NP-hard sub-problems involving costs: examples of applications and Lagrangian based filtering

Hadrien Cambazard

G-SCOP - Université de Grenoble

Plan

1. Context and motivation
   - Illustrative application: the Traveling Purchaser Problem
   - *Optimization* versus *Satisfaction*
   - *Combinatorial* versus *polyhedral* methods

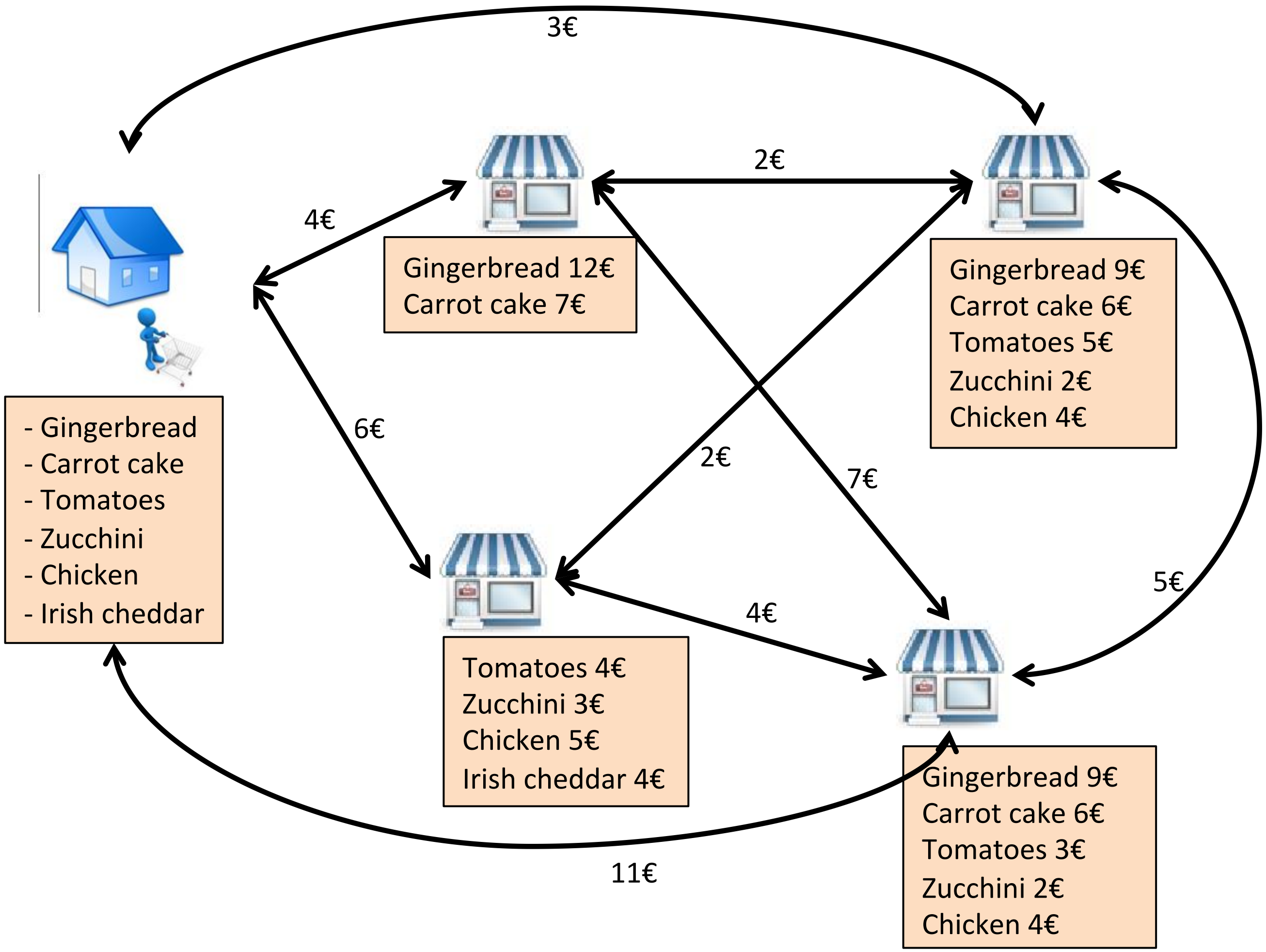
2. Propagation based on Lagrangian Relaxation
   - Lagrangian duality
   - Filtering using Lagrangian reduced costs
   - Let's try on the *Nvalue* global constraint

3. Overview of some NP-Hard Constraints with costs
   - *Multi-cost regular, Weighted-circuit, Weighted-Nvalue, Bin-packing with usage costs*

4. Examples of applications

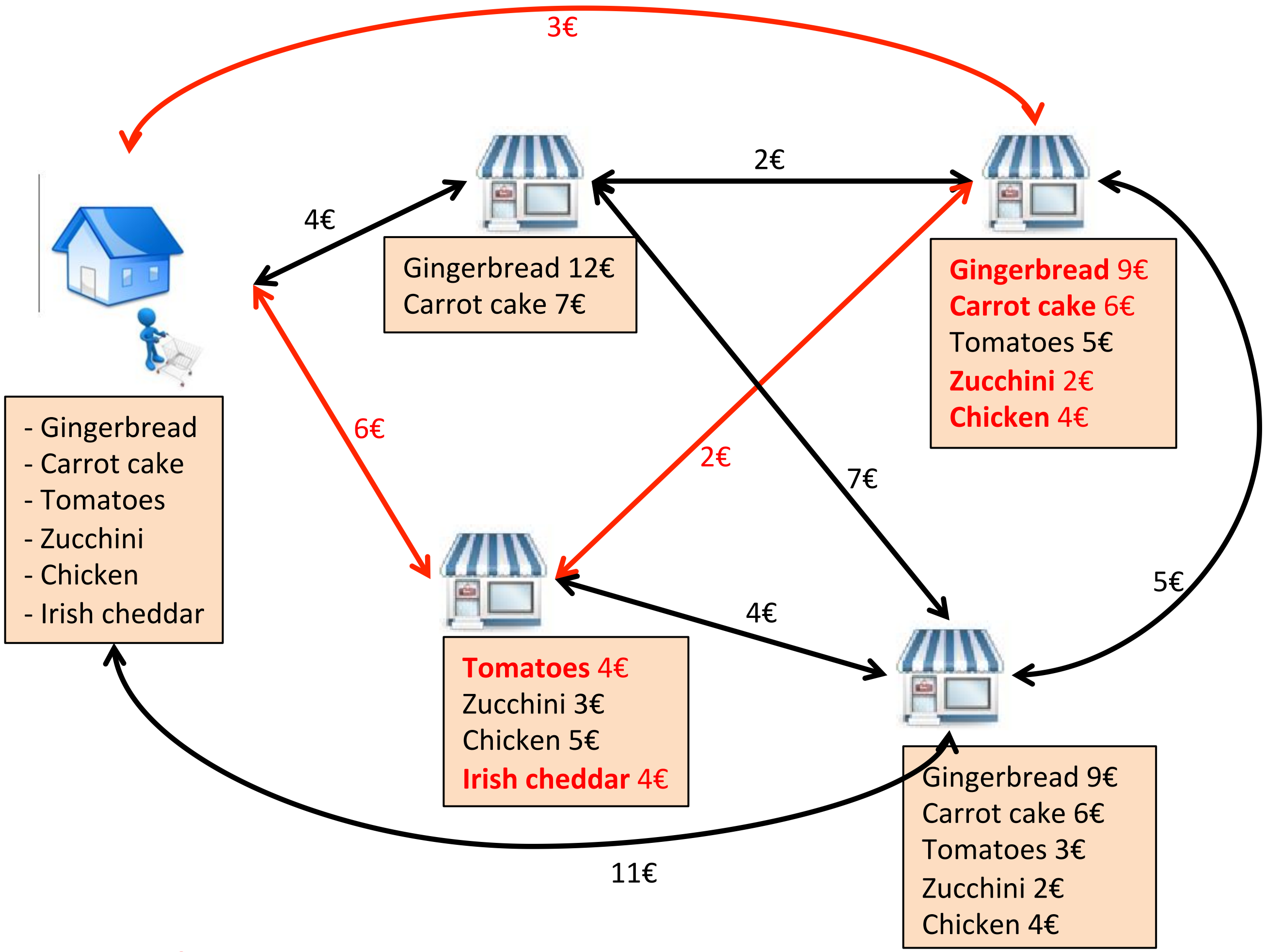
Illustrative Application
The Traveling Purchaser Problem

Traveling Purchaser Problem (TPP)



- **A set of items**
- **A set of markets,** each selling some of the items at different prices
- **The traveling costs** between markets (and from/to home)

Traveling Purchaser Problem (TPP)



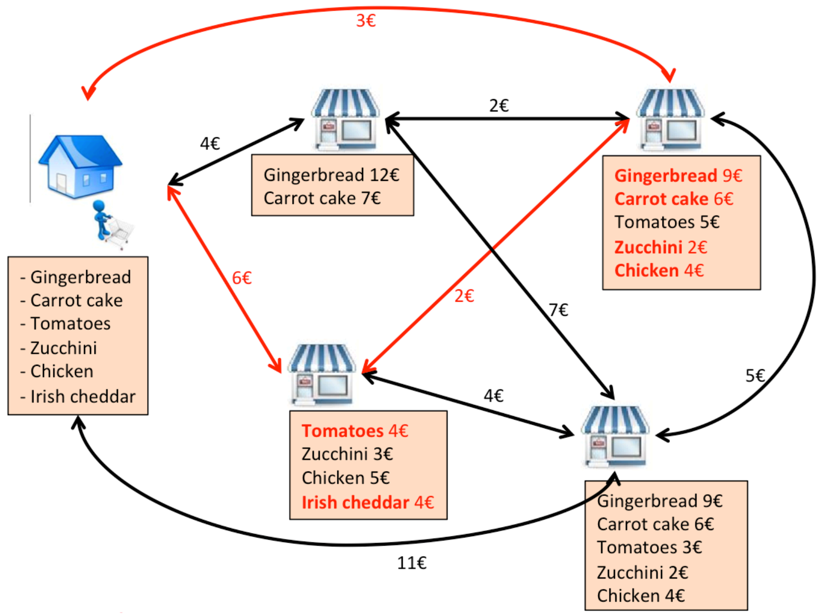
- **A set of items**
- **A set of markets,** each selling some of the items at different prices
- **The traveling costs** between markets (and from/to home)

Traveling cost = 6 + 2 + 3 = 11

Shopping cost = 4 + 4 + 9 + 6 + 2 + 4 = 29

Total cost = 40

Traveling Purchaser Problem (TPP)



Find the route minimizing the sum of traveling and shopping costs to buy all the items

[T. Ramesh, 1981]
[G. Laporte, 2003]
[J. Riera-Ledesma, 2006]
[L.Gouveia, 2011]

- **Generalization of TSP**
- **Numerous heuristics**
- **Best known exact method based on Branch and Cut and Price.** [G. Laporte, 2003]

Let's start modeling

Variables:

$next_i \in \{0, 1, \ldots, n\}$: the successor of market **i** in the shopping trip

$next_i = i$ (i not visited)

$s_k \in \{i | v_i \in M_k\}$: the market where item **k** is bought

Let's start modeling

Variables:

$next_i \in \{0, 1, \ldots, n\}$: the successor of market **i** in the shopping trip
$next_i = i$ (i not visited)

$s_k \in \{i | v_i \in M_k\}$: the market where item **k** is bought

$Cs_k \geq 0$: the price paid for item **k**

$Ct_i \geq 0$: the price paid for traveling from market **i** to its successor

Let's start modeling

Variables:

$next_i \in \{0, 1, \ldots, n\}$: the successor of market **i** in the shopping trip
$next_i = i$ (i not visited)

$s_k \in \{i | v_i \in M_k\}$: the market where item **k** is bought

$Cs_k \geq 0$: the price paid for item **k**

$Ct_i \geq 0$: the price paid for traveling from market **i** to its successor

$$Minimize \sum_i Ct_i + \sum_k Cs_k$$

Let’s start modeling

Variables:

$next_i \in \{0, 1, \ldots, n\}$: the successor of market **i** in the shopping trip
$next_i = i$ (i not visited)

$s_k \in \{i | v_i \in M_k\}$: the market where item **k** is bought

$Cs_k \geq 0$: the price paid for item **k**

$Ct_i \geq 0$: the price paid for traveling from market **i** to its successor

$$Minimize \sum_i Ct_i + \sum_k Cs_k$$

Price of item k in market i

$$\textsc{Element}(Cs_k, [b_{k1}, \ldots, b_{ki}, \ldots, b_{km}], s_k) \ \forall k$$
$$\textsc{Element}(Ct_i, [d_{i1}, \ldots, d_{ij}, \ldots, d_{in}], next_i) \ \forall i$$

Traveling cost from i to j

Let’s start modeling

Variables:

$next_i \in \{0, 1, \ldots, n\}$: the successor of market **i** in the shopping trip
$next_i = i$ (i not visited)

$s_k \in \{i | v_i \in M_k\}$: the market where item **k** is bought

$Cs_k \geq 0$: the price paid for item **k**

$Ct_i \geq 0$: the price paid for traveling from market **i** to its successor

$$Minimize \sum_i Ct_i + \sum_k Cs_k$$

Price of item k in market i

$$\textsc{Element}(Cs_k, [b_{k1}, \ldots, b_{ki}, \ldots, b_{km}], s_k)\ \forall k$$
$$\textsc{Element}(Ct_i, [d_{i1}, \ldots, d_{ij}, \ldots, d_{in}], next_i)\ \forall i$$

Traveling cost from i to j

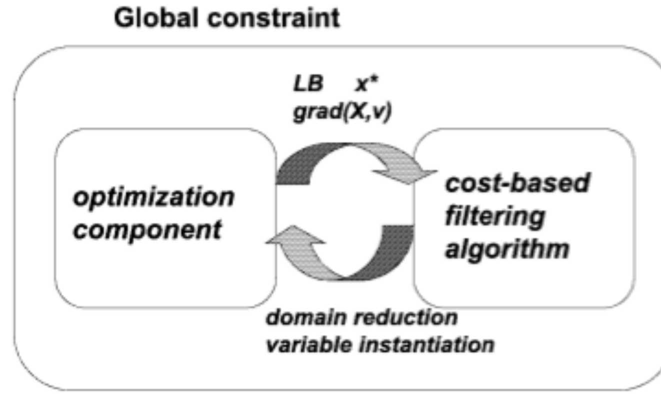
… the $next$ variables must form a circuit + single loops

Optimization in CP

- Objective is decomposed (using Element constraints):
  - Resulting lower bound is often very weak
  - *Infeasible* values are eliminated but not *sub-optimal* ones.

- *sub-optimal = infeasible* regarding the best known upper-bound

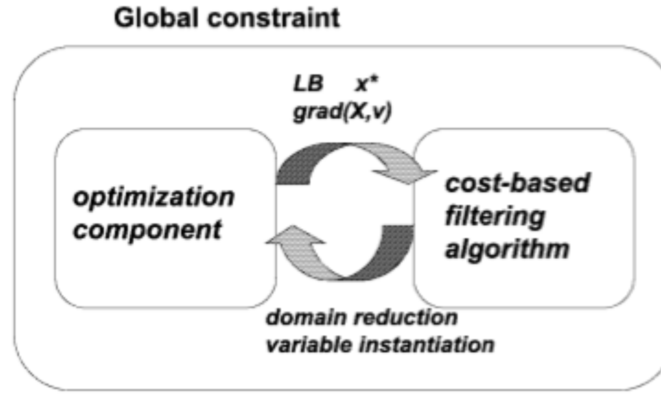
Optimization in CP



(Picture from [Foccaci, 2002])

- Cost-based filtering
  - [Focacci, Lodi, Milano, 2002]: *Embedding relaxations in global constraints for solving TSP and TSPTW*
  - Relaxations based on assignments, spanning tree

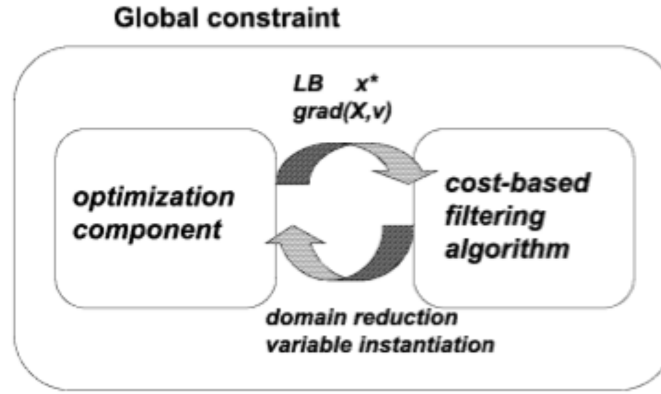
Optimization in CP



(Picture from [Foccaci, 2002])

- Cost-based filtering
  - [Focacci, Lodi, Milano, 2002]: *Embedding relaxations in global constraints for solving TSP and TSPTW*
  - Relaxations based on assignments, spanning tree
- Linear relaxation of global constraints
  - [Refalo, 2000]: *Linear formulation of Constraint Programming models and Hybrid Solvers*

Optimization in CP



(Picture from [Foccaci, 2002])

- Cost-based filtering
  - [Focacci, Lodi, Milano, 2002]: *Embedding relaxations in global constraints for solving TSP and TSPTW*
  - Relaxations based on assignments, spanning tree
- Linear relaxation of global constraints
  - [Refalo, 2000]: *Linear formulation of Constraint Programming models and Hybrid Solvers*
- Back to the TPP: what cost-based filtering can be done ?

TPP: cost based filtering ?

- The traveler has to visit *a minimum number of markets* to buy everything
  - Lower bound of traveling cost

- The traveler *can not visit too many markets* (traveling cost would be too high w.r.t to known upper bound)
  - Lower bound of shopping cost

- Number of markets visited: $Nvisit$

Problem structure 1 : *Hitting set*

- Look only at **feasibility**
- Can we buy everything in less than $\overline{Nvisit}$ markets ?

Problem structure 1 : *Hitting set*

- Look only at **feasibility**
- Can we buy everything in less than $\overline{Nvisit}$ markets ?

→ Hitting Set Problem

$$\overline{Nvisit} = 3$$

Gingerbread:	{M2, M3, M6, M7}
Carrot cake:	{M2, M5}
Organic tomatoes:	{M1, M2, M4, M6}
Zucchini:	{M3, M4, M7}
Chicken:	{M1, M4}
Irish cheddar:	{M8, M9}

Problem structure 1 : *Hitting set*

- Look only at **feasibility**
- Can we buy everything in less than $\overline{Nvisit}$ markets ?

→ Hitting Set Problem

$$\overline{Nvisit} = 3$$

{M2, M4, M8}

Gingerbread:	{**M2**, M3, M6, M7}
Carrot cake:	{**M2**, M5}
Organic tomatoes:	{M1, **M2**, M4, M6}
Zucchini:	{M3, **M4**, M7}
Chicken:	{M1, **M4**}
Irish cheddar:	{**M8,** M9}

Problem structure 1 : *Hitting set*

- Look only at **feasibility**
- Can we buy everything in less than $\overline{Nvisit}$ markets ?

→ Hitting Set Problem

$$\overline{Nvisit} = 3$$

{M2, M4, M8}

Gingerbread: {**M2**, M3, M6, M7}

Carrot cake: {**M2**, M5}

Organic tomatoes: {M1, **M2**, M4, M6}

Zucchini: {M3, **M4**, M7}

Chicken: {M1, **M4**}

Irish cheddar: {**M8,** M9}

Problem structure 1 : *Hitting set*

- Look only at **feasibility**
- Can we buy everything in less than $\overline{Nvisit}$ markets ?

Hitting Set Problem

Gingerbread:	{**M2**, M3, M6, M7}
Carrot cake:	{**M2**, M5}
Organic tomatoes:	{M1, **M2**, M4, M6}
Zucchini:	{M3, **M4**, M7}
Chicken:	{M1, **M4**}
Irish cheddar:	{**M8,** M9}

In CP:
AtMostNValue

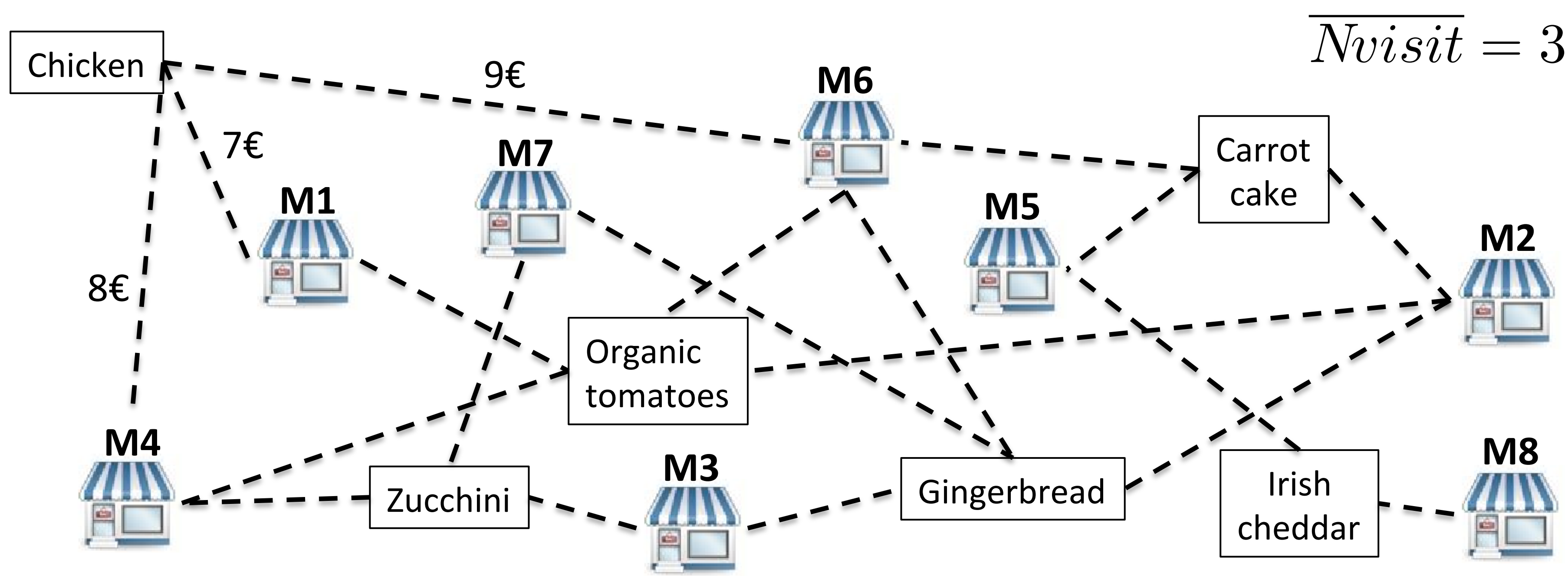
Problem structure 2 : *p-median*

- Look only at **feasibility + shopping cost**
- What is the cheapest way to buy everything in less than $\overline{Nvisit}$ markets ?

