New Filtering algorithm for AtMostNValue and its weighted variant: a Lagrangian approach

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Outline

1. AtMostNValue and AtMostWValue
2. Linear relaxation and reduced cost-based filtering
3. Lagrangian approach
   - Basics of LR: why does it give a lower bound
   - Comparing LP and LR cost based filtering
4. Solving the Lagrangian dual
5. Some results

NValue

$$\textsc{NValue}([X_1, \ldots, X_n], N) \Leftrightarrow |\{j \mid \exists X_i \in \mathcal{X}, X_i = j\}| = N$$

Enforce N to be the number of distinct values appearing in the set X of variables

$$\textsc{AtMostNValue}([X_1, \ldots, X_n], N)$$

$$\left(|\{j \mid \exists X_i \in \mathcal{X}, X_i = j\}| \leq N \right)$$

$$+$$

$$\textsc{AtLeastNValue}([X_1, \ldots, X_n], N)$$

$$\left(|\{j \mid \exists X_i \in \mathcal{X}, X_i = j\}| \geq N \right)$$

AtMostNValue

$$\textsc{AtMostNValue}([X_1, \ldots, X_6], N)$$

$D(X_1)$ = {1, 2, 3, 4, 5, 6}
$D(X_2)$ = {2, 4}
$D(X_3)$ = {1, 2}
$D(X_4)$ = {1, 2, 3}
$D(X_5)$ = {4, 5}
$D(X_6)$ = {4, 5}
$D(N)$ = {1, 2}

A solution:

$$\textsc{AtMostNValue}([2, 2, 2, 2, 4, 4, 2], 2)$$

AtMostNValue

$\textsc{AtMostNValue}([X_1, \ldots, X_6], N)$

$D(X_1)$ = {1, 2, 3, 4, 5, 6}
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$D(X_6)$ = {4, 5}
$D(N)$ = {1, 2}

$D(X_1)$ = {1, 2, $\cancel{3}$, 4, 5, $\cancel{6}$}
$D(X_2)$ = {2, 4}
$D(X_3)$ = {1, 2}
$D(X_4)$ = {1, 2, $\cancel{3}$}
$D(X_5)$ = {4, 5}
$D(X_6)$ = {4, 5}
$D(N)$ = {$\cancel{1}$, 2}

A solution:

$\textsc{AtMostNValue}([2, 2, 2, 2, 4, 4, 2], 2)$

AtMostNValue

$\textsc{AtMostNValue}([X_1, \ldots, X_6], N)$

$D(X_1)$ = {1, 2, 3, 4, 5, 6}
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$D(X_3)$ = {1, 2}
$D(X_4)$ = {1, 2, 3}
$D(X_5)$ = {4, 5}
$D(X_6)$ = {4, 5}
$D(N)$ = {1, 2}

$$D(X_3) \cap D(X_5) = \emptyset$$

$D(X_1)$ = {1, 2, ~~3~~, 4, 5, ~~6~~}
$D(X_2)$ = {2, 4}
$D(X_3)$ = {1, 2}
$D(X_4)$ = {1, 2, ~~3~~}
$D(X_5)$ = {4, 5}
$D(X_6)$ = {4, 5}
$D(N)$ = {~~1~~, 2}

A solution:

$\textsc{AtMostNValue}([2, 2, 2, 2, 4, 4, 2], 2)$

AtMostNValue

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$D(X_5)$ = {4, 5}
$D(X_6)$ = {4, 5}
$D(N)$ = {~~1~~, 2}

- Enforcing GAC is NP-Hard
- [Hebrard et al, 2006], [Beldiceanu et al, 2001]:
  - Lower bound of N obtained by a greedy computing an independent set
  - Best lower bound obtained with a linear relaxation

AtMostNValue - AtMostWValue

- AtMostNValue can be generalized by considering a weight (or cost) for each value:

$$\textsc{AtMostWValue}([X_1, \ldots, X_n], W, w)$$

$$\Updownarrow$$

$$\sum_{\{j \mid \exists X_i \in \mathcal{X}, X_i = j\}} w(j) \leq W$$

- Modeling : a value = a resource
  - The total number of resources available is bounded
  - The total available cost of resources is bounded

