Recoloring sparse graphs

Nicolas Bousquet

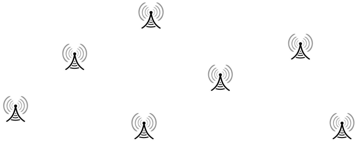
Graphes @ Lyon

ÉCOLE CENTRALE LYON

LIRIS

A first example

orange™



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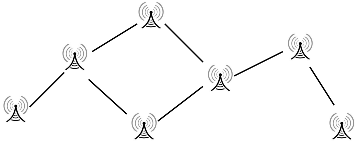

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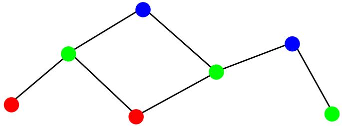

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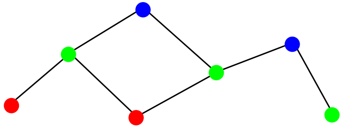
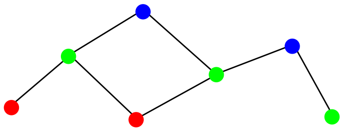
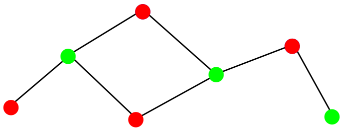


A first example

- Realocate the frequencies of the antennas.
- Condition : no interference at any time.
- Only one antenna reallocation at a time.





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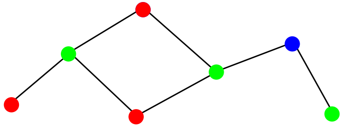

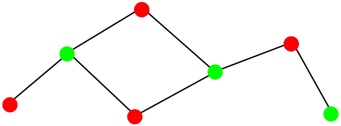

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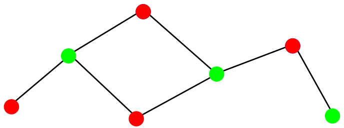

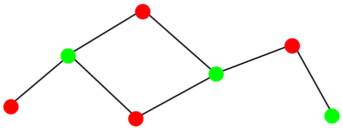

Formally

Definition (k-Reconfiguration graph $\mathcal{C}_k(G)$ of G)

- Vertices : Proper k-colorings of G.
- Create an edge between any two k-colorings which differ on exactly one vertex.

All along the talk k denotes the number of colors.

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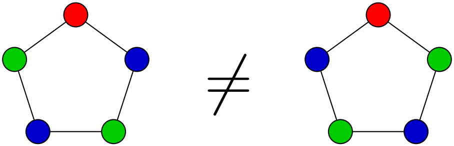
Remark

Two colorings equivalent up to color permutation are distinct.



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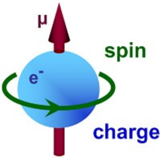
  What is the diameter of the reconfiguration graph ?
- Can we effiently find a short transformation (from an algorithmic point of view) ?
  Can we find a path between two vertices of the reconfiguration graph in polynomial time ?

Anti-ferromagnetic Potts Model



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$H(\sigma)$ is the number of monochromatic edges.

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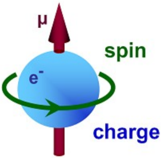

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$$\nu_T(\sigma) = \frac{1}{\lambda} \cdot e^{-\frac{H(\sigma)}{T}}$$

where λ is a normalization constant.

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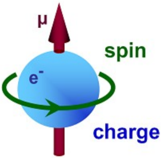

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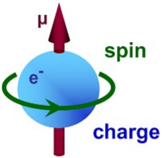

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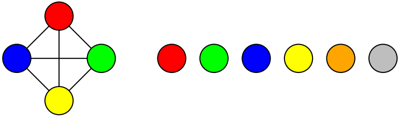
Definition (Glauber dynamics)

Limit of a k-state Potts model when $T \to 0$.

In Glauber dynamics : only proper coloring have positive measure.

Evolution of Glauber dynamics

- At any time t, the state of any spin is modified under some probability rule (e.g. exponential rule).
- The probability that a spin state becomes j is $\frac{1}{\text{number of states that does not appear in the neighborhood of the spin}}$.