Problem structure 2 : *p-median*

- Look only at **feasibility + shopping cost**
- What is the cheapest way to buy everything in less than $\overline{Nvisit}$ markets ?

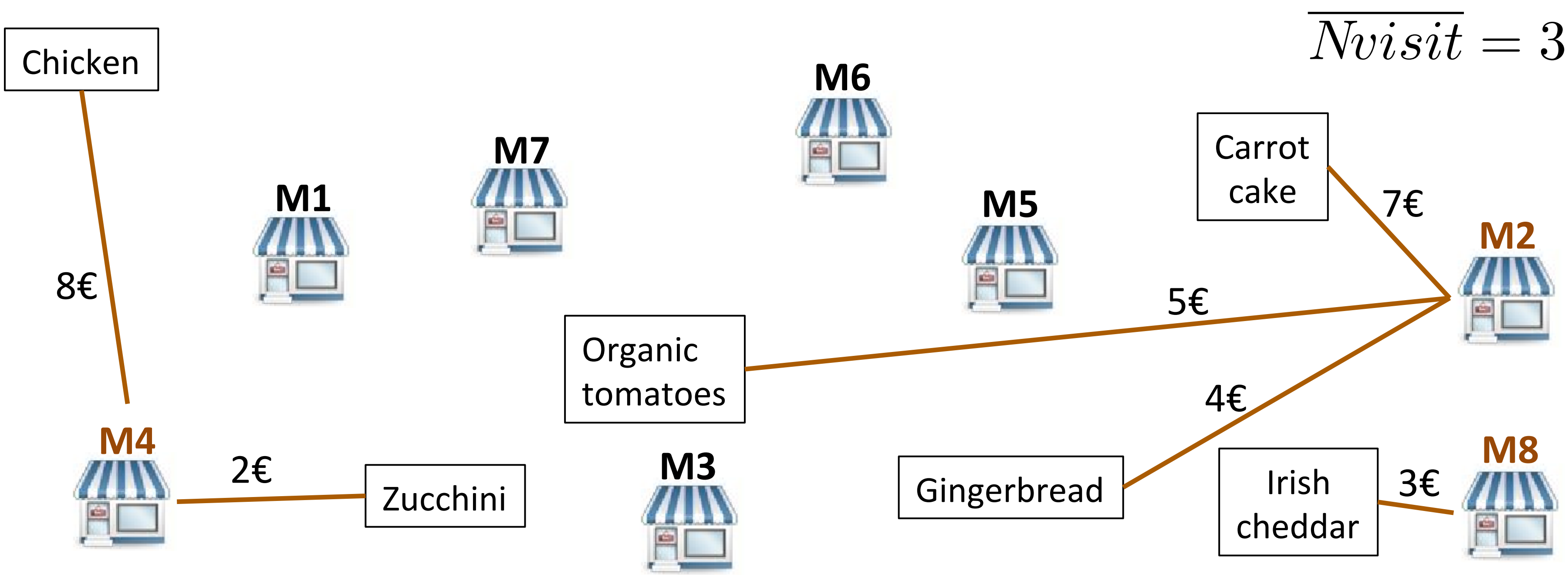
➡ p-median Problem



Problem structure 2 : *p-median*

- Look only at **feasibility + shopping cost**
- What is the cheapest way to buy everything in less than $\overline{Nvisit}$ markets ?

➡ p-median Problem



Problem structure 2 : *p-median*

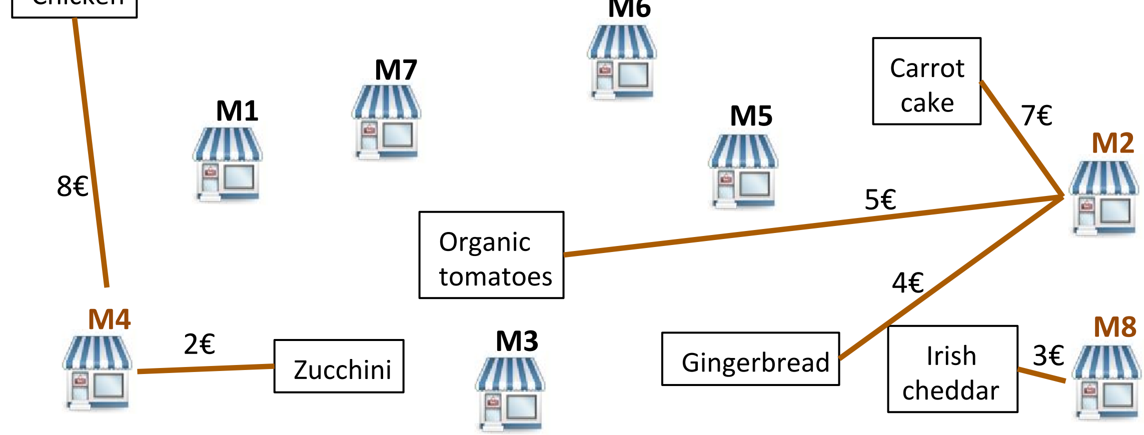
- Look only at **feasibility + shopping cost**
- What is the cheapest way to buy everything in less than $\overline{Nvisit}$ markets ?

p-median Problem

In CP:

AtMostNValue with costs ?

$\overline{Nvisit} = 3$



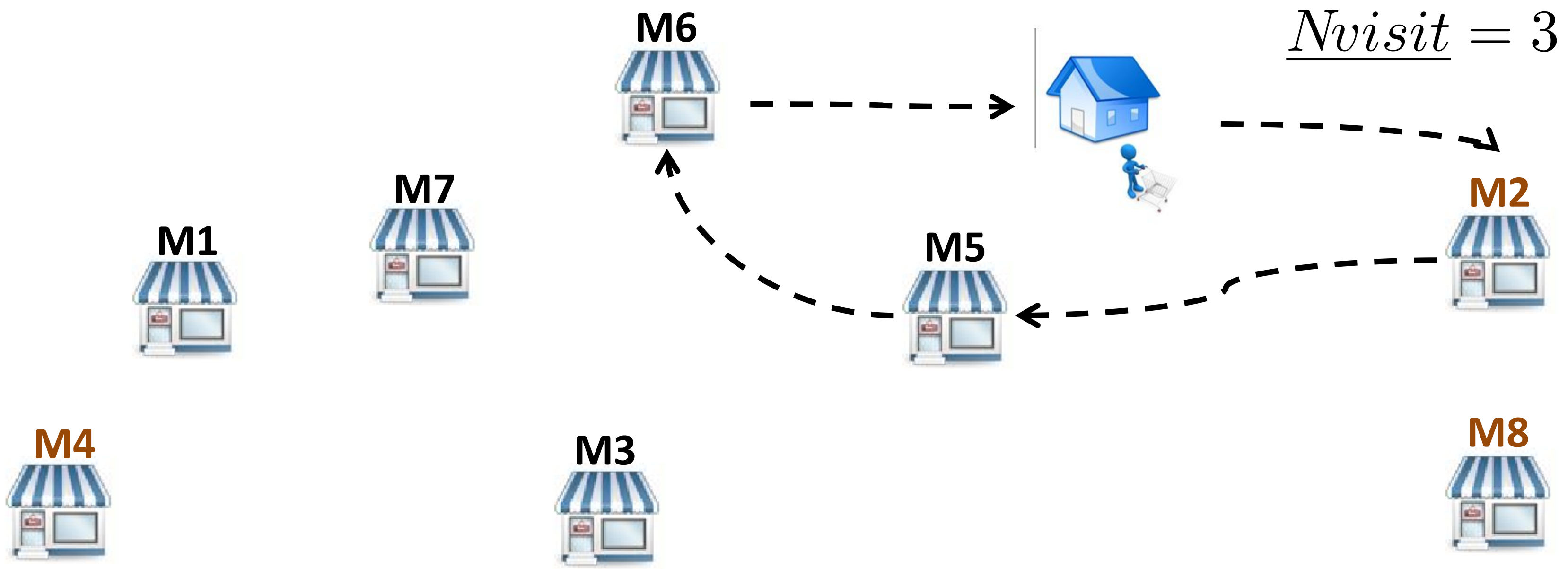
Problem structure 3 : *k-TSP*

- Look only at **traveling cost**
- What is the cheapest way to visit at least $\underline{Nvisit}$ markets ?

Problem structure 3 : *k-TSP*

- Look only at **traveling cost**
- What is the cheapest way to visit at least $\underline{Nvisit}$ markets ?

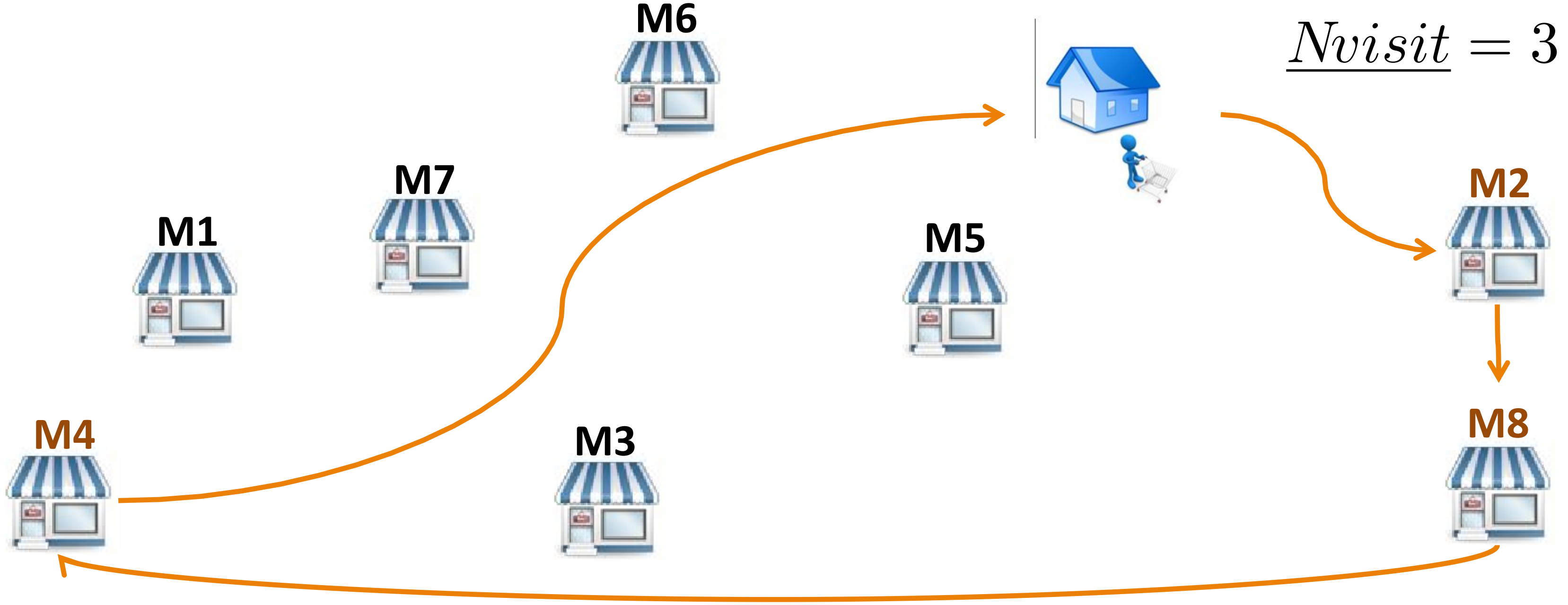
➡ k-TSP problem



Problem structure 3 : *k-TSP*

- Look only at **traveling cost**
- What is the cheapest way to visit at least $\underline{Nvisit}$ markets ?

k-TSP problem



Problem structures

$Nvisit \in \{1, \ldots, B\}$: Number of visited markets

$TotalCost = TravelingCost + ShoppingCost$

Relaxation	Nature of the problem	Value of the parameter	How to solve / propagate it ?	Key propagation
Feasibility	Hitting Set	$\overline{Nvisit}$ (cardinality)		$\underline{Nvisit}$
Feasibility + Shopping cost	p-median	$p = \overline{Nvisit}$		$\underline{ShoppingCost}$ $\underline{Nvisit}$
Traveling Cost	k-TSP	$k = \underline{Nvisit}$		$\underline{TravelingCost}$ $\overline{Nvisit}$

Problem structures

$Nvisit \in \{1, \ldots, B\}$: Number of visited markets
$TotalCost = TravelingCost + ShoppingCost$

Relaxation	Nature of the problem	Value of the parameter	How to solve / propagate it ?	Key propagation
Feasibility	Hitting Set AtMostNValue	$\overline{Nvisit}$ (cardinality)		$\underline{Nvisit}$
Feasibility + Shopping cost	p-median Weighted-NValue	$p = \overline{Nvisit}$		$\underline{ShoppingCost}$ $\underline{Nvisit}$
Traveling Cost	k-TSP Close to Weighted-Circuit	$k = \underline{Nvisit}$		$\underline{TravelingCost}$ $\overline{Nvisit}$

So far on the TPP

- How to reason about NP-Hard sub-problems involving costs ?

So far on the TPP

- How to reason about NP-Hard sub-problems involving costs ?

- Can CP be competitive with "*advanced linear programming methods*" ?

➡ **Best known exact method based on Branch and Cut and Price.** [G. Laporte, 2003]

So far on the TPP

- How to reason about NP-Hard sub-problems involving costs ?
- Can CP be competitive with "*advanced linear programming methods*" ?

➡ **Best known exact method based on Branch and Cut and Price.** [G. Laporte, 2003]

- Branch and Cut and Price is the state of the art exact framework for a large class of problems related to routing :

TSP, TSPTW, TPP, TTP, VRP, …

Can we question that ?

Plan

1. Context and motivation
   - Illustrative application: the Traveling Purchaser Problem
   - *Optimization* versus *Satisfaction*
   - *Combinatorial* versus *polyhedral* methods

2. Propagation based on Lagrangian Relaxation
   - Principles of Lagrangian duality
   - Filtering using Lagrangian reduced costs
   - Let's try on the *Nvalue* global constraint

3. Overview of some NP-Hard Constraints with costs
   - *Multi-cost regular, Weighted-circuit, Weighted-Nvalue, Bin-packing with usage costs*

4. Examples of applications

Propagation based on Lagrangian Relaxation

Principles, filtering, Experimentations with NValue

2- Lagrangian relaxation

Shortest path with resource constraints

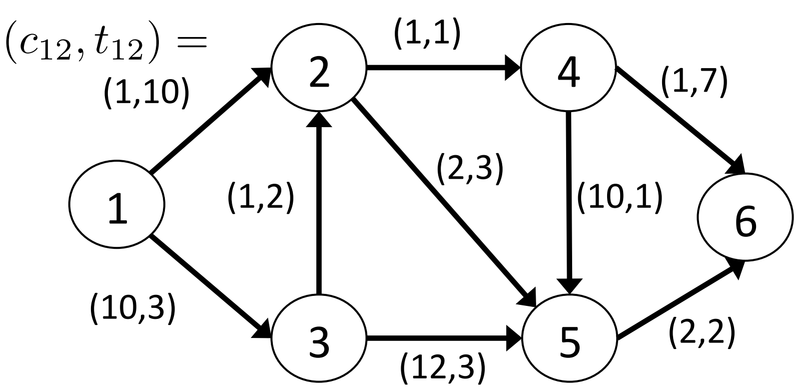
$$\text{Min z} = \sum c_{ij}x_{ij}$$

$$\text{path conservation} \quad (1)$$

$$\sum t_{ij}x_{ij} \leq T \quad (2)$$

$$x_{ij} \in \{0,1\}$$

P



Simplified example taken from *Network flows* of Ahuja, Magnanti, Orlin

2- Lagrangian relaxation

Shortest path with resource constraints

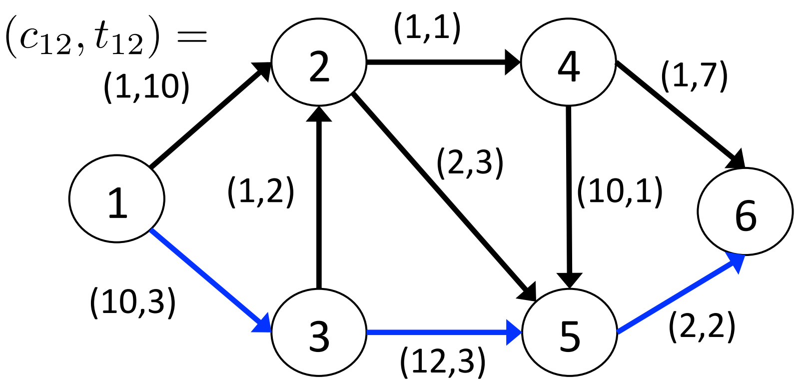
$$\text{Min z} = \sum c_{ij} x_{ij}$$

path conservation (1)

$$\sum t_{ij} x_{ij} \leq T \qquad (2)$$

$$x_{ij} \in \{0, 1\}$$

P



T = 10

A solution:

$$\begin{cases} x_{13} = 1, x_{35} = 1, x_{56} = 1 \\ z = 10 + 12 + 2 = 24 \\ time = 3 + 3 + 2 \leq 10 \end{cases}$$

2- Lagrangian relaxation

Shortest path with resource constraints

$$\text{Min z} = \sum c_{ij}x_{ij}$$

path conservation (1)

$$\sum t_{ij}x_{ij} \leq T \quad (2)$$

$$x_{ij} \in \{0,1\}$$

P

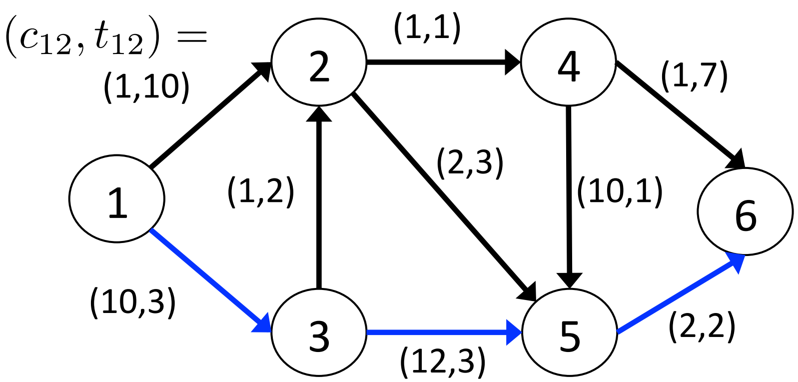
For all $\lambda \geq 0$: **Shortest path**

$$\text{Min } w(\lambda) = \sum c_{ij}x_{ij} - \lambda(T - \sum t_{ij}x_{ij})$$
$$= \sum (c_{ij} + \lambda t_{ij})x_{ij} - \lambda T$$

path conservation (1)

$$x_{ij} \in \{0,1\}$$

$L(\lambda)$



T = 10

A solution:
$$x_{13} = 1, x_{35} = 1, x_{56} = 1$$
$$z = 10 + 12 + 2 = 24$$
$$time = 3 + 3 + 2 \leq 10$$

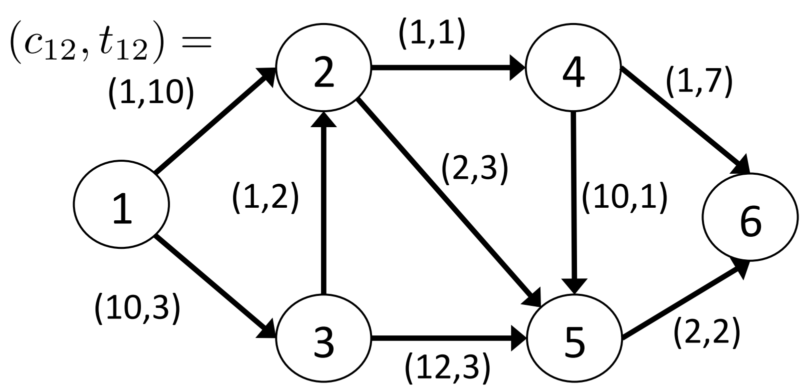
2- Lagrangian relaxation

Shortest path with resource constraints

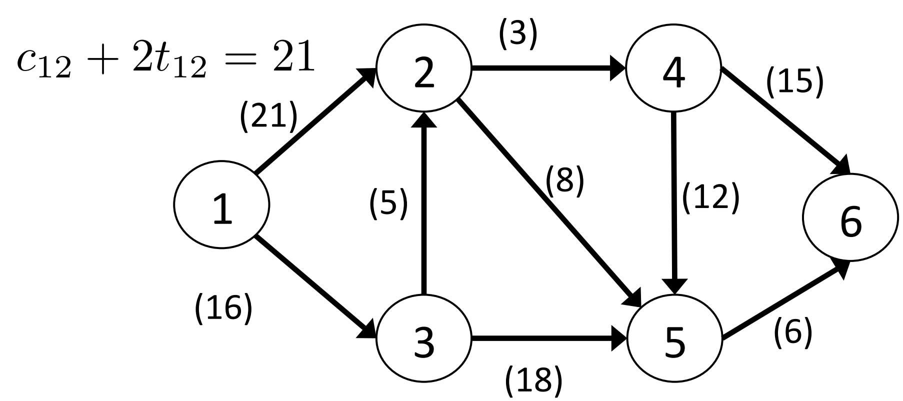
$$\begin{aligned} &\text{Min } z = \sum c_{ij} x_{ij} \\ &\text{path conservation} \quad (1) \\ &\sum t_{ij} x_{ij} \leq T \quad (2) \\ &x_{ij} \in \{0,1\} \end{aligned} \quad P$$