AtMostNValue – Value representation

$\textsc{AtMostNValue}([X_1, \ldots, X_n], [Y_1, \ldots, Y_m], N)$

$Y_j \in \{0, 1\}$: value j occurs at least once

➡ express reasoning on mandatory values

Example:

$$D(X_1) = \{1, 2\}, D(X_2) = \{2, 3\}, D(N) = \{2\}$$
$$D(X_3) = \{2, 4\}$$

➡ propagate $Y_2 = 1$

Linear relaxation

$$\text{AtMostWValue}([X_1, \ldots, X_n], [Y_1, \ldots, Y_m], W, w)$$

$$\begin{aligned}
&\textbf{Minimize:} && w = \sum_{1 \le j \le m} w_j y_j && \\
&\textbf{Subject to:} && \sum_{j \in D(X_i)} y_j \ge 1 && \forall i \in [1, n] \\
& && y_j \ge 0 && \forall j \in [1, m]
\end{aligned}$$

- Assuming all weights (costs) are positive

Reduced-cost filtering with the linear relaxation

$$\text{Minimize:} \quad w = \sum_{1 \leq j \leq m} w_j y_j$$

$$\text{Subject to:} \quad \sum_{j \in D(X_i)} y_j \geq 1 \qquad \forall i \in [1, n] \quad (\alpha_i \geq 0)$$

$$y_j \geq 0 \qquad \forall j \in [1, m]$$

- Linear reduced cost for each y_j such that $y_j^* = 0$

$$r_j^* = w_j - \sum_{i | j \in D(X_i)} \alpha_i^*$$

- In the LP case, reduced-cost based filtering is restricted to forbidden values:

$$\forall j \in [1, m] \text{ s.t } y_j^* = 0 \qquad w^* + r_j^* > \overline{W} \implies Y_j \neq 1$$

Lagrangian relaxation

Primal

$$\text{Min } z = \sum_j w_j y_j$$

$$\sum_{j \in D(X_i)} y_j \geq 1 \ \forall i$$

$$y_j \in \{0, 1\} \ \forall j$$

P

Lagrangian relaxation

Primal

$$\text{Min } z = \sum_j w_j y_j$$

$$\sum_{j \in D(X_i)} y_j \geq 1 \ \forall i \quad (\lambda_i)$$

$$y_j \in \{0, 1\} \ \forall j$$

P

Lagrangian relaxation

Primal

$$\text{Min } z = \sum_j w_j y_j$$

$$\sum_{j \in D(X_i)} y_j \geq 1 \ \forall i \quad (\lambda_i)$$

$$y_j \in \{0,1\} \ \forall j$$

P

For any $\lambda \geq 0$: **Lagrangian subproblem**

$$\text{Min } w(\lambda) = \sum_j w_j y_j + \sum_i \lambda_i (1 - \sum_{j \in D(X_i)} y_j)$$

$$y_j \in \{0,1\} \quad \forall j$$

$L(\lambda)$

Lagrangian relaxation

Primal

$$\text{Min } z = \textstyle\sum_j w_j y_j$$

$$\sum_{j \in D(X_i)} y_j \geq 1 \ \forall i \quad (\lambda_i)$$

$$y_j \in \{0,1\} \ \forall j$$

P

For any $\lambda \geq 0$: **Lagrangian subproblem**

$$\text{Min } w(\lambda) = \textstyle\sum_j w_j y_j + \sum_i \lambda_i (1 - \sum_{j \in D(X_i)} y_j)$$

≥ 0 ≤ 0

$$y_j \in \{0,1\} \quad \forall j$$

$L(\lambda)$

Consider a feasible solution $\overline{y}$ of P with value $\overline{z}$

So we have $\overline{w}(\lambda) \leq \overline{z}$

Lagrangian relaxation

Primal

$$\text{Min } z = \sum_j w_j y_j$$

$$\sum_{j \in D(X_i)} y_j \geq 1 \ \forall i$$

$$y_j \in \{0,1\} \ \forall j$$

P

For any $\lambda \geq 0$: **Lagrangian subproblem**

$$\text{Min } w(\lambda) = \sum_j w_j y_j + \sum_i \lambda_i (1 - \sum_{j \in D(X_i)} y_j)$$

$$y_j \in \{0,1\} \quad \forall j$$

$L(\lambda)$

For any $\lambda \geq 0$:

Any feasible solution $\overline{y}$ of P is also feasible for $L(\lambda)$ and $\overline{w}(\lambda) \leq \overline{z}$

So we have : $w^*(\lambda) \leq z^*$

Lagrangian relaxation

Primal

$$\text{Min } z = \sum_j w_j y_j$$
$$\sum_{j \in D(X_i)} y_j \geq 1 \ \forall i$$
$$y_j \in \{0,1\} \ \forall j$$
P

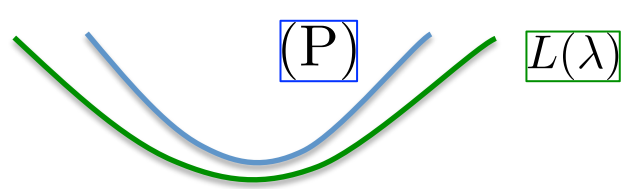
For any $\lambda \geq 0$: **Lagrangian subproblem**

$$\text{Min } w(\lambda) = \sum_j w_j y_j + \sum_i \lambda_i (1 - \sum_{j \in D(X_i)} y_j)$$
$$y_j \in \{0,1\} \quad \forall j$$
$L(\lambda)$

For any $\lambda \geq 0$:

Any feasible solution $\overline{y}$ of P is also feasible for $L(\lambda)$ and $\overline{w}(\lambda) \leq \overline{z}$

So we have : $w^*(\lambda) \leq z^*$



Lagrangian relaxation

Primal

$$\text{Min } z = \sum_j w_j y_j$$
$$\sum_{j \in D(X_i)} y_j \geq 1 \;\; \forall i$$
$$y_j \in \{0,1\} \;\; \forall j$$

P

For any $\lambda \geq 0$: **Lagrangian subproblem**

$$\text{Min } w(\lambda) = \sum_j w_j y_j + \sum_i \lambda_i (1 - \sum_{j \in D(X_i)} y_j)$$
$$= \sum_j \underbrace{(w_j - \sum_{\{i | j \in D(X_i)\}} \lambda_i)}_{q_j} y_j + \sum_i \lambda_i$$
$$y_j \in \{0,1\} \;\; \forall j$$

$L(\lambda)$

- Solving the Lagrangian subproblem optimally for any positive lambda provides a lower bound for W
- Complexity is in O(nm) to check the sign of q_j
  - If q_j is negative, set y_j to 1
  - If q_j is positive, set y_j to 0

$$q_j = w_j - \sum_{\{i | j \in D(X_i)\}} \lambda_i$$

Lagrangian relaxation - filtering

- q_j can be seen as a "Lagrangian reduced cost":

(the increase/decrease of the objective value for setting y_j to 1)

- The filtering rules are simply :

$$y_j^* = 0 \text{ and } w^*(\lambda) + q_j > \overline{W} \implies Y_j \neq 1$$ (forbidden values)

$$y_j^* = 1 \text{ and } w^*(\lambda) - q_j > \overline{W} \implies Y_j \neq 0$$ (mandatory values)

- Filtering can be done from any values of λ

Lagrangian relaxation - filtering

- The filtering rules are simply :

$$w^*(\lambda) + (1 - y_j^*)q_j > \overline{W} \implies Y_j \neq 1 \quad \text{(forbidden values)}$$
$$w^*(\lambda) - y_j^* q_j > \overline{W} \implies Y_j \neq 0 \quad \text{(mandatory values)}$$