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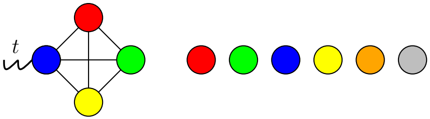

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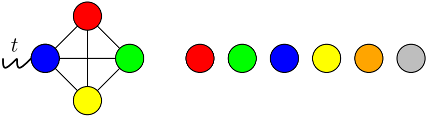

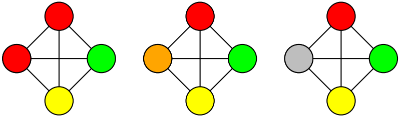

Proba $= 0$ Proba $= 1/2$ Proba $= 1/2$

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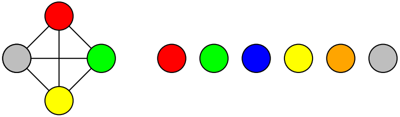

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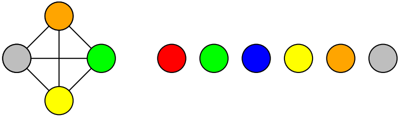

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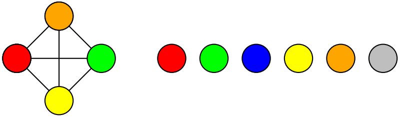

Mixing time

Given a Markov chain :

- Do we converge to the stationary distribution ? i.e. are we ergodic ?
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Theorem

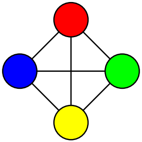
If the diameter of the reconfiguration graph is D then the mixing time is at least $2 \cdot D$.

Main question in Theoretical Physics

When is G $c(\Delta)$-mixing in polynomial time ?

Partial results :

- The chain is not ergodic if $c = \Delta + 1$ (e.g. cliques).



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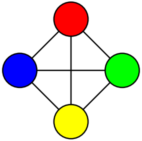

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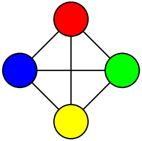


Conjecture

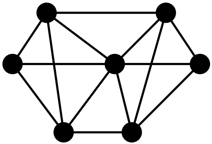
If $c \geq \Delta + 2$, the graph is c-mixing in polynomial time.

Main question in CS

Conjecture (Cereceda)

The $(k+2)$-recoloring diameter of any k-degenerate graph is $\mathcal{O}(n^2)$.

A graph is k-degenerate if there exists an order $v_1, \ldots, v_n$ such that for every i, v_i has at most k neighbors after it in the order.



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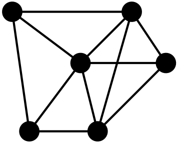

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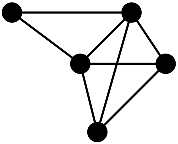

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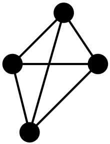

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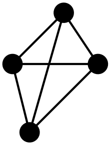

Theorem (Cereceda '07)

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- Delete a vertex of degree at most k.
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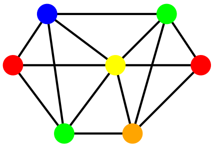

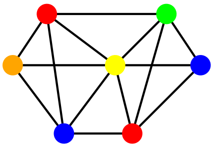

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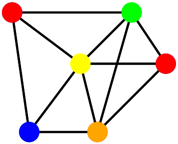

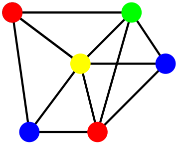

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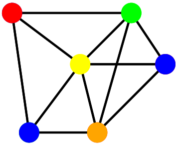

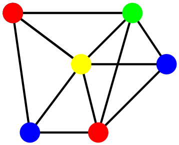

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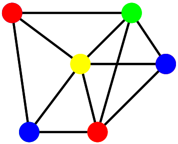

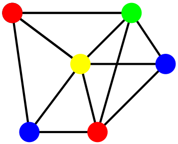

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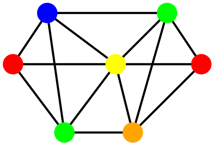

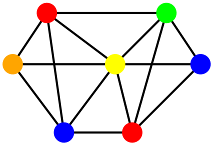

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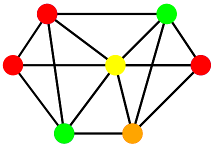

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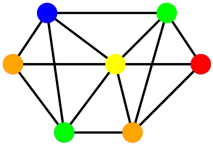

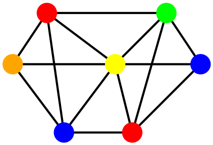

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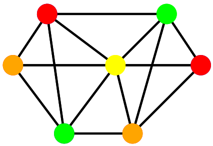

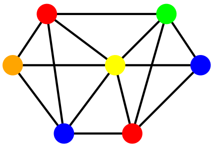

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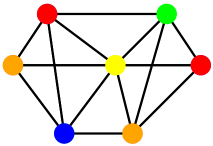

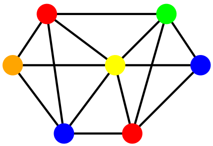

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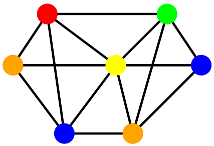

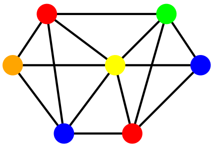

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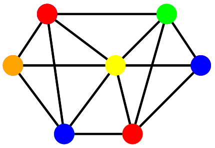

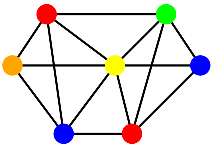

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When do we need to recolor the leftmost vertex ?