For all $\lambda \geq 0$: **Shortest path**

$$\begin{aligned} \text{Min } w(\lambda) &= \sum c_{ij} x_{ij} - \lambda(T - \sum t_{ij} x_{ij}) \\ &= \sum (c_{ij} + \lambda t_{ij}) x_{ij} - \lambda T \\ &\text{path conservation} \quad (1) \\ &x_{ij} \in \{0,1\} \end{aligned} \quad L(\lambda)$$



Lagrangian sub-problem for $\lambda = 2$



2- Lagrangian relaxation

$$\text{Min } z = \sum c_{ij}x_{ij}$$

path conservation (1)

$$\sum t_{ij}x_{ij} \leq T \quad (2)$$

$$x_{ij} \in \{0,1\}$$

P

T = 10

For all $\lambda \geq 0$:

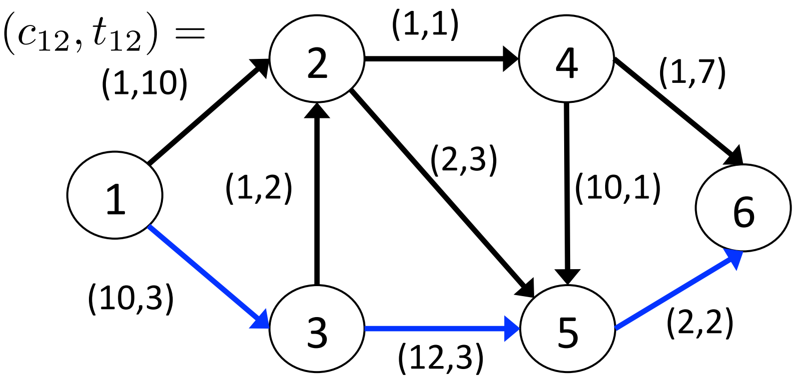
$$\text{Min } w(\lambda) = \sum c_{ij}x_{ij} - \lambda(T - \sum t_{ij}x_{ij}) = \sum(c_{ij} + \lambda t_{ij})x_{ij} - \lambda T$$

path conservation (1)

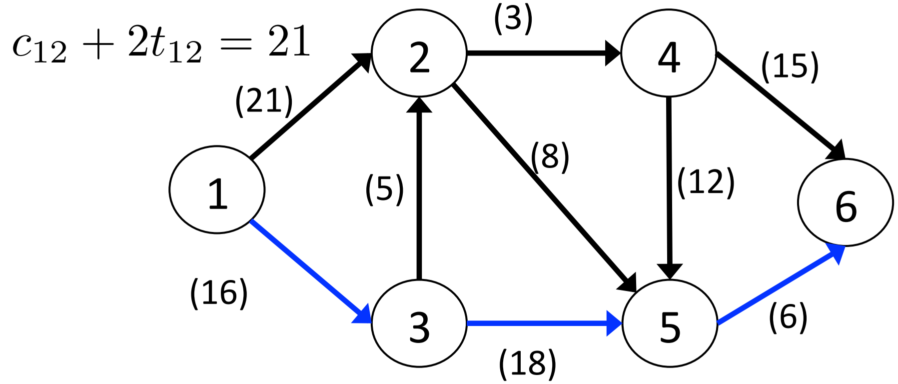
$$x_{ij} \in \{0,1\}$$

$L(\lambda)$

$\lambda = 2$



$\overline{z} = 10 + 12 + 2 = 24$



$\overline{w} = 16 + 18 + 6 - 20 = 20$

For all $\lambda \geq 0$:

Any feasible solution $\overline{x}$ of P is also feasible for $L(\lambda)$ and $\overline{z} \geq \overline{w}(\lambda)$

So we have : $z^* \geq w^*(\lambda)$

2- Lagrangian relaxation

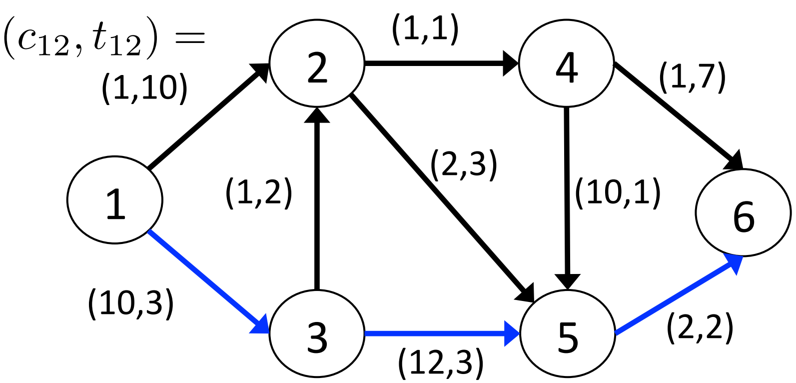
$$\text{Min } z = \sum c_{ij}x_{ij}$$
path conservation (1)
$$\sum t_{ij}x_{ij} \leq T \quad (2)$$
$$x_{ij} \in \{0,1\} \quad P$$

T = 10

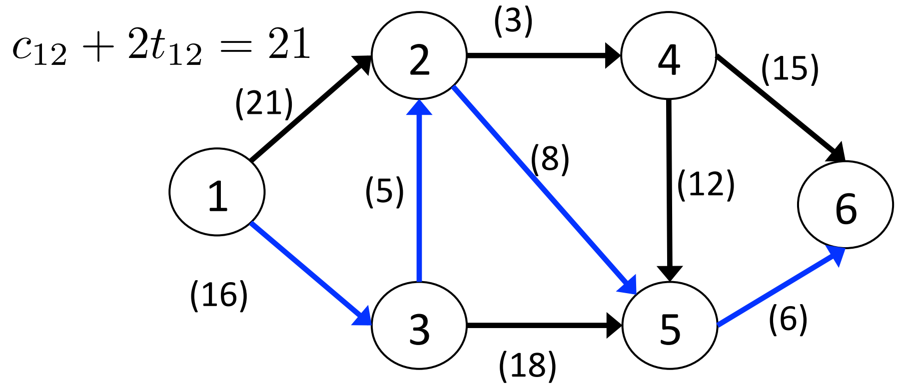
For all $\lambda \geq 0$:

$$\text{Min } w(\lambda) = \sum c_{ij}x_{ij} - \lambda(T - \sum t_{ij}x_{ij}) = \sum (c_{ij} + \lambda t_{ij})x_{ij} - \lambda T$$
path conservation (1)
$$x_{ij} \in \{0,1\} \quad L(\lambda)$$

$\lambda = 2$



$$\overline{z} = 10 + 12 + 2 = 24$$



$$w^*(2) = 35 - 20 = 15$$

For all $\lambda \geq 0$:

Any feasible solution $\overline{x}$ of P is also feasible for $L(\lambda)$ and $\overline{z} \geq \overline{w}(\lambda)$

So we have : $z^* \geq w^*(\lambda)$

2- Lagrangian relaxation

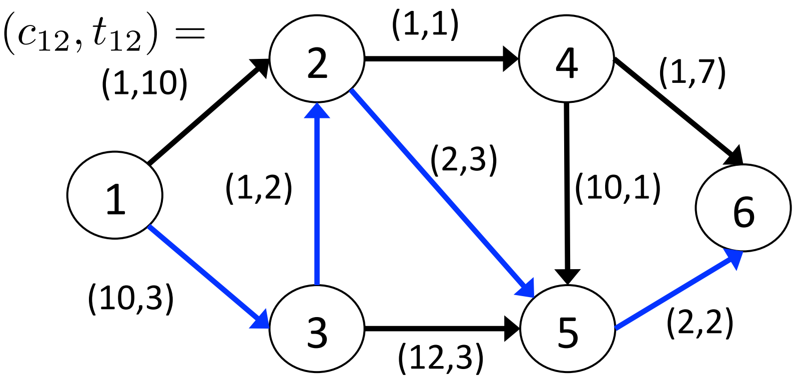
$$\begin{aligned} &\text{Min } z = \sum c_{ij}x_{ij} \\ &\text{path conservation } (1) \\ &\sum t_{ij}x_{ij} \leq T \quad (2) \\ &x_{ij} \in \{0,1\} \end{aligned} \quad P$$

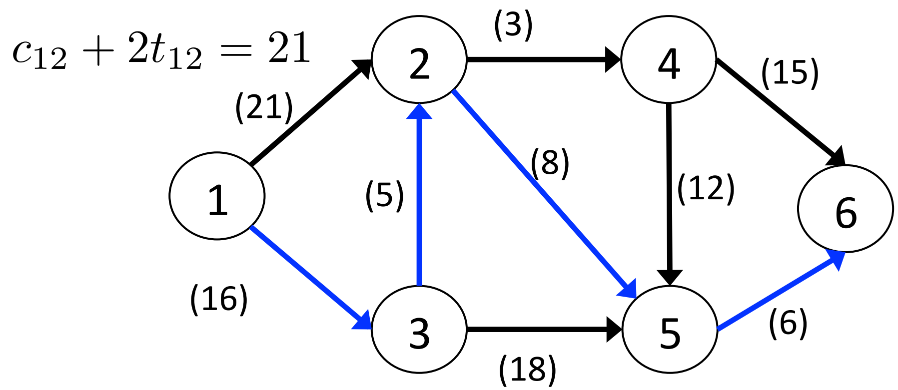
T = 10

For all $\lambda \geq 0$:

$$\begin{aligned} \text{Min } w(\lambda) &= \sum c_{ij}x_{ij} - \lambda(T - \sum t_{ij}x_{ij}) \\ &= \sum (c_{ij} + \lambda t_{ij})x_{ij} - \lambda T \\ &\text{path conservation } (1) \\ &x_{ij} \in \{0,1\} \end{aligned} \quad L(\lambda)$$

$\lambda = 2$





$\overline{z} = 15 = z^*$

$w^*(2) = 35 - 20 = 15$

For all $\lambda \geq 0$:

Any feasible solution $\overline{x}$ of P is also feasible for $L(\lambda)$ and $\overline{z} \geq \overline{w}(\lambda)$

So we have : $z^* \geq w^*(\lambda)$

2- Lagrangian relaxation

$$\text{Min } z = \sum c_{ij}x_{ij}$$
$$\text{path conservation (1)}$$
$$\sum t_{ij}x_{ij} \leq T \quad (2)$$
$$x_{ij} \in \{0,1\}$$
$$P$$

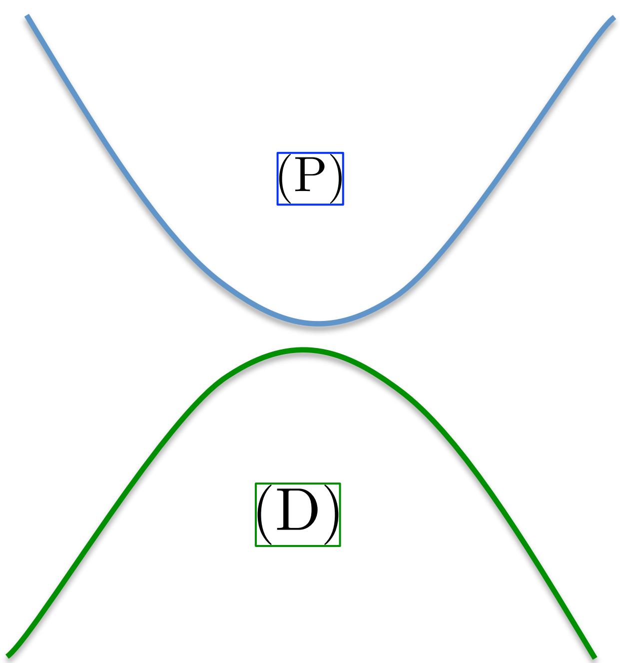
For all $\lambda \geq 0$:

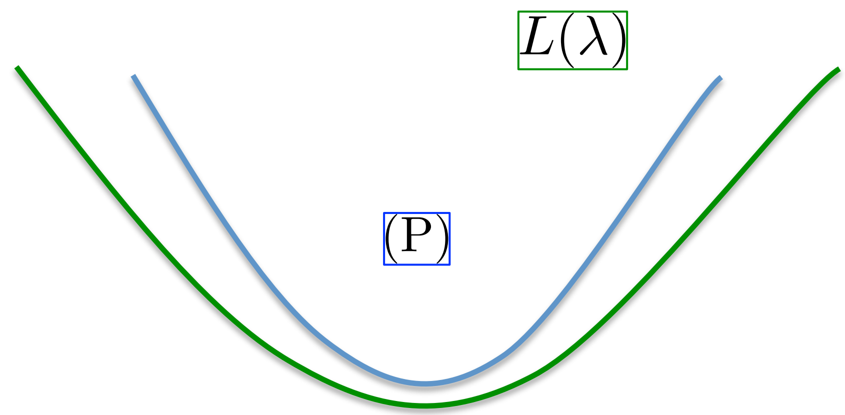
$$\text{Min } w(\lambda) = \sum c_{ij}x_{ij} - \lambda(T - \sum t_{ij}x_{ij})$$
$$= \sum (c_{ij} + \lambda t_{ij})x_{ij} - \lambda T$$
$$\text{path conservation (1)}$$
$$x_{ij} \in \{0,1\}$$
$$L(\lambda)$$

For all $\lambda \geq 0$:

Any feasible solution $\overline{x}$ of P is also feasible for $L(\lambda)$ and $\overline{z} \geq \overline{w}(\lambda)$

So we have : $z^* \geq w^*(\lambda)$





2- Lagrangian relaxation

$$\text{Min } z = \sum c_{ij} x_{ij}$$
$$\text{path conservation (1)}$$
$$\sum t_{ij} x_{ij} \leq T \quad (2)$$
$$x_{ij} \in \{0,1\}$$
P

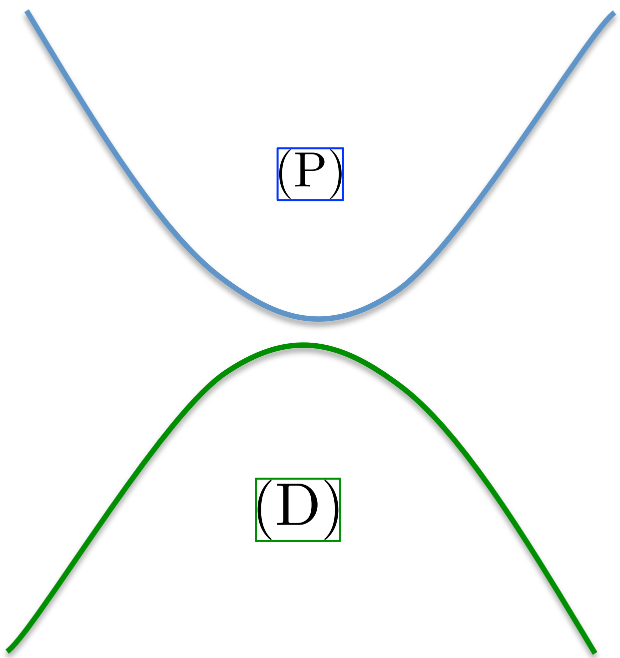
For all $\lambda \geq 0$:

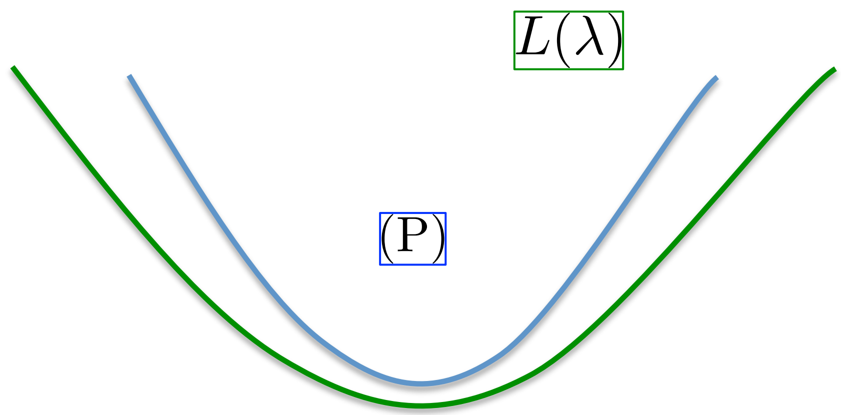
$$\text{Min } w(\lambda) = \sum c_{ij} x_{ij} - \lambda(T - \sum t_{ij} x_{ij})$$
$$= \sum (c_{ij} + \lambda t_{ij}) x_{ij} - \lambda T$$
$$\text{path conservation (1)}$$
$$x_{ij} \in \{0,1\}$$
$L(\lambda)$

For all $\lambda \geq 0$:

Any feasible solution $\overline{x}$ of P is also feasible for $L(\lambda)$ and $\overline{z} \geq \overline{w}(\lambda)$

So we have : $z^* \geq w^*(\lambda)$





Lagrangian Dual:

$$L^* = \max_{\lambda \geq 0} w^*(\lambda)$$

2- Lagrangian relaxation

For all $\lambda \geq 0$:

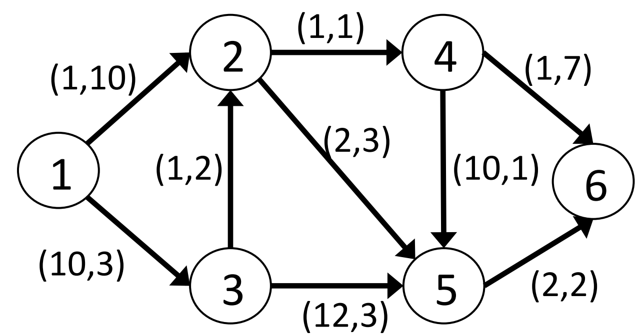
$$\text{Min } w(\lambda) = \sum c_{ij}x_{ij} - \lambda(T - \sum t_{ij}x_{ij})$$
$$= \sum(c_{ij} + \lambda t_{ij})x_{ij} - \lambda T$$

path conservation (1)

$x_{ij} \in \{0,1\}$

$L(\lambda)$

$$L^* = \max_{\lambda \geq 0} w^*(\lambda)$$



- *Note:*
  - Changing λ does not affect the set of feasible solutions of $L(\lambda)$
  - So the cost of given solution of $L(\lambda)$ can be seen as a linear function of λ

2- Lagrangian relaxation

For all $\lambda \geq 0$:

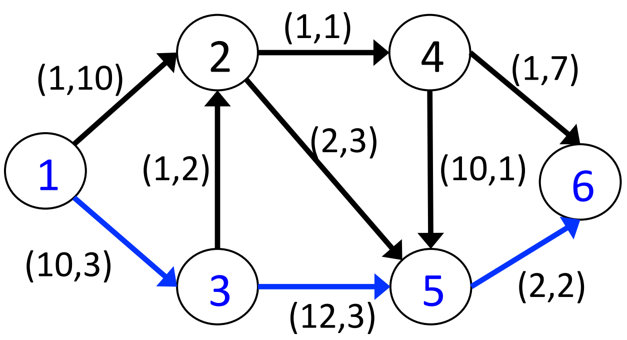
$$\text{Min } w(\lambda) = \sum c_{ij}x_{ij} - \lambda(T - \sum t_{ij}x_{ij})$$
$$= \sum (c_{ij} + \lambda t_{ij})x_{ij} - \lambda T$$

path conservation (1)

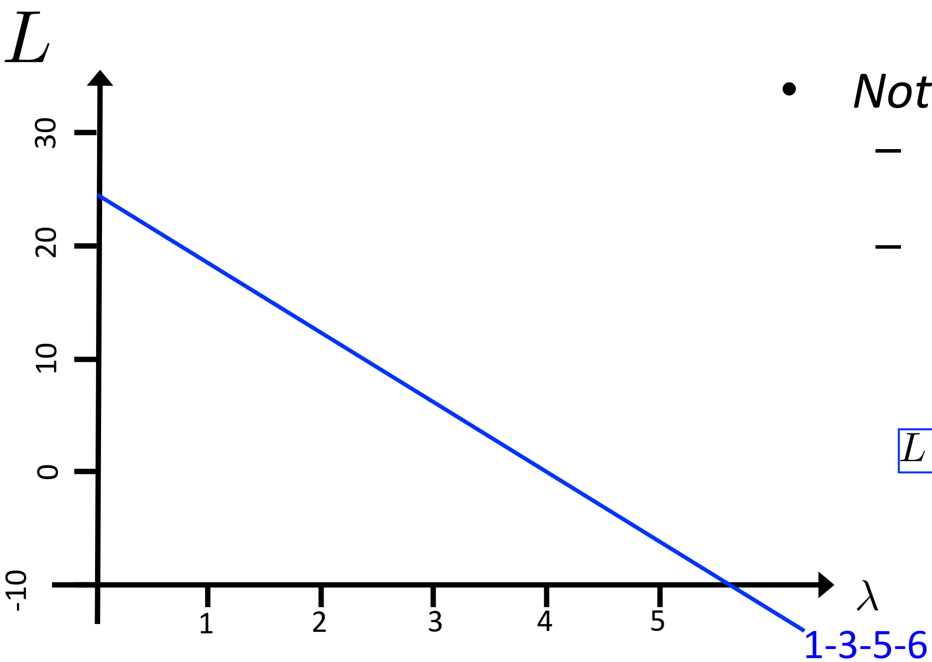
$x_{ij} \in \{0, 1\}$

$L(\lambda)$

$$L^* = \max_{\lambda \geq 0} w^*(\lambda)$$



T = 14



- *Note:*
  - Changing λ does not affect the set of feasible solutions of $L(\lambda)$
  - So the cost of given solution of $L(\lambda)$ can be seen as a linear function of λ

$$L \leq (10 + 3\lambda) + (12 + 3\lambda) + (2 + 2\lambda) - 14\lambda$$
$$= 24 - 6\lambda \quad \text{(1-3-5-6)}$$