- If $\lambda_j = \alpha_j^*$ then $q_j = r_j^*$ (Lagrangian and linear reduced costs are the same)

$$r_j^* = w_j - \sum_{\{i | j \in D(X_i)\}} \alpha_i^*$$

$$q_j = w_j - \sum_{\{i | j \in D(X_i)\}} \lambda_i$$

- Replacing q_j with r_j^* and since $r_j^* y_j^* = 0$ (complementary slackness), we have the LP filtering: $w^* + r_j^* > \overline{W} \implies Y_j \neq 1$

- With the same optimal multipliers, LP and LR perform the same filtering (forbidden values)

Example

$$D(X_1) = \{1, 2\} \quad D(X_3) = \{4, 5\}$$
$$D(X_2) = \{2, 3\} \quad N = \{1, 2\}$$

LP

$$w = y_1 + y_2 + y_3 + y_4 + y_5$$
$$y_1 + y_2 \geq 1$$
$$y_2 + y_3 \geq 1$$
$$y_4 + y_5 \geq 1$$
$$y_j \geq 0 \qquad \forall j \in [1, 5]$$

LR subpb

$$\begin{aligned} w(\lambda_1, \lambda_2, \lambda_3) = \quad & y_1(1 - \lambda_1) + y_2(1 - \lambda_1 - \lambda_2) \\ & + y_3(1 - \lambda_2) + y_4(1 - \lambda_3) + y_5(1 - \lambda_3) \\ & +(\lambda_1 + \lambda_2 + \lambda_3) \\ & y_j \in \{0, 1\} \qquad \forall j \in [1, 5] \end{aligned}$$

Example

$$D(X_1) = \{1, 2\} \qquad D(X_3) = \{4, 5\}$$
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LR subpb

$$\begin{aligned} w(\lambda_1, \lambda_2, \lambda_3) = \quad & y_1(1-\lambda_1) + y_2(1-\lambda_1-\lambda_2) \\ & + y_3(1-\lambda_2) + y_4(1-\lambda_3) + y_5(1-\lambda_3) \\ & +(\lambda_1+\lambda_2+\lambda_3) \\ & y_j \in \{0, 1\} \qquad \forall j \in [1, 5] \end{aligned}$$

λ_1	λ_2	λ_3
0.8	0.8	0.8

Example

$$D(X_1) = \{1, 2\} \qquad D(X_3) = \{4, 5\}$$
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λ_1	λ_2	λ_3
0.8	0.8	0.8

q_1	q_2	q_3	q_4	q_5
0.2	-0.6	0.2	0.2	0.2

Example

$$D(X_1) = \{1, 2\} \qquad D(X_3) = \{4, 5\}$$
$$D(X_2) = \{2, 3\} \qquad N = \{1, 2\}$$

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$$w = y_1 + y_2 + y_3 + y_4 + y_5$$
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y_1^*	y_2^*	y_3^*	y_4^*	y_5^*
0	1	0	0	0

Example

$$D(X_1) = \{1,2\} \qquad D(X_3) = \{4,5\}$$
$$D(X_2) = \{2,3\} \qquad N = \{1,2\}$$

LP

$$w = y_1 + y_2 + y_3 + y_4 + y_5$$
$$y_1 + y_2 \geq 1$$
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$$y_j \geq 0 \qquad \forall j \in [1,5]$$

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$$\begin{aligned} w(\lambda_1, \lambda_2, \lambda_3) = \quad & y_1(1-\lambda_1) + y_2(1-\lambda_1-\lambda_2) \\ & + y_3(1-\lambda_2) + y_4(1-\lambda_3) + y_5(1-\lambda_3) \\ & +(\lambda_1+\lambda_2+\lambda_3) \\ & y_j \in \{0,1\} \qquad \forall j \in [1,5] \end{aligned}$$

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$$w^*(\lambda) = 1.8$$

Example

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$$y_j \geq 0 \qquad \forall j \in [1,5]$$

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y_1^*	y_2^*	y_3^*	y_4^*	y_5^*
0	1	0	0	0

$$w^*(\lambda) = 1.8$$

$$w^*(\lambda) - q_2 = 2.4 > \overline{N} \qquad \Longrightarrow Y_2 \neq 0$$

Example

$D(X_1) = \{1,2\}$ $D(X_3) = \{4,5\}$

$D(X_2) = \{2,3\}$ $N = \{1,2\}$

LP

$$w = y_1 + y_2 + y_3 + y_4 + y_5$$
$$y_1 + y_2 \geq 1$$
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$$y_j \geq 0 \qquad \forall j \in [1,5]$$