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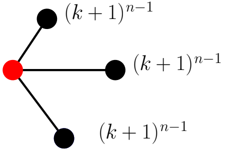

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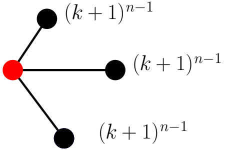

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- In the last round : $+1$ recoloring.

The total number of recolorings is at most $(k+1)^n$.

On specific classes of graphs

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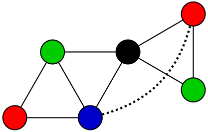
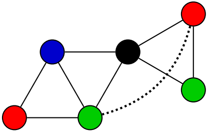
-  

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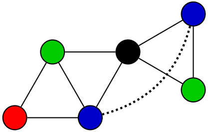
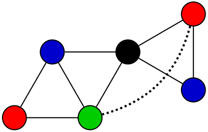
-  

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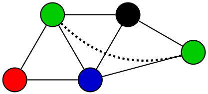
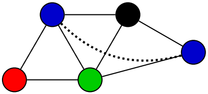
-  

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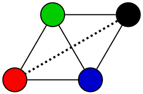
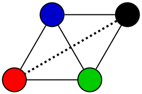
-  

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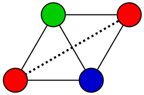
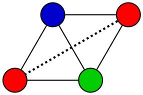
-  

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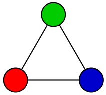
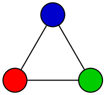
-  

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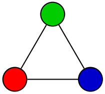
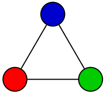
-  
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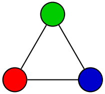
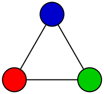
-  
- Cliques are k-mixing in $\mathcal{O}(n)$.
- We can find the two vertices that have to be identified using the clique-tree.

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Theorem (Bonamy, B.)

The $(tw(G)+2)$-reconfiguration graph of G has diameter $\mathcal{O}(n^2)$.

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Key argument :

- Peel the graph according to a tree decomposition.
- "Interactions" between remaining vertices and deleted vertices are reduced to the vertices of a leaf of a tree decomposition.
  $\Rightarrow$ Recoloring a vertex has a limited impact on the graph.

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Theorem (B., Perarnau '14)

If the maximum average degree of G is at most $d - \epsilon$ then the diameter of the $(d+1)$-reconfiguration graph is polynomial.

Maximum average degree = maximum density of a subgraph of G.

$$mad(G) = \max_{S \subseteq V} \left(\frac{\text{number of edges induced by } S}{|S|} \right)$$

General framework

We want to recolor the graph from α to β.

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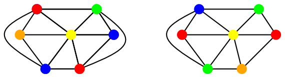

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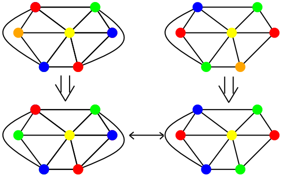

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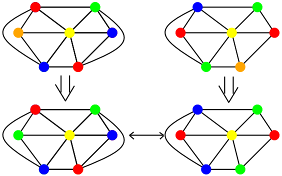

Standard method :

- Eliminate one color from the current coloring.
- Use this additional color to color a well-chosen stable set.
- Apply induction with $\chi(G) - 1$.

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Lemma

If G has maximum average degree at most $d - \epsilon$ then a linear fraction of the vertices has degree at most $d - 1$.

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Corollary (layer partition)

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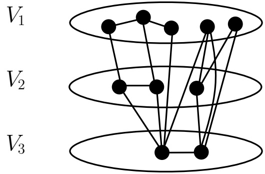

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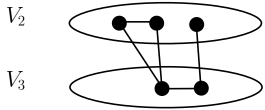

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- Select a vertex of color • in the largest possible layer.
- Recolor it without recoloring any vertex of a largest (or equal) layer.
- Repeat until there is a vertex of color •.