2- Lagrangian relaxation

For all $\lambda \geq 0$:

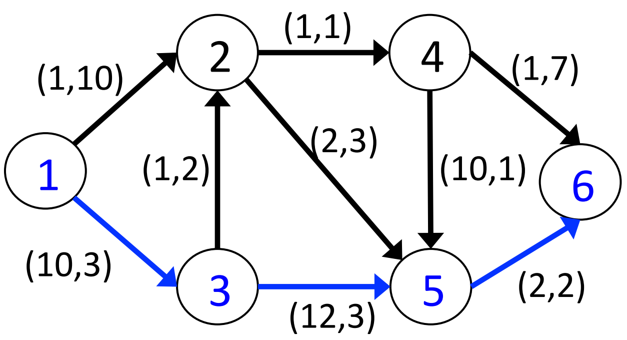
$$\text{Min } w(\lambda) = \sum c_{ij}x_{ij} - \lambda(T - \sum t_{ij}x_{ij}) = \sum(c_{ij} + \lambda t_{ij})x_{ij} - \lambda T$$

path conservation (1)

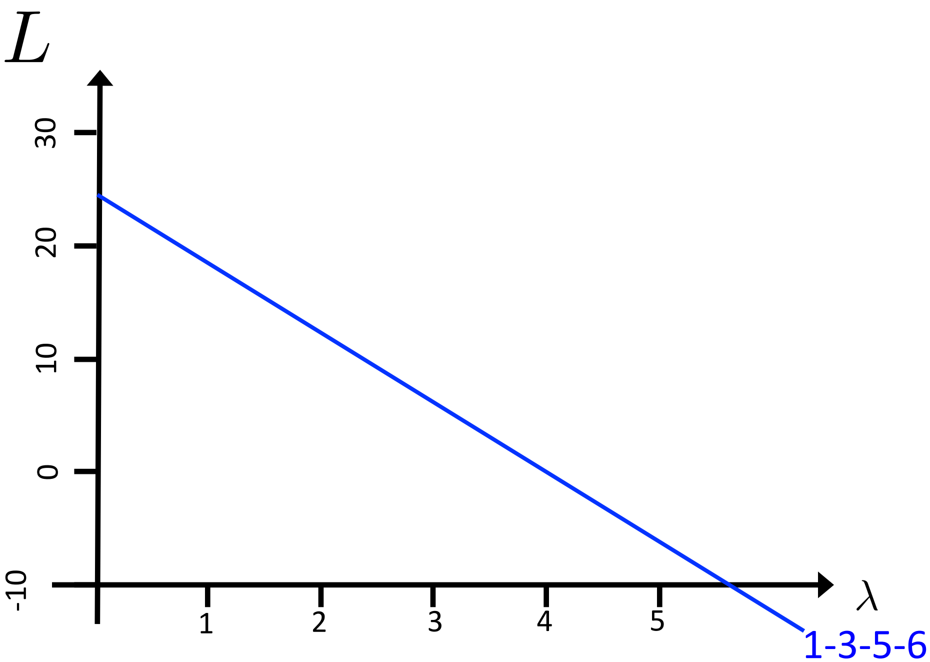
$x_{ij} \in \{0,1\}$

$L(\lambda)$

$$L^* = \max_{\lambda \geq 0} w^*(\lambda)$$



T = 14



Max L

$$L \leq (10 + 3\lambda) + (12 + 3\lambda) + (2 + 2\lambda) - 14\lambda$$

$$= 24 - 6\lambda$$ (1-3-5-6)

2- Lagrangian relaxation

For all $\lambda \geq 0$:

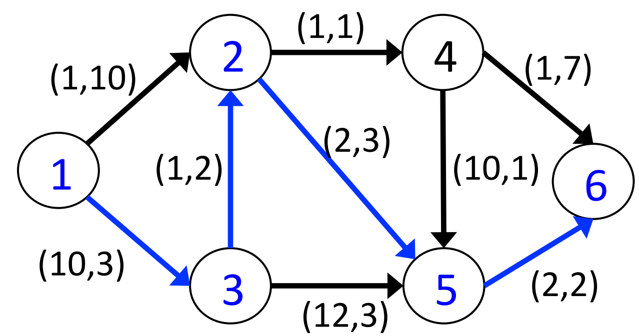
$$\text{Min } w(\lambda) = \sum c_{ij}x_{ij} - \lambda(T - \sum t_{ij}x_{ij}) = \sum (c_{ij} + \lambda t_{ij})x_{ij} - \lambda T$$

path conservation (1)

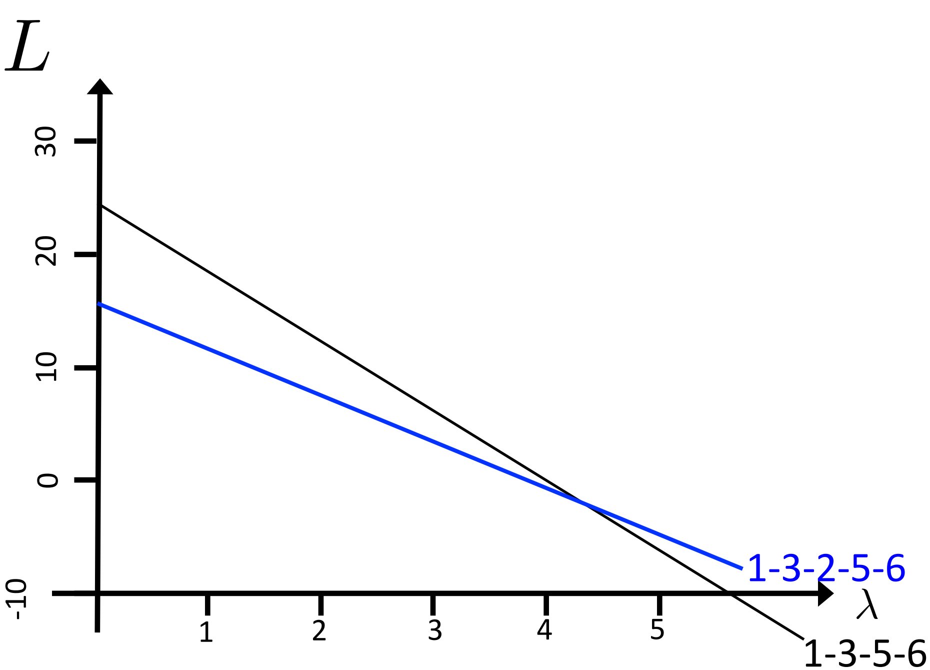
$x_{ij} \in \{0,1\}$

$L(\lambda)$

$$L^* = \max_{\lambda \geq 0} w^*(\lambda)$$



T = 14



Max L

$$L \leq (10 + 3\lambda) + (12 + 3\lambda) + (2 + 2\lambda) - 14\lambda = 24 - 6\lambda$$ (1-3-5-6)

$L \leq 15 - 4\lambda$ (1-3-2-5-6)

2- Lagrangian relaxation

For all $\lambda \geq 0$:

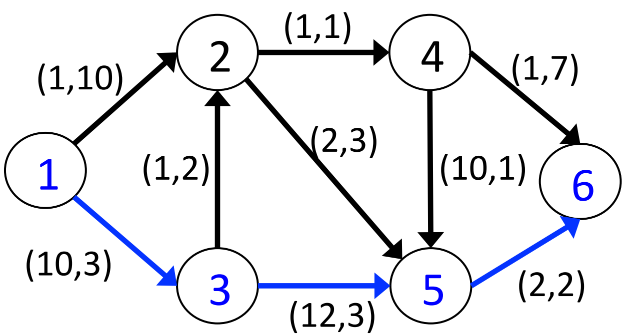
$$\text{Min } w(\lambda) = \sum c_{ij}x_{ij} - \lambda(T - \sum t_{ij}x_{ij}) = \sum(c_{ij} + \lambda t_{ij})x_{ij} - \lambda T$$

path conservation (1)

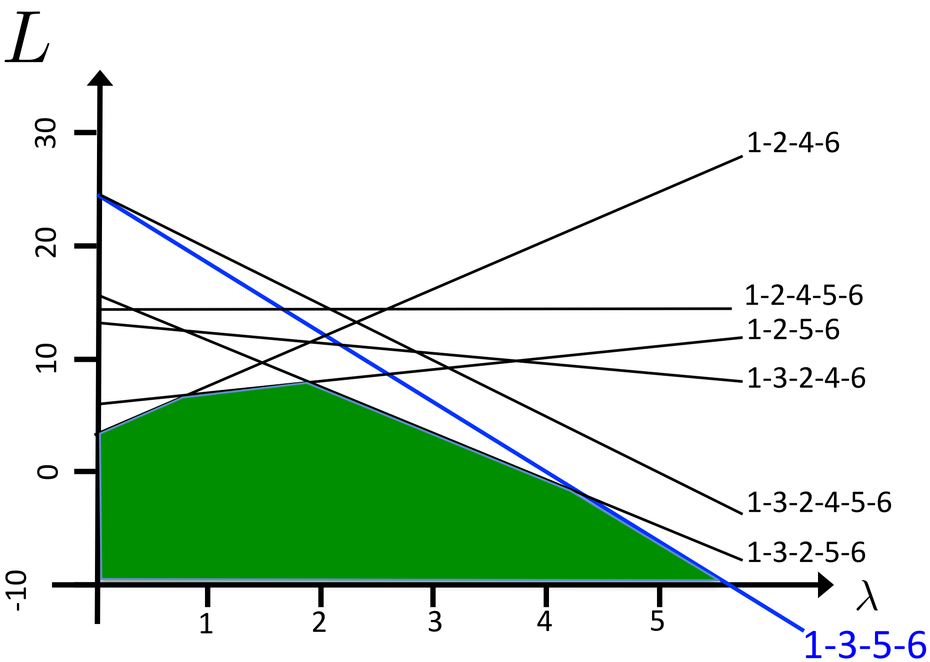
$x_{ij} \in \{0,1\}$

$L(\lambda)$

$$L^* = \max_{\lambda \geq 0} w^*(\lambda)$$



T = 14



$\text{Max } L$

$L \leq 3 + 4\lambda$ (1-2-4-6)

$L \leq 14$ (1-2-4-5-6)

$L \leq 5 + \lambda$ (1-2-5-6)

$L \leq 13 - \lambda$ (1-3-2-4-6)

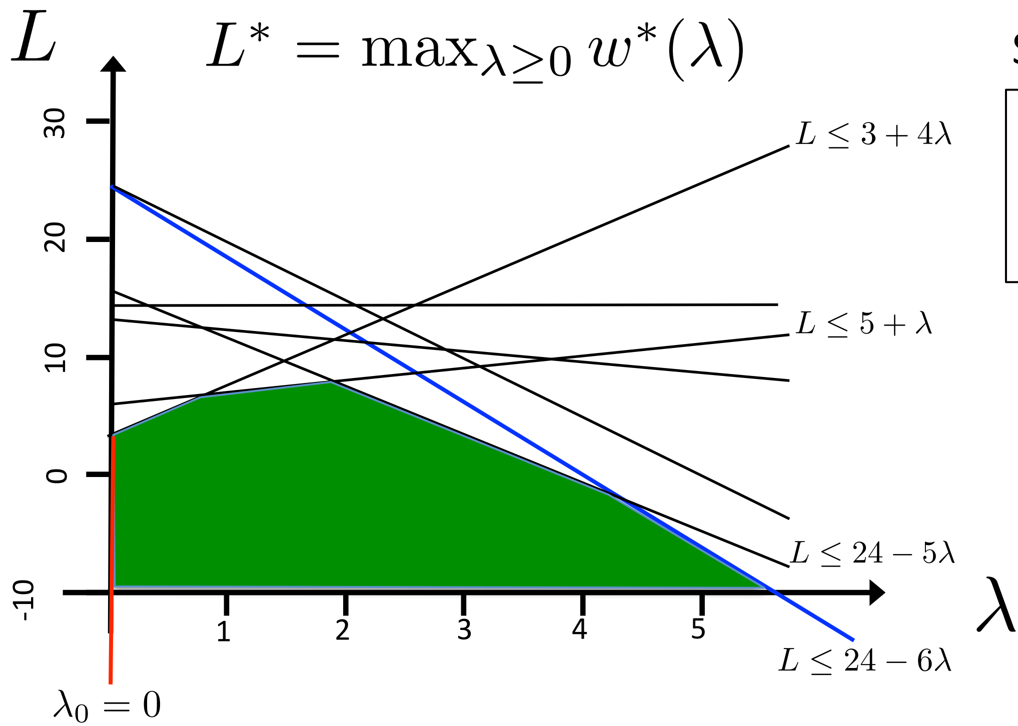
$L \leq 24 - 5\lambda$ (1-3-2-4-5-6)

$L \leq 15 - 4\lambda$ (1-3-2-5-6)

$L \leq (10 + 3\lambda) + (12 + 3\lambda) + (2 + 2\lambda) - 14\lambda$

$= 24 - 6\lambda$ (1-3-5-6)

2- Lagrangian relaxation

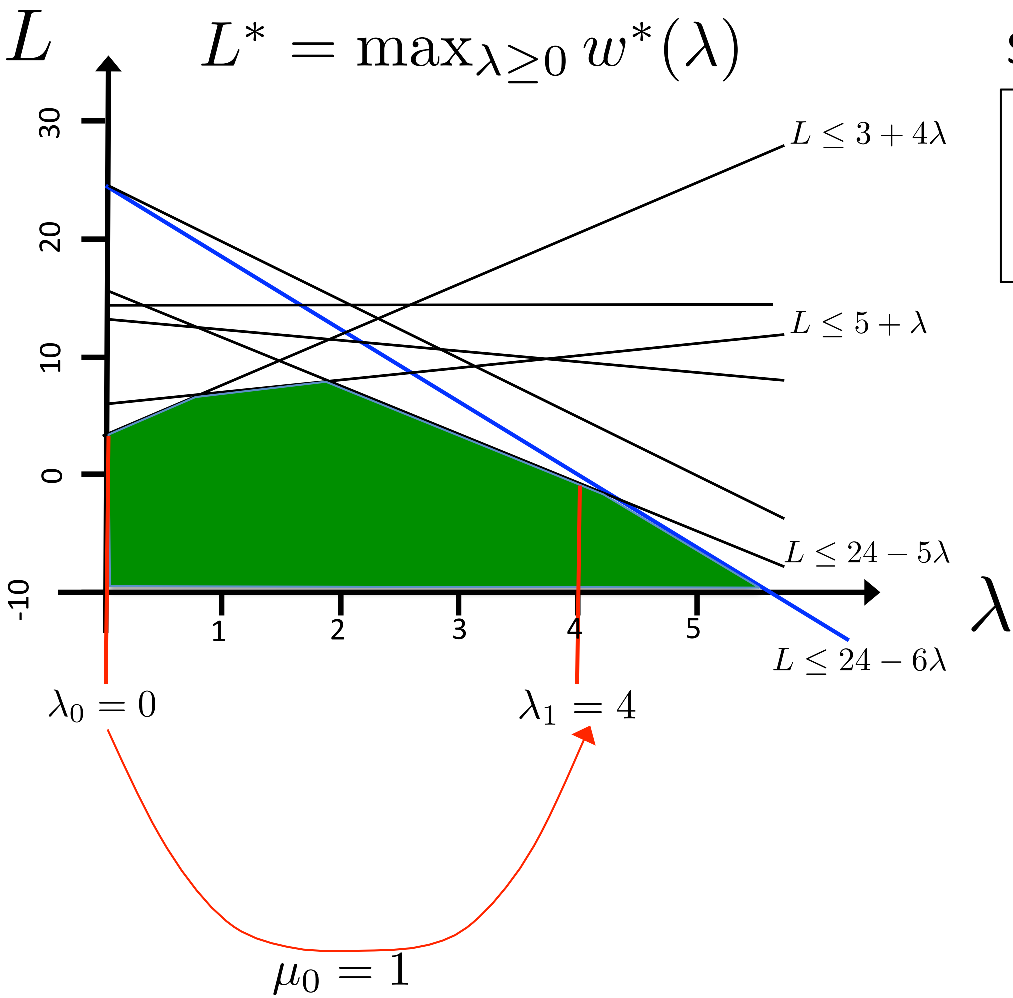


Subgradient algorithm:

$$\lambda_{k+1} \leftarrow \max(0, \lambda_k + \mu(\sum t_{ij} x^k - T))$$
$$\mu_{k+1} = \mu_0 (3/5)^k$$

$$\lambda_0 = 0$$
$$\mu_0 = 1$$

2- Lagrangian relaxation



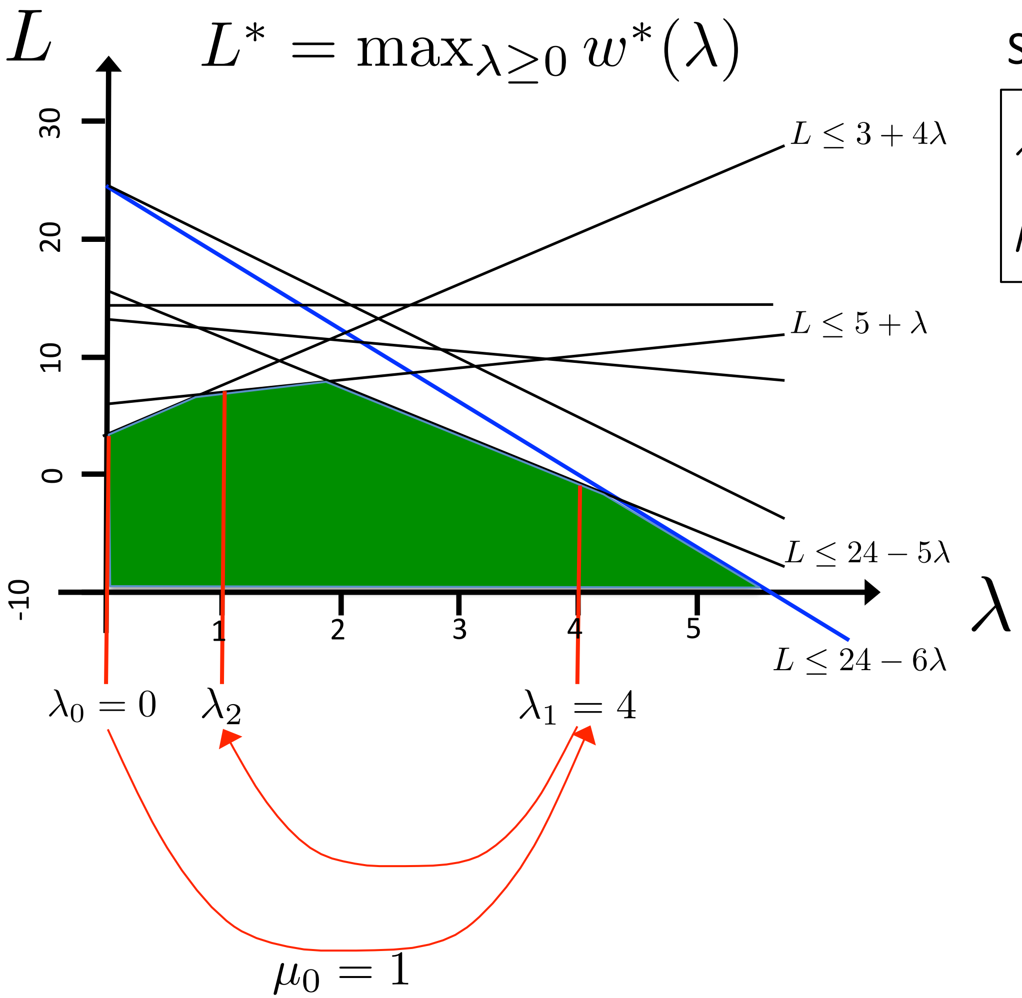
Subgradient algorithm:

$$\lambda_{k+1} \leftarrow \max(0, \lambda_k + \mu(\sum t_{ij} x^k - T))$$
$$\mu_{k+1} = \mu_0 (3/5)^k$$

$\lambda_0 = 0$
$\mu_0 = 1$

$\lambda_1 = 4$
$\mu_1 = 0.6$

2- Lagrangian relaxation



Subgradient algorithm:

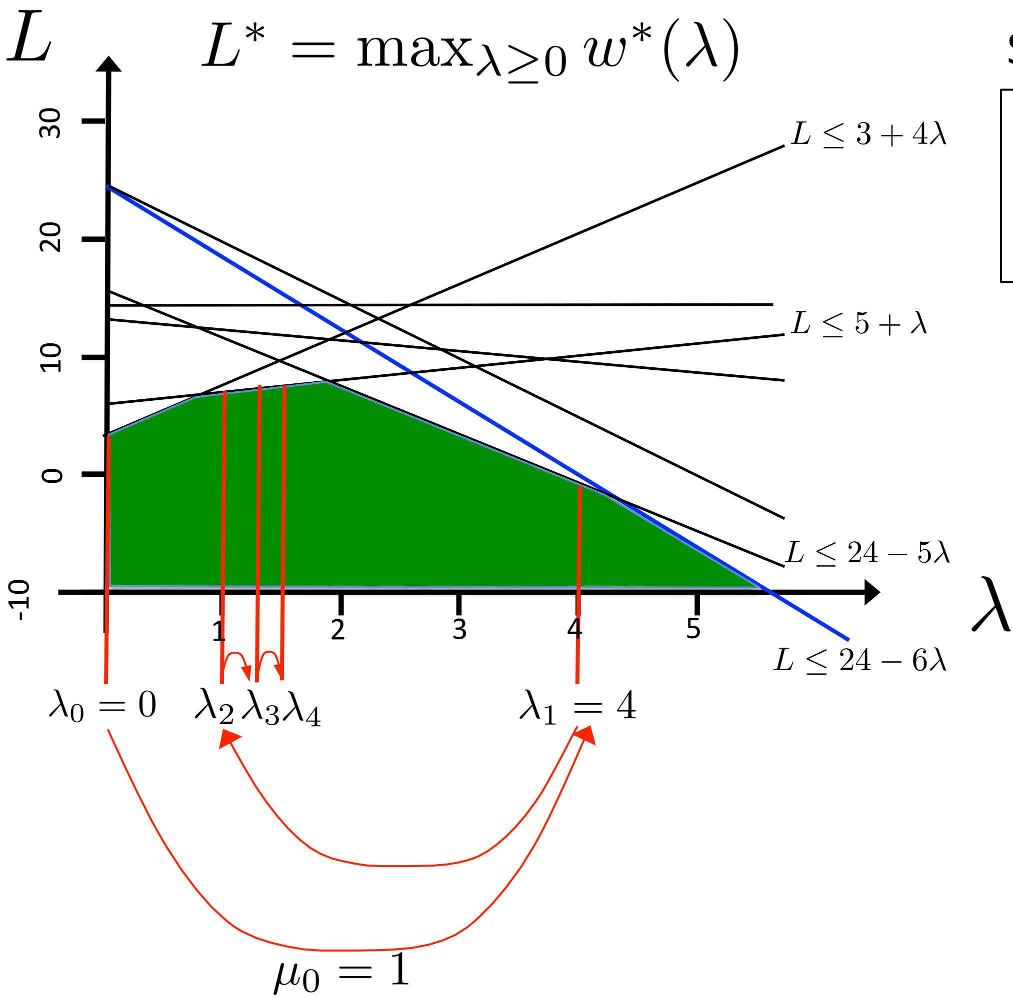
$$\lambda_{k+1} \leftarrow \max(0, \lambda_k + \mu(\sum t_{ij} x^k - T))$$
$$\mu_{k+1} = \mu_0 (3/5)^k$$

$\lambda_0 = 0$
$\mu_0 = 1$

$\lambda_1 = 4$
$\mu_1 = 0.6$

$\lambda_2 = 1$
$\mu_2 = 0.6^2 = 0.36$

2- Lagrangian relaxation



Subgradient algorithm:

$$\lambda_{k+1} \leftarrow \max(0, \lambda_k + \mu(\sum t_{ij} x^k - T))$$
$$\mu_{k+1} = \mu_0 (3/5)^k$$

$\lambda_0 = 0$
$\mu_0 = 1$

$\lambda_1 = 4$
$\mu_1 = 0.6$

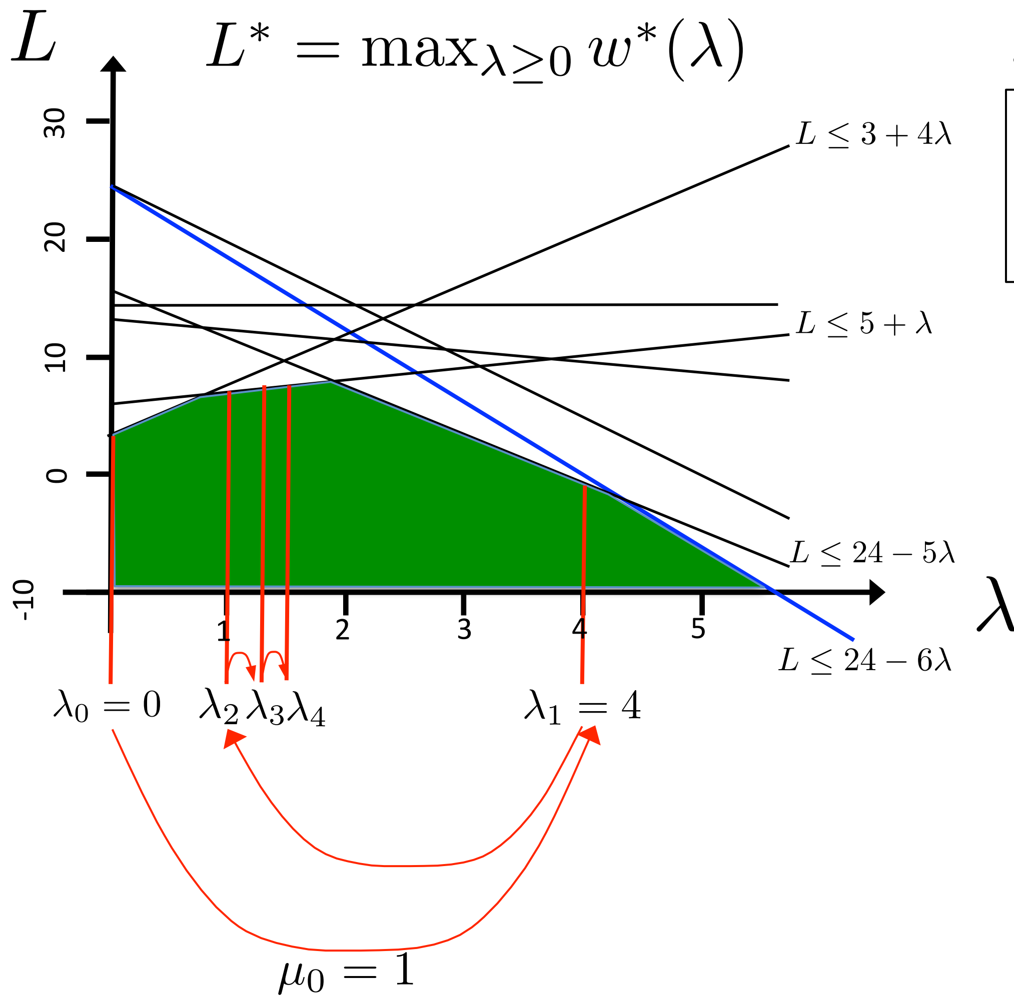
$\lambda_2 = 1$
$\mu_2 = 0.6^2 = 0.36$

$\lambda_3 = 1.36$
$\mu_3 = 0.6^3 = 0.216$

$\lambda_4 = 1.57$

...

2- Lagrangian relaxation



Subgradient algorithm:

$$\lambda_{k+1} \leftarrow \max(0, \lambda_k + \mu(\sum t_{ij} x^k - T))$$
$$\mu_{k+1} = \mu_0 (3/5)^k$$

$\lambda_0 = 0$
$\mu_0 = 1$

$\lambda_1 = 4$
$\mu_1 = 0.6$

$\lambda_2 = 1$
$\mu_2 = 0.6^2 = 0.36$

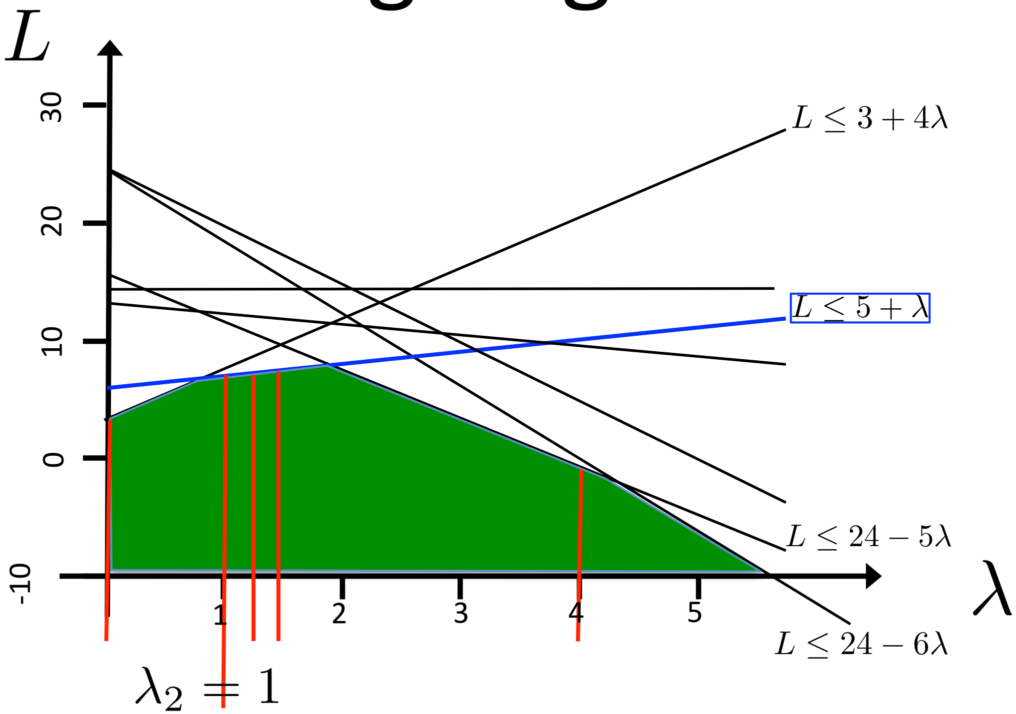
$\lambda_3 = 1.36$
$\mu_3 = 0.6^3 = 0.216$

$\lambda_4 = 1.57$
$\cdots$

To ensure convergence, **we should have**:

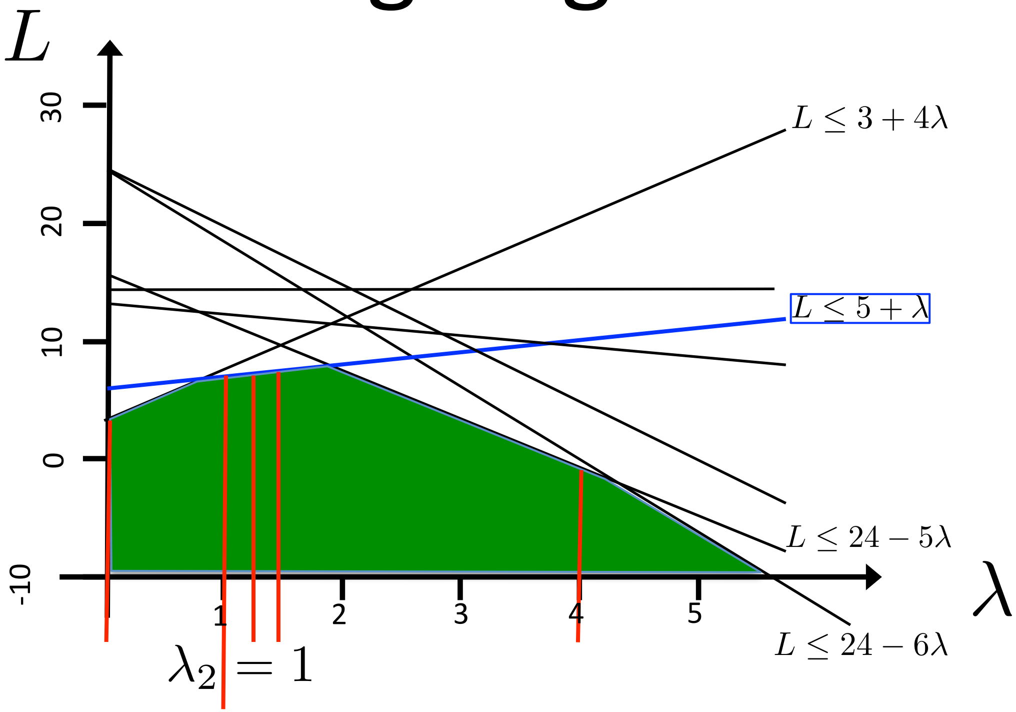
$$\mu_k \to 0 \text{ and } \sum_{j=1}^{k} \mu_j \to \infty$$

2- Lagrangian relaxation - Filtering

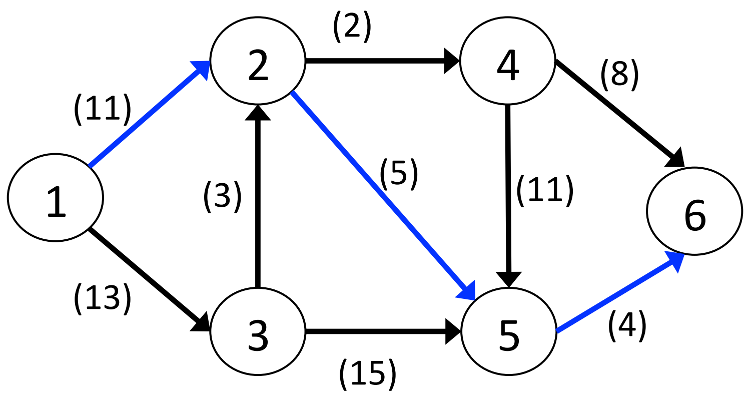


– We can filter at any iteration of this algorithm using the current Lagrangian subproblem and its $w^*(\lambda)$