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0.8	0.8	0.8

q_1	q_2	q_3	q_4	q_5
0.2	-0.6	0.2	0.2	0.2

y_1^*	y_2^*	y_3^*	y_4^*	y_5^*
0	1	0	0	0

$w^*(\lambda) = 1.8$

$$w^*(\lambda) - q_2 = 2.4 > \overline{N} \implies Y_2 \neq 0$$

λ_1	λ_2	λ_3
0.8	0.8	0.9

q_1	q_2	q_3	q_4	q_5
0.2	-0.6	0.2	0.1	0.1

y_1^*	y_2^*	y_3^*	y_4^*	y_5^*
0	1	0	0	0

$w^*(\lambda) = 1.9$

$$w^*(\lambda) + q_1 = 2.1 > \overline{N} \implies Y_1 \neq 1$$
$$w^*(\lambda) + q_3 = 2.1 > \overline{N} \implies Y_3 \neq 1$$

Example

$$D(X_1) = \{1, 2\} \quad D(X_3) = \{4, 5\}$$
$$D(X_2) = \{2, 3\} \quad N = \{1, 2\}$$

LP

$$w = y_1 + y_2 + y_3 + y_4 + y_5$$
$$y_1 + y_2 \geq 1$$
$$y_2 + y_3 \geq 1$$
$$y_4 + y_5 \geq 1$$
$$y_j \geq 0 \qquad \forall j \in [1, 5]$$

LR subpb

$$w(\lambda_1, \lambda_2, \lambda_3) = y_1(1 - \lambda_1) + y_2(1 - \lambda_1 - \lambda_2) + y_3(1 - \lambda_2) + y_4(1 - \lambda_3) + y_5(1 - \lambda_3) + (\lambda_1 + \lambda_2 + \lambda_3)$$
$$y_j \in \{0, 1\} \qquad \forall j \in [1, 5]$$

λ_1	λ_2	λ_3
0.8	0.8	0.8

q_1	q_2	q_3	q_4	q_5
0.2	-0.6	0.2	0.2	0.2

y_1^*	y_2^*	y_3^*	y_4^*	y_5^*
0	1	0	0	0

$w^*(\lambda) = 1.8$

$$w^*(\lambda) - q_2 = 2.4 > \overline{N} \implies Y_2 \neq 0$$

λ_1	λ_2	λ_3
0.8	0.8	0.9

q_1	q_2	q_3	q_4	q_5
0.2	-0.6	0.2	0.1	0.1

y_1^*	y_2^*	y_3^*	y_4^*	y_5^*
0	1	0	0	0

$w^*(\lambda) = 1.9$

$$w^*(\lambda) + q_1 = 2.1 > \overline{N} \implies Y_1 \neq 1$$
$$w^*(\lambda) + q_3 = 2.1 > \overline{N} \implies Y_3 \neq 1$$

Optimal LP multipliers

λ_1	λ_2	λ_3
0	1	1

q_1	q_2	q_3	q_4	q_5
1	0	0	0	0

y_1^*	y_2^*	y_3^*	y_4^*	y_5^*
0	0	0	0	0

$w^*(\lambda) = 2$

$$w^*(\lambda) + q_1 = 2.1 > \overline{N} \implies Y_1 \neq 1$$

Lagrangian relaxation

Primal

$$\text{Min } z = \sum_j w_j y_j$$

$$\sum_{j \in D(X_i)} y_j \geq 1 \ \forall i$$

$$y_j \in \{0,1\} \ \forall j$$

P

For any $\lambda \geq 0$: **Lagrangian subproblem**

$$\text{Min } w(\lambda) = \sum_j w_j y_j + \sum_i \lambda_i (1 - \sum_{j \in D(X_i)} y_j)$$

$$y_j \in \{0,1\} \quad \forall j$$

$L(\lambda)$

Lagrangian Dual:

$$L^* = \max_{\lambda \geq 0} w^*(\lambda)$$

Lagrangian relaxation

Primal

$$\text{Min } z = \sum_j w_j y_j$$
$$\sum_{j \in D(X_i)} y_j \geq 1 \ \forall i$$
$$y_j \in \{0,1\} \ \forall j$$

