

A vertex set \(X\) is **tight** if \(\rho_D(X) = r_M(S) - r_M(S_X)\).
A vertex set \(Y\) **dominates** a vertex set \(X\) if \(S_X \subseteq Span(S_Y)\).
An arc \(uv\) is **good** if \(v\) does NOT dominate \(u\).

**Claim**

Every vertex \(v\) of a tight set \(X\) inducing no good arc dominates \(X\).

If there exists no good arc, the packing in which each arborescence has no arc is \(M\)-basic. So we may assume that there exists good arcs and that each good arc \(uv\) enters a tight set \(X\) that dominates \(u\).

- Choose \((uv, X)\) with \(X\) minimal
- \(X\) induces a good arc \(u'v'\)
- \(u'v'\) enters a tight set \(Y\) that dominates \(u'\)
\( \pi' \) is independent \iff \( s \not\in \text{Span}(S_v) \).

\( D' \) is NOT \( \mathcal{M}' \)-connected \iff \( uv \) enters a vertex set \( X \) such that
\[
\rho_D(X) = r_{\mathcal{M}}(S) - r_{\mathcal{M}}(S_X) \quad \text{and} \quad s \in \text{Span}(S_X).
\]

A vertex set \( X \) is **tight** if \( \rho_D(X) = r_{\mathcal{M}}(S) - r_{\mathcal{M}}(S_X) \).

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**Claim**

Every vertex \( v \) of a tight set \( X \) inducing no good arc dominates \( X \).

If there exists no good arc, the packing in which each arborescence has no arc is \( \mathcal{M} \)-basic. So we may assume that there exists good arcs and that each good arc \( uv \) enters a tight set \( X \) that dominates \( u \).

- Choose \( (uv, X) \) with \( X \) minimal
- \( X \) induces a good arc \( u'v' \)
- \( u'v' \) enters a tight set \( Y \) that dominates \( u' \)
- \( u'v' \) enters the tight set \( Y \cap X \) that dominates \( u' \)
\[ \pi' \text{ is independent } \iff s \notin \text{Span}(S_v). \]

\[ D' \text{ is NOT } \mathcal{M}'\text{-connected } \iff uv \text{ enters a vertex set } X \text{ such that } \rho_D(X) = r_{\mathcal{M}}(S) - r_{\mathcal{M}}(S_X) \text{ and } s \in \text{Span}(S_X). \]

A vertex set \( X \) is **tight** if \( \rho_D(X) = r_{\mathcal{M}}(S) - r_{\mathcal{M}}(S_X) \).

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Every vertex \( v \) of a tight set \( X \) inducing no good arc dominates \( X \).

If there exists no good arc, the packing in which each arborescence has no arc is \( \mathcal{M} \)-basic. So we may assume that there exists good arcs and that each good arc \( uv \) enters a tight set \( X \) that dominates \( u \).

- Choose \( (uv, X) \) with \( X \) minimal
- \( X \) induces a good arc \( u'v' \)
- \( u'v' \) enters a tight set \( Y \) that dominates \( u' \)
- \( u'v' \) enters the tight set \( Y \cap X \) that dominates \( u' \)
- \( (u'v', Y) \) contradicts the minimality of \( X \).
Thank you for your attention.