2- Lagrangian relaxation - Filtering

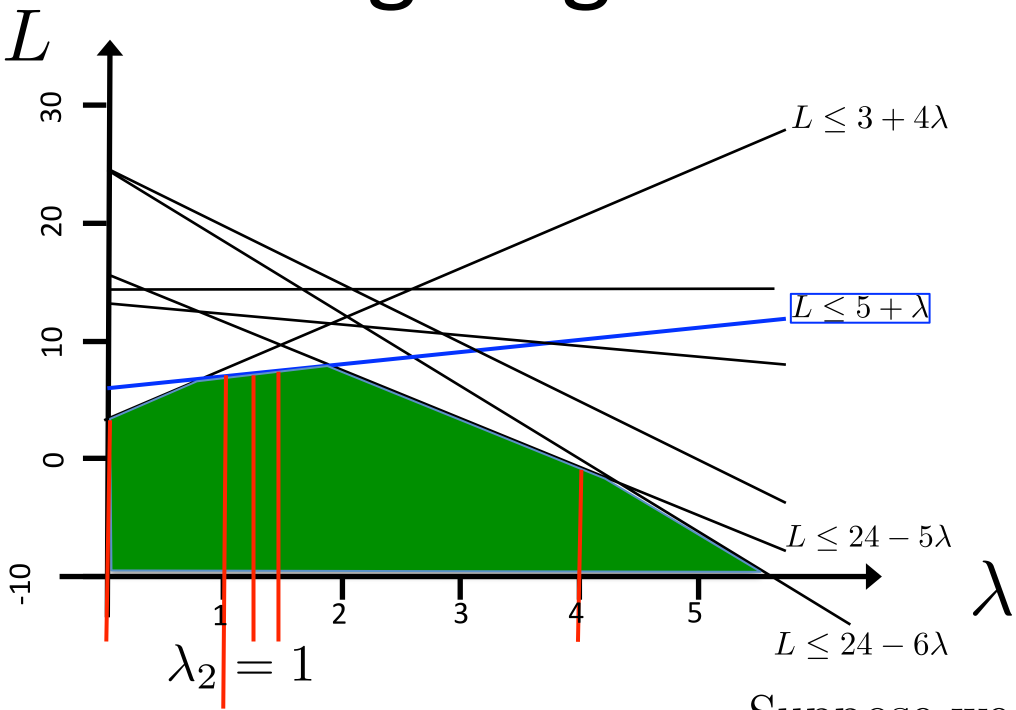


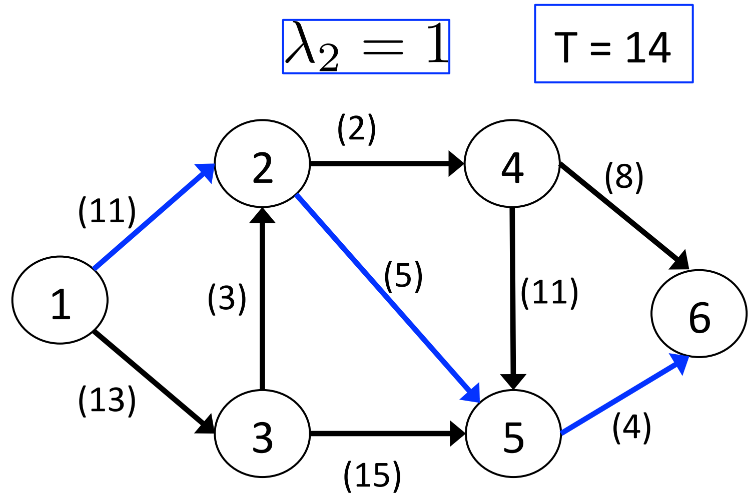
$\lambda_2 = 1$ T = 14



$$w^*(1) = 20 - 14 = 6 \quad \leq z^*$$

2- Lagrangian relaxation - Filtering

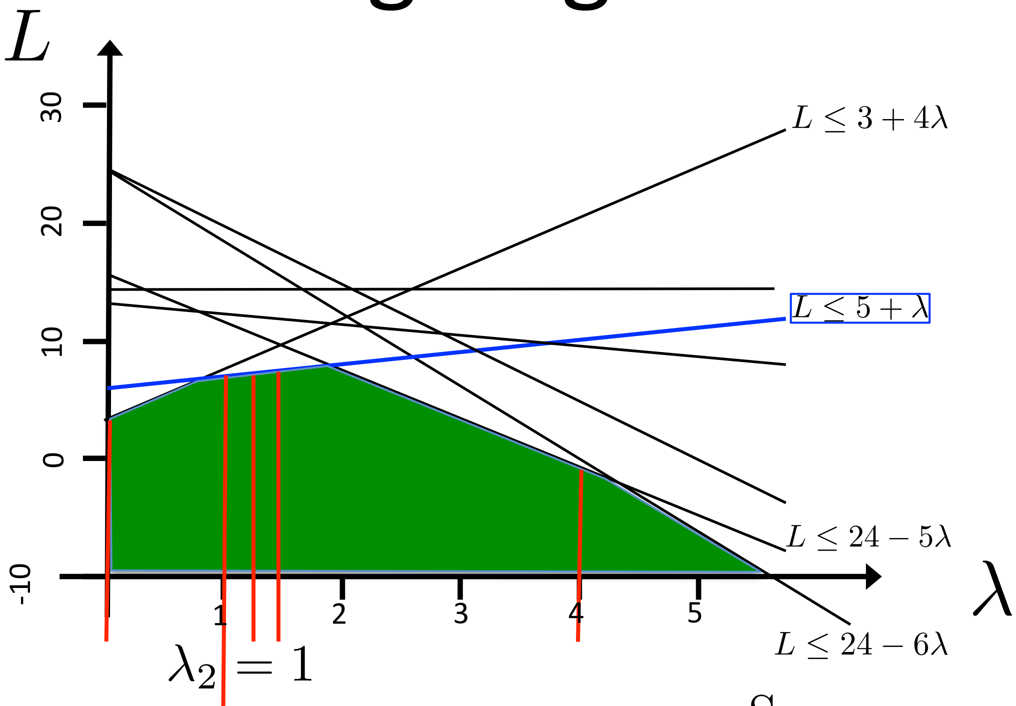


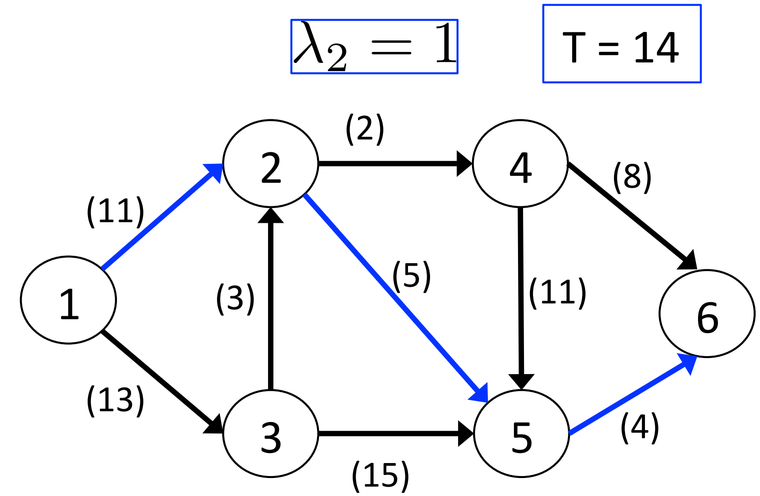


$$w^*(1) = 20 - 14 = 6 \quad \leq z^*$$

Suppose we know an upper bound of $\overline{z} = 15$

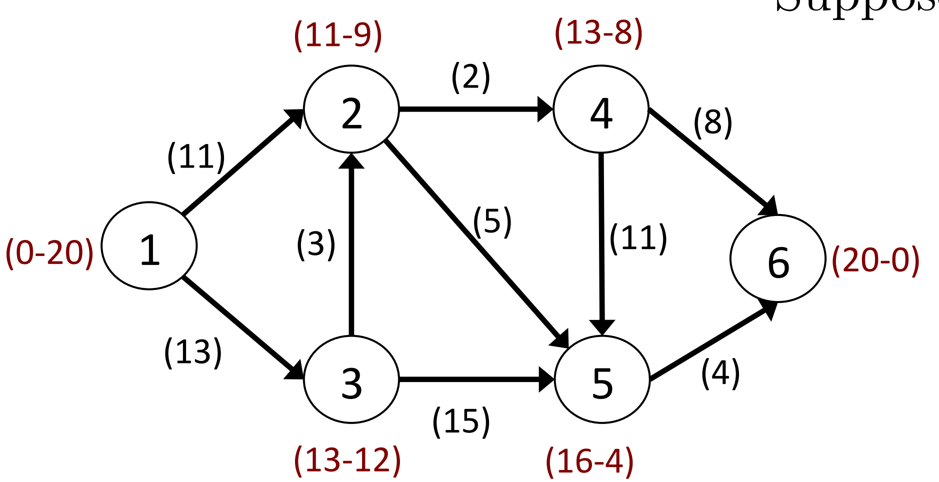
2- Lagrangian relaxation - Filtering





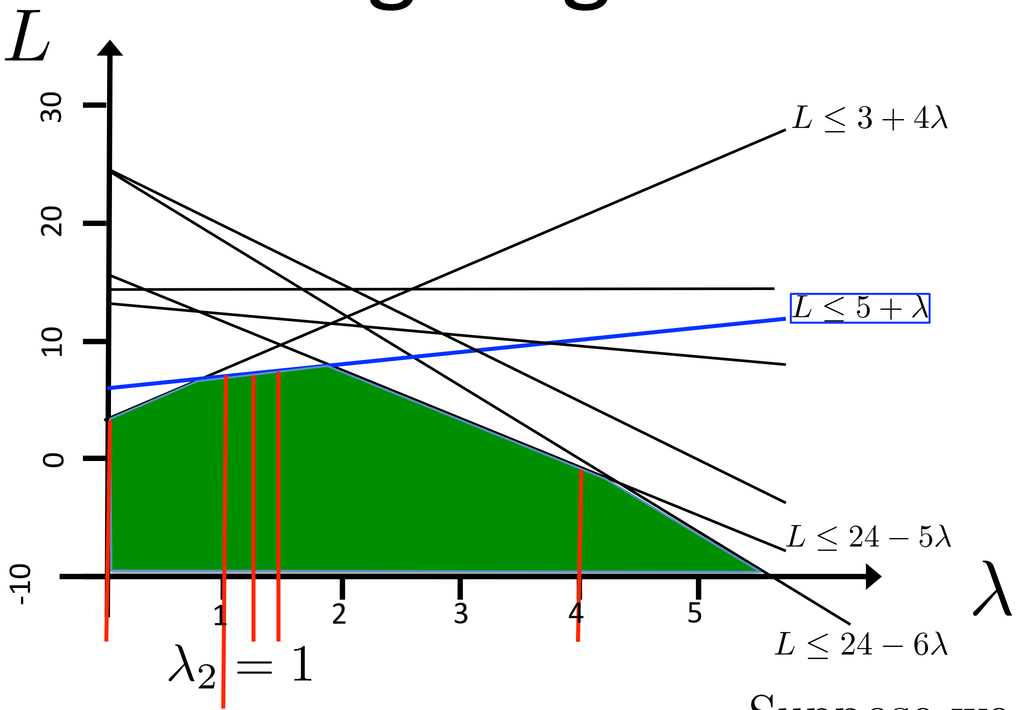
$$w^*(1) = 20 - 14 = 6 \quad \leq z^*$$

Suppose we know an upper bound of $\overline{z} = 15$

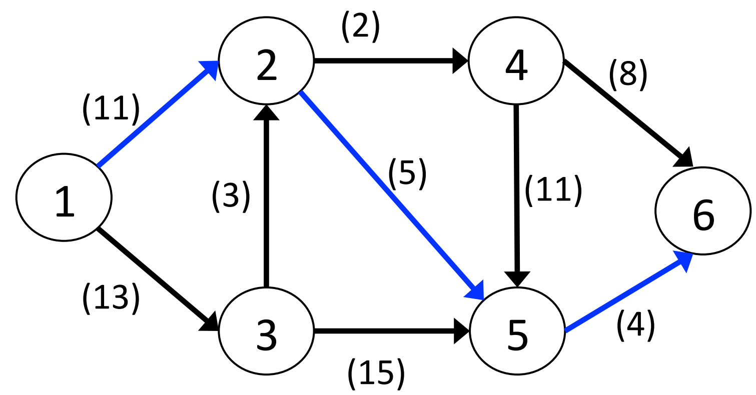


We compute shortest path from source to all other nodes and from all other nodes to sink

2- Lagrangian relaxation - Filtering



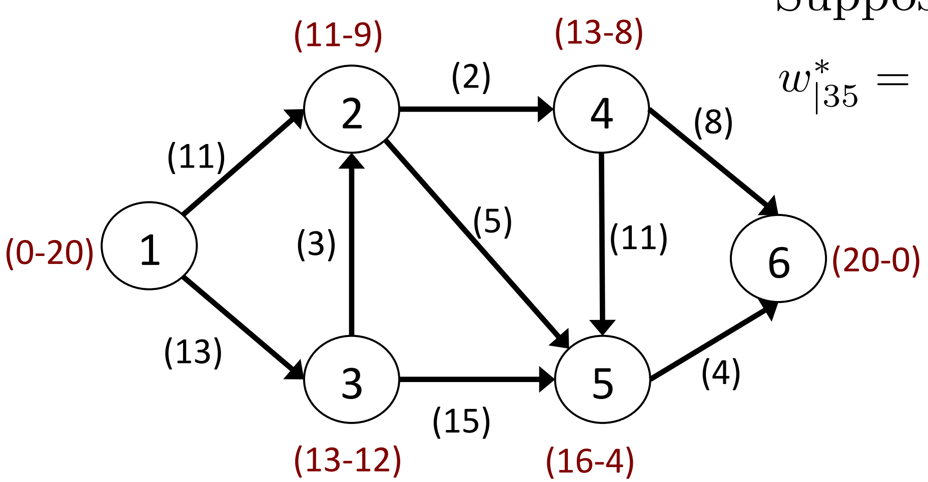
$\lambda_2 = 1$ T = 14



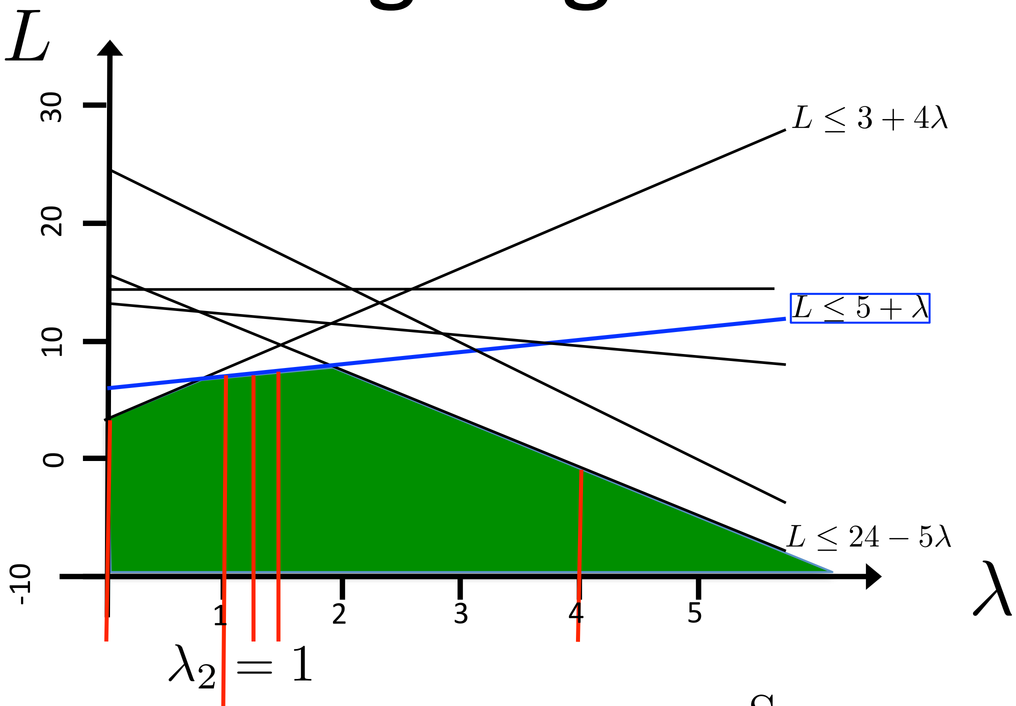
$w^*(1) = 20 - 14 = 6 \quad \leq z^*$

Suppose we know an upper bound of $\overline{z} = 15$

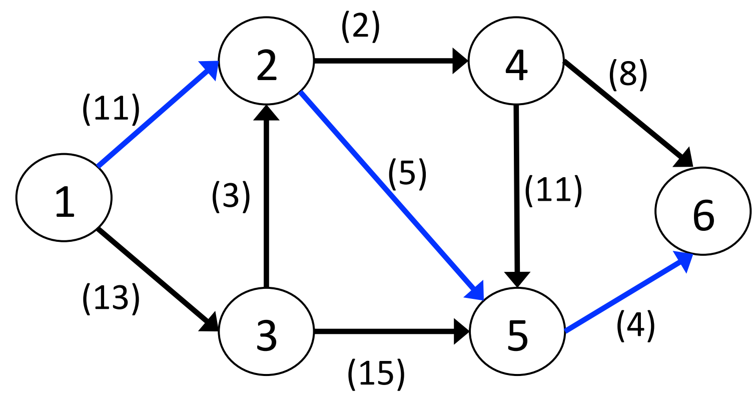
$w^*_{|35} = 13 + (15) + 4 - 14 = 18 > \overline{z} = 15 \Rightarrow x_{35} = 0$



2- Lagrangian relaxation - Filtering



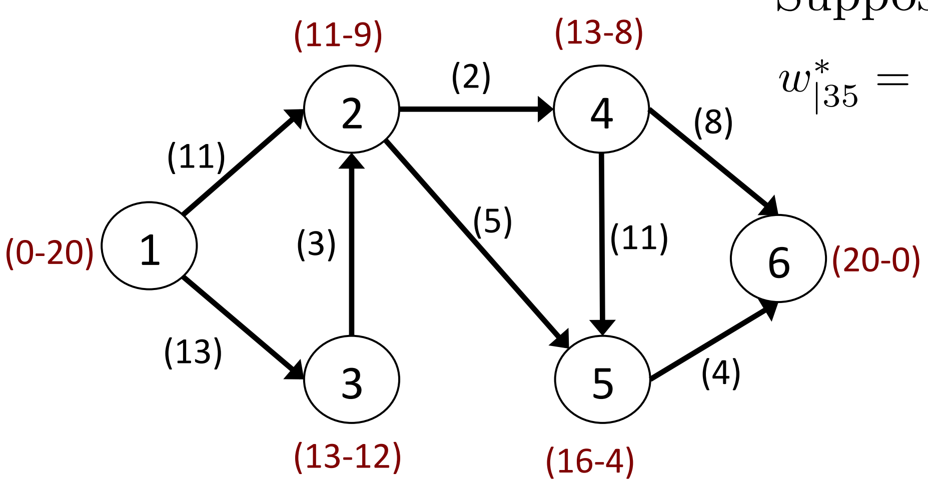
$\lambda_2 = 1$ T = 14



$$w^*(1) = 20 - 14 = 6 \;\; \leq z^*$$

Suppose we know an upper bound of $\overline{z} = 15$

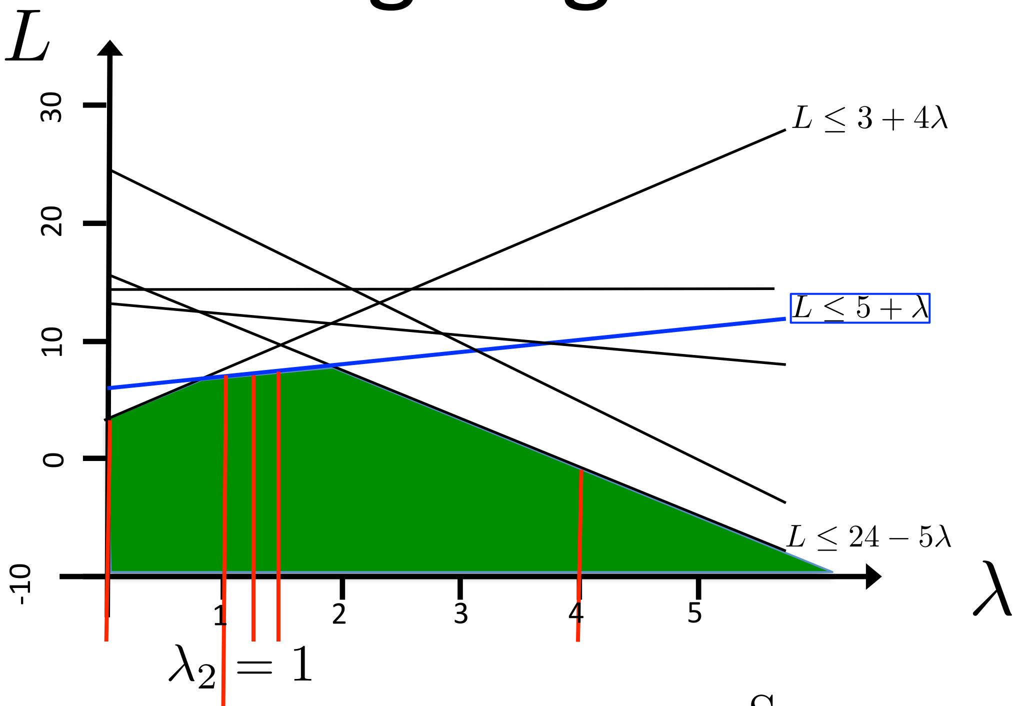
$$w^*_{|35} = 13 + (15) + 4 - 14 = 18 > \overline{z} = 15 \Rightarrow x_{35} = 0$$

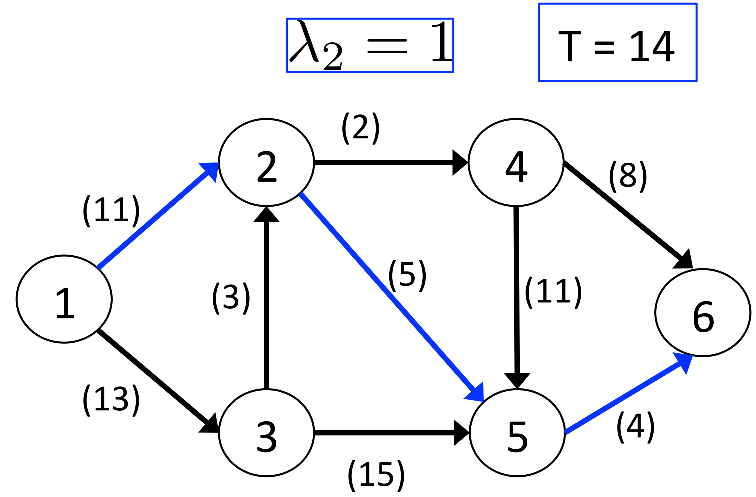


[Sellmann, 2004]

– Lagrangian dual is changed ! does it affect convergence ?

2- Lagrangian relaxation - Filtering

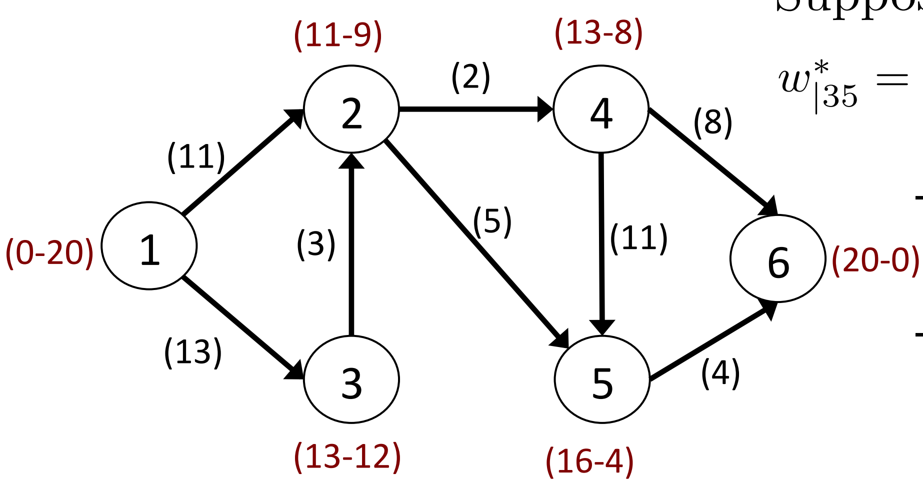




$$w^*(1) = 20 - 14 = 6 \quad \leq z^*$$

Suppose we know an upper bound of $\overline{z} = 15$

$$w^*_{|35} = 13 + (15) + 4 - 14 = 18 > \overline{z} = 15 \Rightarrow x_{35} = 0$$



– Lagrangian dual is changed [Sellmann, 2004]
does it affect convergence ?

– Filtering takes place near L* most of the time but not necessarily

What values of λ are good for filtering ?

Plan

1. Context and motivation
   - Illustrative application: the Traveling Purchaser Problem
   - *Optimization* versus *Satisfaction*
   - *Combinatorial* versus *polyhedral* methods

2. Propagation based on Lagrangian Relaxation
   - Lagrangian duality
   - Filtering using Lagrangian reduced costs
   - Let's try on the *Nvalue* global constraint

3. Overview of some NP-Hard Constraints with costs
   - *Multi-cost regular, Weighted-circuit, Weighted-Nvalue, Bin-packing with usage costs*

4. Examples of applications

3- NValue

$$\text{NV}\textsc{alue}(N, [X_1, \ldots, X_n])$$

- Enforce N to be the number of distinct values appearing in the set X of variables

$D(X_1)$ = {1, 2, 3, 4, 5, 6}

$D(X_2)$ = {2, 4}

$D(X_3)$ = {1, 2}

$D(X_4)$ = {1, 2, 3}

$D(X_5)$ = {4,5}

$D(X_6)$ = {4,5}

$D(N)$ = {1, 2}

$\text{NV}\textsc{alue}(2, [2, 2, 2, 2, 4, 4, 2])$

3- NValue

$$\text{NVALUE}(N, [X_1, \ldots, X_n])$$

– Enforce N to be the number of distinct values appearing in the set X of variables

$D(X_1)$ = {1, 2, 3, 4, 5, 6}
$D(X_2)$ = {2, 4}
$D(X_3)$ = {1, 2}
$D(X_4)$ = {1, 2, 3}
$D(X_5)$ = {4,5}
$D(X_6)$ = {4,5}
$D(N)$ = {1, 2}

$\text{NVALUE}(2, [2, 2, 2, 2, 4, 4, 2])$

$D(X_1)$ = {1, 2, 3, 4, 5, 6}
$D(X_2)$ = {2, 4}
$D(X_3)$ = {1, 2}
$D(X_4)$ = {1, 2, 3}
$D(X_5)$ = {4,5}
$D(X_6)$ = {4,5}
$D(N)$ = {1, 2}

3- NValue

$$\text{NValue}(N, [X_1, \ldots, X_n])$$

– Enforce N to be the number of distinct values appearing in the set X of variables

$D(X_1)$ = {1, 2, 3, 4, 5, 6}
$D(X_2)$ = {2, 4}
$D(X_3)$ = {1, 2}
$D(X_4)$ = {1, 2, 3}
$D(X_5)$ = {4,5}
$D(X_6)$ = {4,5}
$D(N)$ = {1, 2}

$$\text{NValue}(2, [2, 2, 2, 2, 4, 4, 2])$$

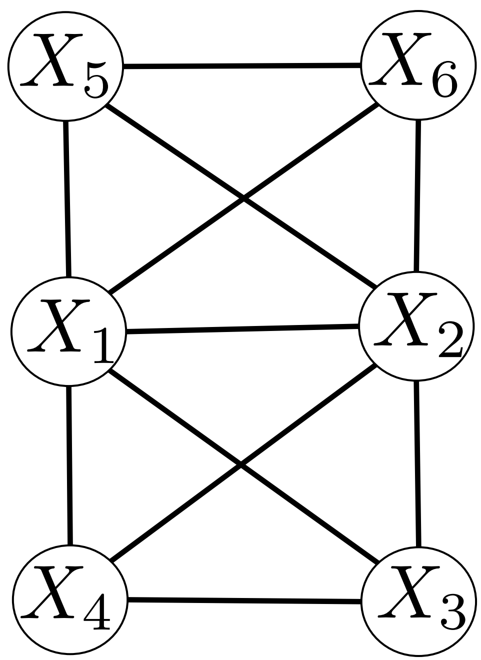
$D(X_1)$ = {1, 2, 3, 4, 5, 6}
$D(X_2)$ = {2, 4}
$D(X_3)$ = {1, 2}
$D(X_4)$ = {1, 2, 3}
$D(X_5)$ = {4,5}
$D(X_6)$ = {4,5}
$D(N)$ = {1, 2}

- Enforcing GAC is NP-Hard
- Several lower bounds proposed by [Hebrard et al, 2006]

3- NValue

$$\text{NVALUE}(N, [X_1, \ldots, X_n])$$

– Enforce N to be the number of distinct values appearing in the set X of variables



$D(X_1)$ = {1, 2, 3, 4, 5, 6}

$D(X_2)$ = {2, 4}

$D(X_3)$ = {1, 2}

$D(X_4)$ = {1, 2, 3}

$D(X_5)$ = {4,5}

$D(X_6)$ = {4,5}

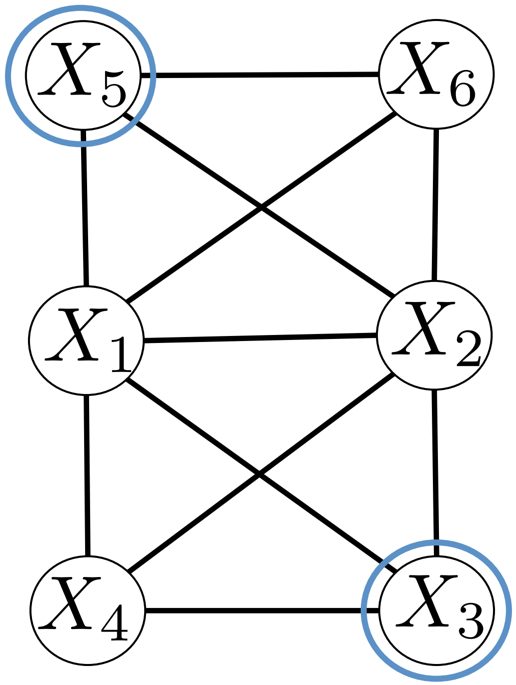
$D(N)$ = {1, 2}

- Enforcing GAC is NP-Hard
- Lower bound of N obtained by a greedy computing an independent set [Hebrard et al, 2006]