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For any $\lambda \geq 0$: **Lagrangian subproblem**

$$\text{Min } w(\lambda) = \sum_j w_j y_j + \sum_i \lambda_i (1 - \sum_{j \in D(X_i)} y_j)$$
$$y_j \in \{0,1\} \quad \forall j$$

$L(\lambda)$

Lagrangian Dual:

$$L^* = \max_{\lambda \geq 0} w^*(\lambda)$$

It is known that L* is equal to the optimal value of the LP

Filtering algorithm based on Lagrangian relaxation

- The objective function of the Lagrangian dual is known to be concave piecewise linear
- A popular approach to solve it is to use a sub-gradient algorithm for updating the multipliers

Algorithm 1: Outline of the ATMOSTWVALUE LR-based propagator

```
for (i ∈ [1, n]) do
    λ_i ← 0 // Lagrangian multipliers initialization
k ← 0
while (not subgradient.isStopCriterionMet ()) do
    SolveSubProblem()
    FilterFromSubProblem()
    subgradient.UpdateMultipliers(k)
    k ← k + 1
```

Solving the Lagrangian dual
And filtering along the way

Sub-gradient algorithms

Algorithm 4: subgradient algorithm

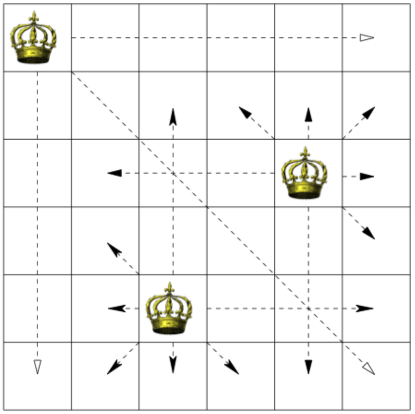
Function `UpdateMultipliers` (*int* k)
for ($i \in [1, n]$) **do**
$$\lambda_i \leftarrow max(0, \lambda_i + \mu^k(1 - \sum_{j \in D(X_i)} y_j))$$

Step-size rule: how far the direction is followed

Gives a direction (decrease or increase) the multiplier

- Violation/over-satisfaction of the relaxed constraints give a direction for improving the multipliers.
- The direction is followed using a step-size μ^k that decreases over time.

Harmonic	Geometric	Newton
$\mu^k = \frac{1}{k}$	$\mu^k = \mu^0 \times (0.95)^k$	....



Dominating queens

Slow subgradient (better convergence)

Fast subgradient

LP + Reduced-costs

Instance			CP+NVAL(Harmonic)			CP+NVAL(*Newton*)			CP		CP + NVAL(LP)	
n	v	Feas	Cpu	Fails	Iters	Cpu	Fails	Iters	Cpu	Fails	Cpu	Fails
6	3	Yes	0.2	7	25k	**0.1**	9	**3k**	0.2	15	0.5	9
7	4	Yes	0.5	52	128k	0.1	31	**9k**	0.1	353	4.4	93
8	5	Yes	1.0	86	210k	**0.2**	79	**17k**	0.7	2275	18.8	259
8	4	No	23.3	**1796**	2767k	**5.1**	6260	**299k**	155.6	1074789	222.8	2596
9	5	Yes	14.7	**862**	1426 k	**3.4**	1593	**157k**	186.8	920666	153.7	884

- The filtering of LR outperforms the state-of-the art propagator
- LR can be fast (faster than LP) although the bounds are the same
- LR can filter more than LP

Conclusion

- Simple and powerful propagator for useful NP-Hard global constraints
- A propagator not meant to be used by default but really make sense when (weighted) independent set is at the heart of the problem:
  - Example: facility location
- CP models up to being competitive with MIP on facility location benchmark

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