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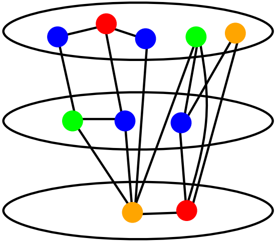

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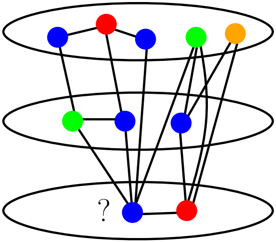

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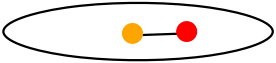

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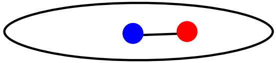

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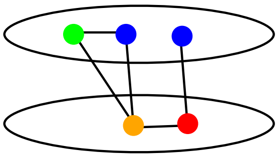

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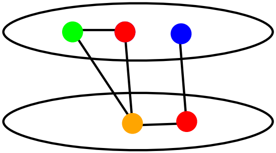

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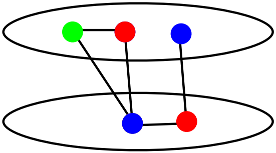

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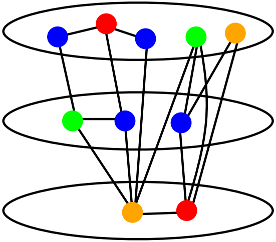

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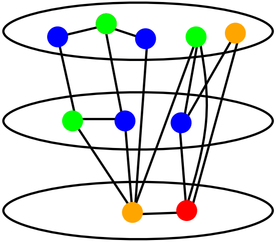

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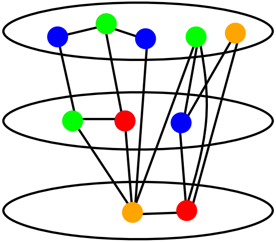

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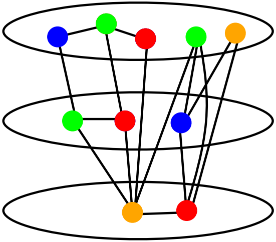

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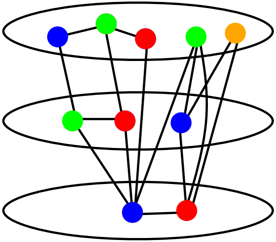

Counting trick

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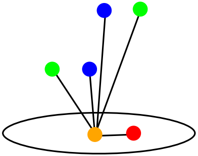
This recoloring process eliminates color • in a polynomial number of steps.



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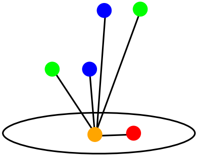

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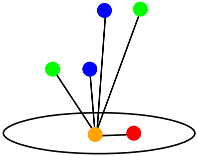

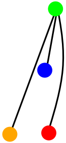

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- But each vertex has degree at most $(d-1)$ in larger layers. So its recoloring is asked a bounded number of times.

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Since this equation holds for every vertex, the total number of recoloring needed to modify the color of • is polynomial.

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Treewidth proof :

- Select a small degree vertex v and delete it.

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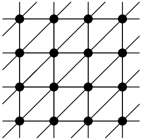
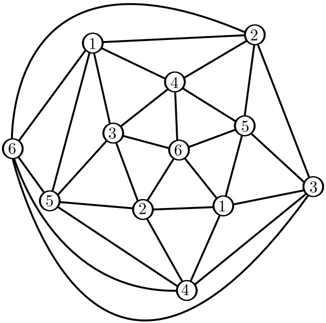


- There exist planar graphs such that $mad(G) \to 6$.
- We need $mad(G) < 6 - \epsilon$ to obtain this layer partition.

Lower bound



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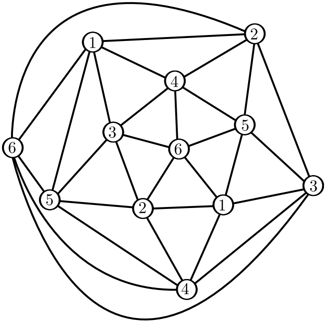

Similarly, we can show

> **Corollary** (B., Perarnau '14)
>
> The k-recoloring diameter of any triangle-free planar graph is polynomial if $k \geq 6$.

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We know that :

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Definition (frozen coloring)

A coloring is frozen if none of the vertices of the graph can be recolored.

Theorem (Feghali, Johnson, Paulusma '14)

One can recolor any non-frozen $(\Delta + 1)$-coloring of G into any other (in $\mathcal{O}(n^2)$ steps).

Structure of the $(\Delta + 1)$-colorings

So the $(\Delta + 1)$-reconfiguration graph consists in :

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Corollary

If two $(\Delta + 1)$ colorings are chosen at random, then one can recolor the first into the second with probability $\to 1$ if $n \to +\infty$.

Conclusion

- Solve the Cereceda conjecture.
- Extend / improve recoloring results to other graph classes.
- Existence of a k-reconfiguration graph with bounded but exponential diameter.
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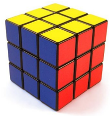

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- Stable sets : applications to discrete fields in mathematics.
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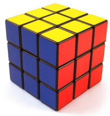

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Thanks for your attention !