3- NValue

$$\text{NVALUE}(N, [X_1, \ldots, X_n])$$

– Enforce N to be the number of distinct values appearing in the set X of variables



$D(X_1)$ = {1, 2, 3, 4, 5, 6}

$D(X_2)$ = {2, 4}

$D(X_3)$ = {1, 2}

$D(X_4)$ = {1, 2, 3}

$D(X_5)$ = {4,5}

$D(X_6)$ = {4,5}

$D(N)$ = {1, 2}

- Enforcing GAC is NP-Hard
- Lower bound of N obtained by a greedy computing an independent set [Hebrard et al, 2006]

3- NValue

$$\text{NVALUE}(N, [X_1, \ldots, X_n])$$

- Propagating **a sharp lower bound of N** is NP-Hard
- The best lower bound proposed in [Bessière et al, 2006] is based on LP-relaxation of:

$$\begin{array}{ll} \text{Min } \sum_{i=1}^{m} y_i & \\ \sum_{i \in D(X_j)} y_i \geq 1 & \forall j = 1, \ldots, n \\ y_i \in \{0, 1\} & \forall i \in V \end{array}$$

m: number of values
n: number of variables

3- NValue

$$\textsc{NValue}(N, [X_1, \ldots, X_n])$$

- Propagating **a sharp lower bound of N** is NP-Hard
- The best lower bound proposed in [Bessière et al, 2006] is based on LP-relaxation of:

$$\begin{array}{ll} \text{Min } \sum_{i=1}^{m} y_i & \\ \sum_{i \in D(X_j)} y_i \geq 1 & \forall j = 1, \ldots, n \\ y_i \in \{0,1\} & \forall i \in V \end{array}$$

For all $(\lambda_1, \ldots, \lambda_n) \geq 0$

$$\begin{array}{l} \text{Min } w_\lambda = \sum_{i=1}^{m} y_i + \sum_{j=1}^{n} \lambda_j (1 - \sum_{i \in D(X_j)} y_i) \\ \quad = \sum_{i=1}^{m} (1 - \sum_{j | i \in D(X_j)} \lambda_j) y_i + \sum_{j=1}^{n} \lambda_j \\ y_i \in \{0,1\} \qquad \forall i \in V \end{array}$$

m: number of values
n: number of variables

3- NValue

$$\text{NVALUE}(N, [X_1, \ldots, X_n])$$

- Propagating **a sharp lower bound of N** is NP-Hard
- The best lower bound proposed in [Bessière et al, 2006] is based on LP-relaxation of:

$$\begin{aligned} &\text{Min } \textstyle\sum_{i=1}^{m} y_i \\ &\textstyle\sum_{i \in D(X_j)} y_i \geq 1 \qquad \forall j = 1, \ldots, n \\ &y_i \in \{0,1\} \qquad \forall i \in V \end{aligned}$$

For all $(\lambda_1, \ldots, \lambda_n) \geq 0$

$$\begin{aligned} \text{Min } w_\lambda &= \textstyle\sum_{i=1}^{m} y_i + \sum_{j=1}^{n} \lambda_j (1 - \sum_{i \in D(X_j)} y_i) \\ &= \textstyle\sum_{i=1}^{m} (1 - \sum_{j | i \in D(X_j)} \lambda_j) y_i + \sum_{j=1}^{n} \lambda_j \\ y_i &\in \{0,1\} \qquad \forall i \in V \end{aligned}$$

m: number of values
n: number of variables

- No constraints in the Lagrangian subproblem
- Easily solved by inspection :

$$\text{Set } y_i \text{ to } 1 \text{ if } (1 - \textstyle\sum_{j | i \in D(X_j)} \lambda_j) < 0$$

- Filtering is also done "for free"

3- NValue

$$\text{NVALUE}(N, [X_1, \ldots, X_n])$$

- Propagating **a sharp lower bound of N** is NP-Hard
- The best lower bound proposed in [Bessière et al, 2006] is based on LP-relaxation of:

$$\begin{aligned} &\text{Min } \textstyle\sum_{i=1}^{m} y_i \\ &\textstyle\sum_{i \in D(X_j)} y_i \geq 1 \qquad \forall j = 1, \ldots, n \\ &y_i \in \{0,1\} \qquad \forall i \in V \end{aligned}$$

For all $(\lambda_1, \ldots, \lambda_n) \geq 0$

$$\begin{aligned} \text{Min } w_\lambda &= \textstyle\sum_{i=1}^{m} y_i + \sum_{j=1}^{n} \lambda_j (1 - \sum_{i \in D(X_j)} y_i) \\ &= \textstyle\sum_{i=1}^{m} (1 - \sum_{j | i \in D(X_j)} \lambda_j) y_i + \sum_{j=1}^{n} \lambda_j \\ y_i &\in \{0,1\} \qquad \forall i \in V \end{aligned}$$

m: number of values
n: number of variables

- No constraints in the Lagrangian subproblem
- Easily solved by inspection :

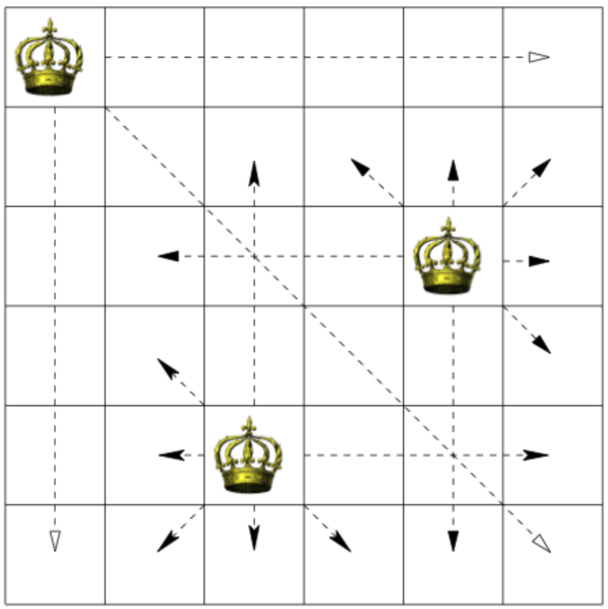
$$\text{Set } y_i \text{ to 1 if } (1 - \textstyle\sum_{j | i \in D(X_j)} \lambda_j) < 0$$

- Filtering is also done "for free"

[Mouthy, Deville, Dooms, JFPC 2007]
A global constraint for the set covering problem

3- NValue

$$\textsc{NValue}(N, [X_1, \ldots, X_n])$$



dominating set of queens

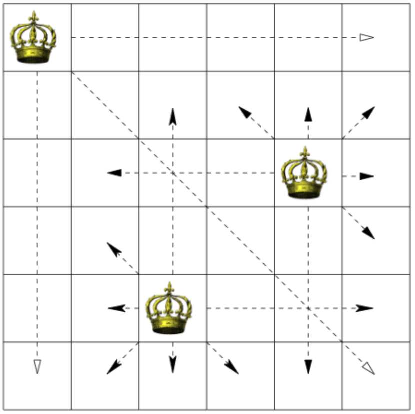
(picture from [Hebrard et al, 2006])

$x_i \in S_i \subset \{1, \ldots, n^2\}$: the queen attacking cell i

Minimize z

$\textsc{NValue}(z, [x_1, \ldots, x_{n^2}]),$

$x_i \in S_i \subset \{1, \ldots, n^2\}$



3- NValue

$$\text{NVALUE}(N, [X_1, \ldots, X_n])$$

$precision : 10^{-4}$
$maxIter : 1000$

$\mu_k = 1/k$

$\mu_0 = 10^3$
$\mu_k = \mu_0 (0.95)^k$

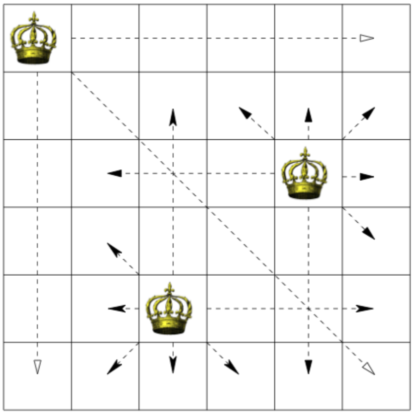
Greedy bound + filtering (MD)

Solve the Linear relaxation + reduced cost filtering (LP)

	Q	N	MD		LR1		LR2		LP	
			Back	Time (s)	Back	Time (s)	Back	Time (s)	Back	Time (s)
SAT	6	3	15	**0.01**	**7**	0.15	12	0.1	10	0.4
SAT	7	4	386	**0.13**	**55**	0.6	128	0.3	120	3.5
SAT	8	5	2541	**0.6**	**97**	0.9	233	**0.6**	287	13.5
UNSAT	8	4	1273232	70.5	**1791**	28.9	2948	**9.8**	2656	167
SAT	9	5	1076891	81.4	**862**	15.8	1862	**7.2**	894	123

Branching lexicographic

- LR can be fast (faster than LP)
- LR can filter a more than LP (even if the bound is theoretically the same)



3- NValue

$$\text{NVALUE}(N, [X_1, \ldots, X_n])$$

$precision : 10^{-4}$
$maxIter : 1000$

Greedy bound + filtering

$\mu_k = 1/k$

$\mu_0 = 10^3$
$\mu_k = \mu_0(0.95)^k$

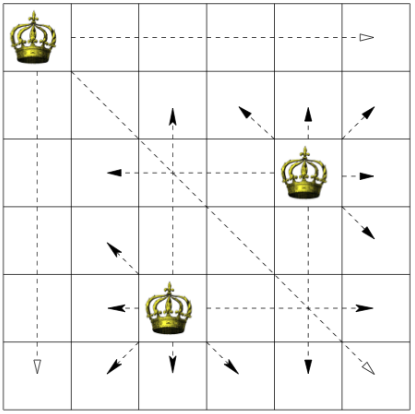
Solve the Linear relaxation + reduced cost filtering

	Q	N	MD		LR1		LR2		LP	
			Back	Time (s)	Back	Time (s)	Back	Time (s)	Back	Time (s)
SAT	6	3	15	**0.01**	**7**	0.15	12	0.1	10	0.4
SAT	7	4	386	**0.13**	**55**	0.6	128	0.3	120	3.5
SAT	8	5	2541	**0.6**	**97**	0.9	233	**0.6**	287	13.5
UNSAT	8	4	2546241	70.5 26.7	**1791**	28.9	2948	**9.8**	2656	167
SAT	9	5	2153565	81.4 22.5	**862**	15.8	1862	**7.2**	894	123

Branching lexicographic

Jean-Guillaume, 1h30 du mat

- LR can be fast (faster than LP)
- LR can filter a more than LP (even if the bound is theoretically the same)



3- NValue

$$\text{NVALUE}(N, [X_1, \ldots, X_n])$$

Perform "singleton" filtering

Solve the Linear relaxation + "singleton" filtering

$\mu_0 = 1$
$\mu_k = 1/k$

$\mu_0 = 10^3$
$\mu_k = \mu_0(0.95)^k$

Q	N	MD		Strong LR1		Strong LR2		Strong LP	
		Back	Time (s)	Back	Time (s)	Back	Time (s)	Back	Time (s)
6	3	15	**0.01**	**0**	0.2	**0**	0.1	**0**	0.1
7	4	386	**0.1**	4	4.4	4	0.9	**3**	0.8
8	5	2541	**0.6**	3	8.6	7	2.5	**2**	1.7
8	4	1273232	70.5	**20**	14.2	21	**4.2**	**20**	5.1
9	5	1076891	81.4	**5**	18.2	**5**	**4.3**	**5**	5.7

- Instead of using Lagrangian/linear reduced costs, we fix the assignment and recompute the bound in a "singleton" manner
- LP has a better incremental behaviour

3- Multi-cost regular

- Regular : $\textsc{Regular}([X_1, \ldots, X_n], A)$ [Pesant, 2004]
  - Propagation based on breath-first-search in the unfolded automaton

3- Multi-cost regular

- Regular : $\textsc{Regular}([X_1, \ldots, X_n], A)$ → Automaton [Pesant, 2004]
  - Propagation based on breath-first-search in the unfolded automaton

3- Multi-cost regular

- Regular : $\textsc{Regular}([X_1, \ldots, X_n], A)$ [Pesant, 2004]
  - Propagation based on breath-first-search in the unfolded automaton
- Cost regular : $\textsc{Regular}([X_1, \ldots, X_n], A) \wedge \sum_{i=1}^{n} c_{iX_i} = Z$
  - Propagation based on shortest/longest path in the unfolded automaton

3- Multi-cost regular

- Regular : $\textsc{Regular}([X_1, \ldots, X_n], A)$ [Pesant, 2004]
  - Propagation based on breath-first-search in the unfolded automaton

- Cost regular : $\textsc{Regular}([X_1, \ldots, X_n], A) \ \wedge \ \sum_{i=1}^{n} c_{iX_i} = Z$
  - Propagation based on shortest/longest path in the unfolded automaton

- Multi-cost regular : $\textsc{Multi-cost Regular}([X_1, \ldots, X_n], [Z^1, \ldots, Z^R], A)$

$$\textsc{Regular}([X_1, \ldots, X_n], A) \ \wedge \ (\sum_{i=1}^{n} c^r_{iX_i} = Z^r, \forall\, r = 0, \ldots, R)$$

  - Propagation based on **resource constrained shortest/longest path**
  - Sequencing and counting at the same time
    - Personnel scheduling
    - Routing

[Menana, Demassey, 2009]

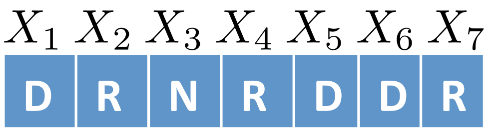
  - Example: combine Regular and GCC

3- Multi-cost regular

- Multi-cost regular :

  $$\textsc{Regular}([X_1,\ldots,X_n],A)\ \wedge\ (\sum_{i=1}^{n} c^r_{iX_i} = Z^r, \forall\, r = 0,\ldots,R)$$

- Example:
  - Schedule 7 shifts of type: **night (N), day (D), rest (R)**
  - (1) "A **Rest must follow a Night** shift"
  - (2) "**Exactly 3 day shifts** and **1 night shift** must take place in the week"



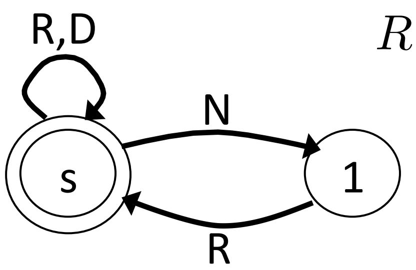
3- Multi-cost regular

- Multi-cost regular :

$$\textsc{Regular}([X_1, \ldots, X_n], A) \;\wedge\; (\sum_{i=1}^{n} c^r_{iX_i} = Z^r, \forall\, r = 0, \ldots, R)$$

- Example:
  - Schedule 7 shifts of type: **night (N), day (D), rest (R)**
  - (1) “A **Rest must follow a Night** shift”
  - (2) “**Exactly 3 day shifts** and **1 night shift** must take place in the week”

X_1	X_2	X_3	X_4	X_5	X_6	X_7
D	R	N	R	D	D	R



$R = 2$

	X_1	X_2	X_3	X_4	X_5	X_6	X_7
c^1_D	1	1	1	1	1	1	1
c^1_N	0	0	0	0	0	0	0
c^1_R	0	0	0	0	0	0	0

	X_1	X_2	X_3	X_4	X_5	X_6	X_7
c^2_D	0	0	0	0	0	0	0
c^2_N	1	1	1	1	1	1	1
c^2_R	0	0	0	0	0	0	0

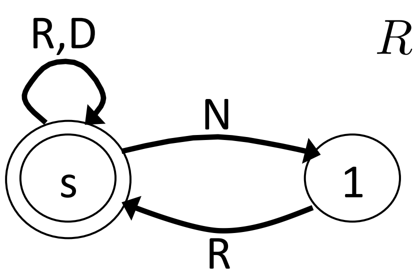
3- Multi-cost regular

- Multi-cost regular :

$$\textsc{Regular}([X_1,\ldots,X_n],A)\ \wedge\ (\sum_{i=1}^{n} c^r_{iX_i} = Z^r, \forall\, r = 0,\ldots,R)$$

- Example:
  - Schedule 7 shifts of type: **night (N), day (D), rest (R)**
  - (1) “A **Rest must follow a Night** shift”
  - (2) “**Exactly 3 day shifts** and **1 night shift** must take place in the week”

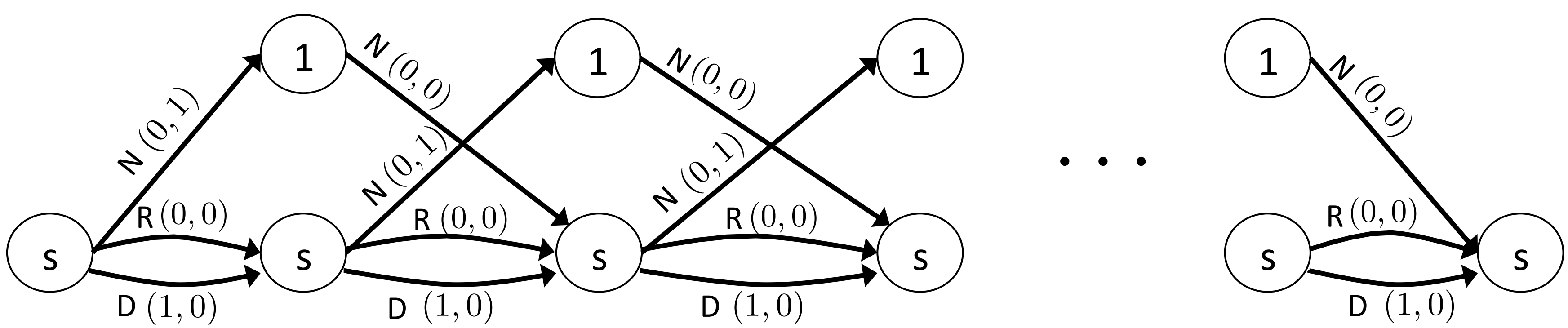
X_1	X_2	X_3	X_4	X_5	X_6	X_7
D	R	N	R	D	D	R



$R = 2$

	X_1	X_2	X_3	X_4	X_5	X_6	X_7
c^1_D	1	1	1	1	1	1	1
c^1_N	0	0	0	0	0	0	0
c^1_R	0	0	0	0	0	0	0

	X_1	X_2	X_3	X_4	X_5	X_6	X_7
c^2_D	0	0	0	0	0	0	0
c^2_N	1	1	1	1	1	1	1
c^2_R	0	0	0	0	0	0	0



3- Weighted-circuit

$$\text{WEIGHTED-CIRCUIT}(X = [X_1, \ldots, X_n], Z)$$

$$X_i = j \text{ means } j \text{ is the successor of } i$$

- Enforce X to be a Hamiltonian tour of weight at most Z

$$\text{CIRCUIT}(X = [X_1, \ldots, X_n]) \wedge \sum_{i=1}^{n} c_{iX_i} \leq Z$$

3- Weighted-circuit

$$\text{WEIGHTED-CIRCUIT}(X = [X_1, \ldots, X_n], Z)$$

$$X_i = j \text{ means } j \text{ is the successor of } i$$

- Enforce X to be a Hamiltonian tour of weight at most Z

$$\text{CIRCUIT}(X = [X_1, \ldots, X_n]) \wedge \sum_{i=1}^{n} c_{iX_i} \leq Z$$

- Filtering based on graph structure [Fages et al, 2012] [Caseau et al, 1997]

3- Weighted-circuit

$$\textsc{weighted-circuit}(X = [X_1, \ldots, X_n], Z)$$

$$X_i = j \text{ means } j \text{ is the successor of } i$$

- Enforce X to be a Hamiltonian tour of weight at most Z

$$\textsc{circuit}(X = [X_1, \ldots, X_n]) \wedge \sum_{i=1}^{n} c_{iX_i} \leq Z$$

[Fages et al, 2012]

[Caseau et al, 1997]

[Benchimol et al, 2012]

- Filtering based on graph structure
- Filtering based on the Held and Karp 1-Tree relaxation
  - Relax the **tour** into a **1-tree** (a tree over all nodes except one + 2 edges connected to the ignored node)
  - Lagrangian subproblem based on a minimum spanning tree

3- Weighted-circuit

$$\text{WEIGHTED-CIRCUIT}(X = [X_1, \ldots, X_n], Z)$$

$$X_i = j \text{ means } j \text{ is the successor of } i$$

- Enforce X to be a Hamiltonian tour of weight at most Z

$$\text{CIRCUIT}(X = [X_1, \ldots, X_n]) \wedge \sum_{i=1}^{n} c_{iX_i} \leq Z$$

[Fages et al, 2012]

[Caseau et al, 1997]

[Benchimol et al, 2012]

- Filtering based on graph structure
- Filtering based on the Held and Karp 1-Tree relaxation
  - Relax the **tour** into a **1-tree** (a tree over all nodes except one + 2 edges connected to the ignored node)
  - Lagrangian subproblem based on a minimum spanning tree

- Use "Lagrangian reduced-cost" to identify:
  - Edges that must be in the tour
  - Edges that can not be in a "better" tour

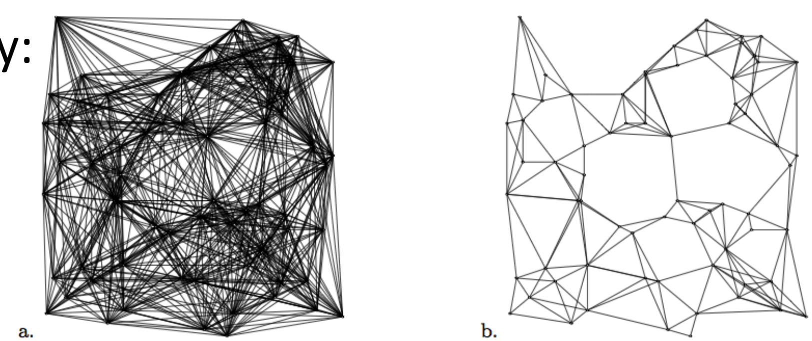


Fig. 3 The filtered graph for **st70** with respect to an upper bound of 700 (a) and 675 (b).

[Benchimol et al, 2012]

3- Weighted-circuit

$$\text{WEIGHTED-CIRCUIT}(X = [X_1, \ldots, X_n], Z)$$

$$X_i = j \text{ means } j \text{ is the successor of } i$$

- Enforce X to be a Hamiltonian tour of weight at most Z

$$\text{CIRCUIT}(X = [X_1, \ldots, X_n]) \wedge \sum_{i=1}^{n} c_{iX_i} \leq Z$$

[Fages et al, 2012]

[Caseau et al, 1997]

[Benchimol et al, 2012]

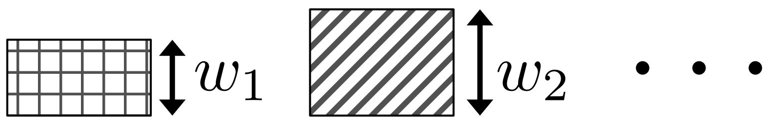
- Filtering based on graph structure
- Filtering based on the Held and Karp 1-Tree relaxation
  - Relax the **tour** into a **1-tree** (a tree over all nodes except one + 2 edges connected to the ignored node)
  - Lagrangian subproblem based on a minimum spanning tree

- Strong filtering based on dynamic programming when the number of visited nodes is small (around 15-20 : very common in wide a range of applications) [Cambazard et al, 2012]

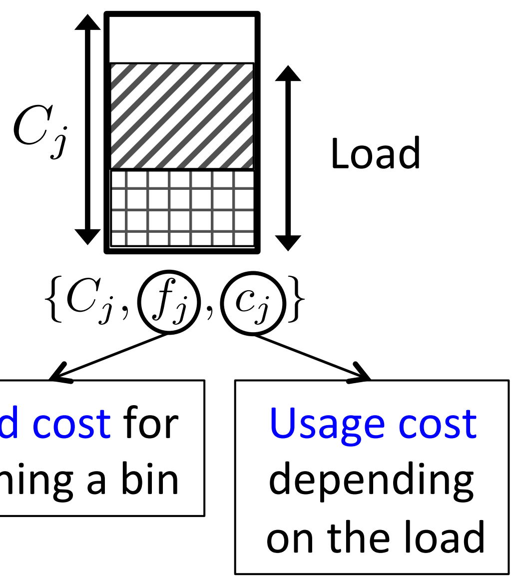
3- Bin Packing with Usage Costs

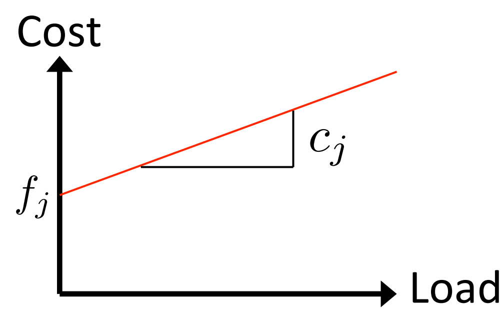
$\textsc{BinPackingUsageCost}([X_1, \ldots, X_n], [L_1, \ldots, L_m], [Y_1, \ldots, Y_m], T, B, S)$

- A set of items $\mathrm{S} = \{w_1, \ldots, w_n\}$



- A set of bins $B = \{\{C_1, f_1, c_1\}, \ldots, \{C_m, f_m, c_m\}\}$





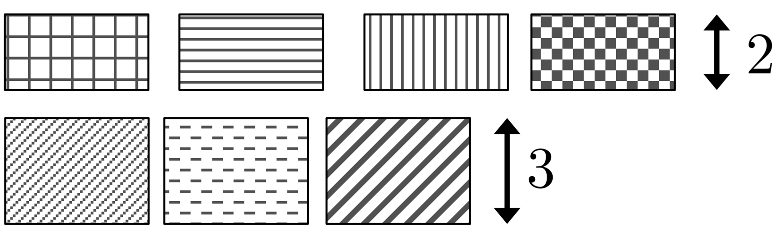
$$cost_j = f_j + Load_j c_j$$

- Minimize $\sum_{j=1|Load_j>0}^{m} cost_j$

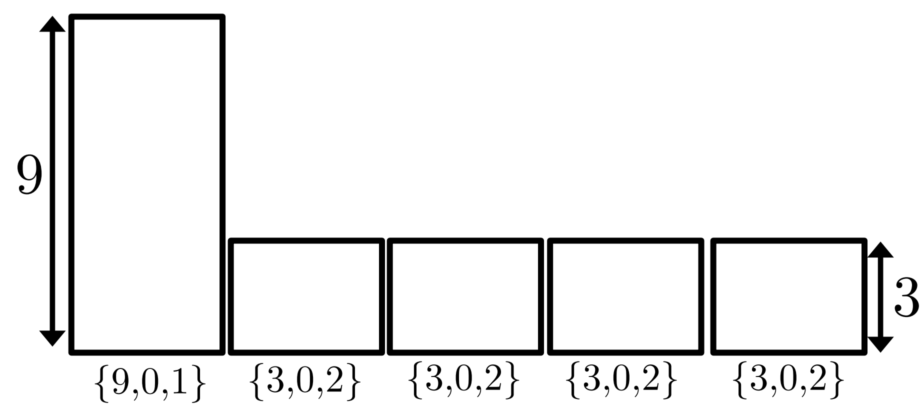
3- Bin Packing with Usage Costs

$\textsc{BinPackingUsageCost}([X_1,\ldots,X_n],[L_1,\ldots,L_m],[Y_1,\ldots,Y_m],T,B,S)$

- A set of items $\mathrm{S} = \{w_1,\ldots,w_n\}$
- A set of bins $B = \{\{C_1,f_1,c_1\},\ldots,\{C_m,f_m,c_m\}\}$
- Minimize the sum of the costs of the used bins



Items

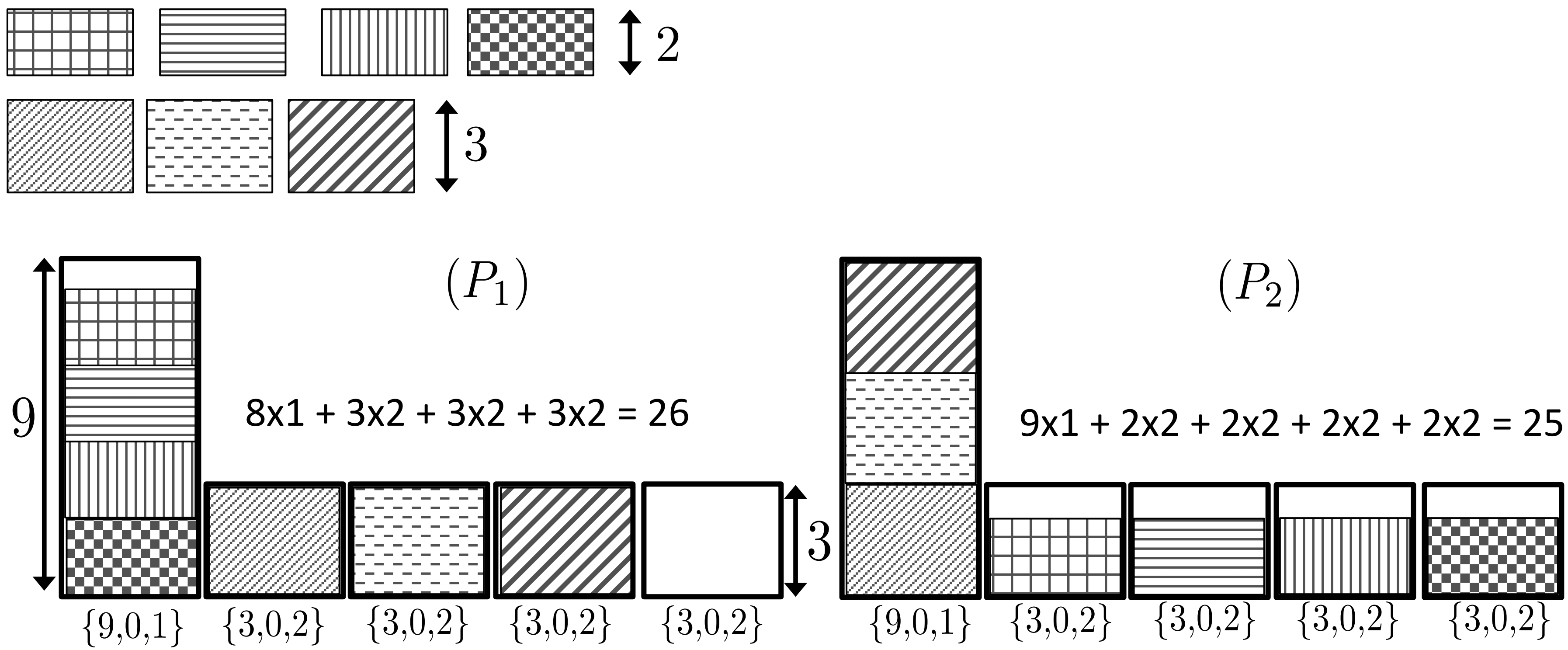


Bins

3- Bin Packing with Usage Costs

$\textsc{BinPackingUsageCost}([X_1, \ldots, X_n], [L_1, \ldots, L_m], [Y_1, \ldots, Y_m], T, B, S)$

- A set of items $\mathrm{S} = \{w_1, \ldots, w_n\}$
- A set of bins $B = \{\{C_1, f_1, c_1\}, \ldots, \{C_m, f_m, c_m\}\}$
- Minimize the sum of the costs of the used bins



3- Bin Packing with Usage Costs

$$\textsc{BinPackingUsageCost}([X_1,\ldots,X_n],[L_1,\ldots,L_m],[Y_1,\ldots,Y_m],T,B,S)$$

- LP relaxation easy to characterize and fast cost filtering can be done [Cambazard et al, 2013]
- Stronger filtering can be achieved using Lagrangian relaxation
  - Relax the constraint enforcing an item to occur in exactly one bin.
  - Lagrangian sub-problem is a knapsack and dynamic Programming provides the reduced costs.

Overview of Lagrangian based filtering for NP-Hard global constraints

Constraint	Lagrangian Subproblem	Examples of applications	References
Multi-cost-regular	Shortest/Longest Path	Personnel Scheduling	[Menana et al, 2009]
Weighted-circuit	1-Tree (Spanning Tree)	Traveling Salesman Problem Traveling Purchaser Problem Traveling Tournament	[Caseau et al, 1997] [Benoist et al, 2001] [Benchimol et al, 2012] [Fages et al, 2012] [Cambazard et al, 2012]
Weighted - atMostNValue	Sorting	Traveling Purchaser Problem Warehouse location	[Cambazard et al, 2012]
atMostNValue	Inspection	…	
Bin-Packing with usage costs	Knapsack	Energy optimization in data-centers	
Shortest Path in DAG with resource constraints	Shortest Path	Multileaf collimator sequencing	[Sellmann, 2005] [Cambazard et al, 2010]

Other applications:

- Golomb rulers [Van Hoove, 2013],
- Automated Recording Problem [Sellmann, 2003]
- Capacitated Network Design [Sellmann, 2002]

Plan

1. Context and motivation
   - Illustrative application: the Traveling Purchaser Problem
   - *Optimization* versus *Satisfaction*
   - *Combinatorial* versus *polyhedral* methods

2. Propagation based on Lagrangian Relaxation
   - Lagrangian duality
   - Filtering using Lagrangian reduced costs
   - Let's try on the *Nvalue* global constraint

3. Overview of some NP-Hard Constraints with costs
   - *Multi-cost regular, Weighted-circuit, Weighted-Nvalue, Bin-packing with usage costs*

4. Examples of applications

Back to the Traveling Purchaser Problem

Problem structures

$Nvisit \in \{1, \ldots, B\}$: Number of visited markets

$TotalCost = TravelingCost + ShoppingCost$

Relaxation	Nature of the problem	Value of the parameter	How to solve / propagate it ?	Key propagation
Feasibility	Hitting Set AtMostNValue	$\overline{Nvisit}$ (cardinality)	[Bessière et al, 2006]	$\underline{Nvisit}$
Feasibility + Shopping cost	p-median Weighted-NValue	$p = \overline{Nvisit}$	Lagrangian relaxation	$\frac{ShoppingCost}{\underline{Nvisit}}$
Traveling Cost	k-TSP Close to Weighted-Circuit	$k = \underline{Nvisit}$	Dynamic Programming ? Lagrangian relaxation	$\frac{TravelingCost}{\overline{Nvisit}}$

CP Model for the TPP

$Nvisit \in \{1, \ldots, B\}$: Number of visited markets
$y_i \in \{0, 1\}$: do we visit market **i** ?
$s_k \in \{i | v_i \in M_k\}$: the market where item **k** is bought
$Cs_k \geq 0$: the price paid for item **k**

$$Minimize\ TravelingCost + ShoppingCost$$

$$Cs_k = \text{ELEMENT}([b_{k1}, \ldots, b_{ki}, \ldots, b_{km}], s_k)$$

$$\exists i \mid s_k = i \Leftrightarrow y_i = 1$$ (channeling s_k and y_i)

$$\text{NVALUE}([s_1, \ldots, s_m], Nvisit)$$

$$\text{TSP}([y_1, \ldots, y_n], Nvisit, TravelingCost, \ldots)$$

$$\text{WEIGHTED-NVALUE}([s_1, \ldots, s_m], Nvisit, ShoppingCost, \ldots)$$

CP Model for the TPP

$Nvisit \in \{1, \ldots, B\}$: Number of visited markets
$y_i \in \{0, 1\}$: do we visit market **i** ?
$s_k \in \{i | v_i \in M_k\}$: the market where item **k** is bought
$Cs_k \geq 0$: the price paid for item **k**

$$Minimize\ TravelingCost + ShoppingCost$$

$$Cs_k = \text{ELEMENT}([b_{k1}, \ldots, b_{ki}, \ldots, b_{km}], s_k)$$

$\exists i \mid s_k = i \Leftrightarrow y_i = 1$ (channeling s_k and y_i)

$\text{NVALUE}([s_1, \ldots, s_m], Nvisit)$

$\text{TSP}([y_1, \ldots, y_n], Nvisit, TravelingCost, \ldots)$ → Close to WEIGHTED-CIRCUIT

$\text{WEIGHTED-NVALUE}([s_1, \ldots, s_m], Nvisit, ShoppingCost, \ldots)$

Overview of results on TPP

- Benchmark (Laporte class3):
  - 100 instances: up to 250 markets and 200 items
  - 11 open instances

- Very efficient when the optimal solution contains few markets
- Very complementary to [Laporte and al]

n	m	Nvisit	Obj BCP	Time BCP	Obj CP	Time CP
250	50	5	3161	17399 s	3161	**0.6 s**
250	150	18	2121	> 18000 s	1531	**417 s**
150	200	25	2594	**1317 s**	2594	5677 s
200	100	28	3161	**8599 s**	3178	> 7200 s

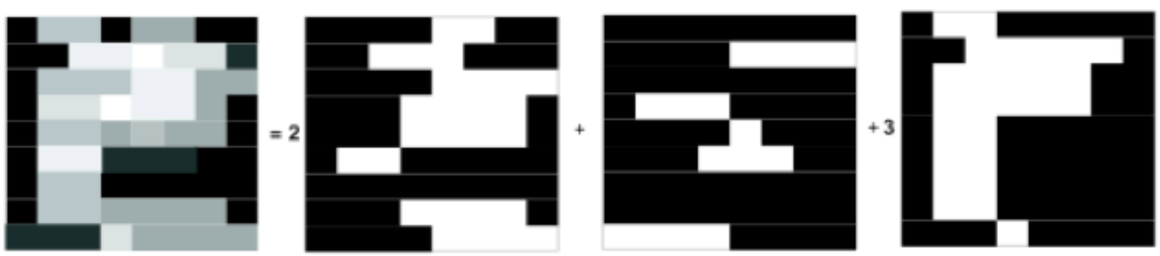
- CP only fails to prove optimality on 10 instances
- Closes 8 instances out of the 11 open instances (improves 10 best known solutions)

The Multileaf Collimator Sequencing Problem

Data : A matrix of integers (**The intensities**)

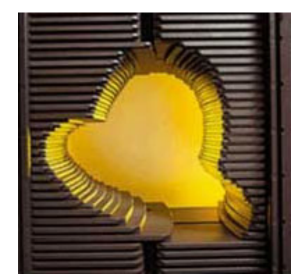
Question : Find a decomposition into a weighted sum of Boolean matrices such that,

- The matrices have the **consecutive ones** property
- The sum of the coefficients (**Beam on time B**) is minimum
- The number of matrices (**Cardinality K**) is minimum



B= 6

K= 3



$$\begin{bmatrix} 0 & 3 & 3 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 5 & 5 & 6 & 4 & 4 & 1 \\ 0 & 3 & 3 & 3 & 5 & 5 & 2 & 2 \\ 0 & 4 & 4 & 6 & 5 & 5 & 2 & 0 \\ 0 & 3 & 3 & 2 & 3 & 2 & 2 & 0 \\ 0 & 5 & 5 & 1 & 1 & 1 & 0 & 0 \\ 0 & 3 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 2 & 2 & 2 & 2 & 0 \\ 1 & 1 & 1 & 4 & 2 & 2 & 2 & 2 \end{bmatrix} = 2 \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} + 3 \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$minimise \;\; w_1 K + w_2 B$$

Overview of results

- Some results using **CP**:
  - Counter model: 20 x 20 with max intensity 10 *[Baatar, Boland, Brand, Stuckey 07], [Brand 08]*
  - Path model: 40 x 40 with max intensity 10 *[Cambazard, O'Mahony, O'Sullivan 09]*
- ***Dedicated*** *algorithm:*
  - 15 x 15 with max intensity 10 (up to 10h of computation) *[Kalinowski 08]*
- Using ***Benders decomposition***: *[Taskin, Smith, Romeijn, Dempsey ANOR'09]*
  Clinical instances (around 20x20 with max intensity 20) solved optimally with up to 5.8 h of computation
- Results can be improved using **Lagrangian Relaxation** when intensity remains small
- Significant improvement using ***Branch and Price and constraint propagation*** : *[Cambazard, O'Mahony, O'Sulllivan, 2012]*
  - 80 x 80 with max intensity 10
  - 20 x 20 with max intensity 20
  - 12 x 12 with max intensity 25
  - Clinical instances with up to 10 min of computation

Conclusion

- Some applications require strong reasoning involving costs (and key NP-Hard sub-problems).

- Lagrangian relaxation (LR) can provide a suitable filtering mechanism without the need of an LP solver:
  - LR can be faster than LP to compute the bound
  - LR can provide more filtering
  - Drawbacks of LR:
    - It needs parameters (when using a sub-gradient algorithm)
    - It can experience issues for converging

- Can we (CP) question the domination (exact algorithms) of **Branch and Cut and Price** for a large class of routing problems ?

References

- Y. Caseau, F. Laburthe*, Solving Small TSPs with Constraints*, ICLP 1997
- P. Refalo, *Linear formulation of Constraint Programming models and Hybrid Solvers* CP 2000
- T. Benoist, F. Laburthe, B. Rottembourg *Lagrange relaxation and constraint programming collaborative schemes for travelling tournament problems,* CP-AI-OR 2001
- F. Focacci, A. Lodi, M. Milano, *Embedding relaxations in global constraints for solving TSP and TSPTW*, Ann. Math. Artif. Intell., 2002
- M. Sellmann, T. Fahle: *Constraint Programming Based Lagrangian Relaxation for the Automatic Recording Problem*. Annals OR 118(1-4): 17-33, 2003
- M. Sellmann: *Theoretical Foundations of CP-Based Lagrangian Relaxation*. CP 2004: 634-647
- T. Gellermann, M. Sellmann, R. Wright: *Shorter Path Constraints for the Resource Constrained Shortest Path Problem*. CPAIOR 2005: 201-216
- J. Menana, S. Demassey: *Sequencing and Counting with the multicost-regular Constraint*. CPAIOR 2009: 178-192
- Hadrien Cambazard, Eoin O'Mahony, Barry O'Sullivan: *Hybrid Methods for the Multileaf Collimator Sequencing Problem*. CPAIOR 2010: 56-70
- P. Benchimol, W.J. van Hoeve, J.C. Régin, L.M. Rousseau, M. Rueher, Improved filtering for weighted circuit constraints. Constraints 17(3): 205-233, 2012
- J.G. Fages, X. Lorca*: Improving the Asymmetric TSP by Considering Graph Structure*. CoRR abs/1206.3437, 2012
- H. Cambazard, B. Penz: A Constraint Programming Approach for the Traveling Purchaser Problem. CP 2012: 735-749
- H. Cambazard, E. O'Mahony, B. O'Sullivan: *A shortest path-based approach to the multileaf collimator sequencing problem*. Discrete Applied Mathematics 160(1-2): 81-99 (2012)
- M. R. Slusky, W.J. van Hoeve*: A Lagrangian Relaxation for Golomb Rulers*. CPAIOR 2013: 251-267