SIAM J. COMPUT.
Vol. 26, No. 2, pp. 582–603, March 1997

013

POTENTIALS IN UNDIRECTED GRAPHS AND PLANAR MULTIFLOWS*

ANDRÁS SEBŐ†

Abstract. The duality relation between shortest paths and potentials in *directed graphs* and the significance of both of these in the theory of network flows is well known. In this paper, we work out the analogous *undirected* notions, which neither are contained in nor contain their directed counterpart. They are more related to matching theory than to network flows: the corresponding min-path-max-potential theorem can be considered a weighted generalization of the Gallai–Edmonds structure theorem for matchings.

In our earlier work [*J. Combin. Theory Ser. B*, 49 (1990), pp. 10–39], the corresponding theorems are proved in the special case of ± 1 bipartite weightings, and this special case already contains the main points of the general proof. The goal of the present paper is to extrapolate from this ± 1-weighted bipartite special case the arbitrarily weighted general min-path-max-potential theorem and to show some algorithmic consequences related to planar multiflows, the Chinese postman problem, the weighted and unweighted matching structure, etc. In order to make this paper self-contained, we also include a compact, revised variant of earlier proofs, adapted to the present context. In addition to good characterization theorems and polynomial algorithms, efficient (logarithmic polynomial) parallel algorithms follow for some of these problems.

Key words. T-joins, T-cuts, multicommodity flows, Chinese postman, matching, structure, parallel algorithm

AMS subject classifications. 05C38, 05C45, 90B10

PII. S0097539790186704

1. Introduction. In this section, we explain the background of the paper and introduce the main tools that we will use. In particular, we present potentials in the ± 1-weighted bipartite special case, which was developed in Sebő [1990]. Since this constitutes the kernel of our results, we fully include a compact proof of the main theorem concerning this case.

Then in section 2, we define potentials in arbitrary weighted undirected graphs and prove (extrapolate from the bipartite special case) a minimax theorem on minimum weight paths and maximum potentials. In section 3, we point out the algorithmic consequences of our results and apply them to, for example, planar multiflows.

If G is a graph and $w : E(G) \to \mathbb{R}$, define the ***distance*** of $x, y \in V(G)$ as

$$\lambda_{G,w}(x,y) = \lambda_w(x,y) = \lambda(x,y) = \min\{w(P) : P \text{ is an } (x,y) \text{ path}\}.$$

In this paper, ***paths*** are considered to be sets of edges, or subgraphs. (For instance, $V(P)$ will denote the set of vertices of the path P.) They can have a repetition of vertices, but no repetition of edges is allowed. An (x,y) path is a path whose endpoints are $x, y \in V(G)$. The definition of $\lambda_{G,w}(x,y)$ is meant to be ∞ if x and y are not in the same component of G.

A path without repetition of vertices will be called ***simple***. If the two endpoints of a (simple) path coincide, it is a ***cycle*** (***circuit***). $w(P)$ denotes the sum $\sum_{e \in P} w(e)$. A ***shortest*** (w-***shortest***) ***path*** is an (a,b) path P with $w(P) = \lambda_w(a,b)$. If $a, b \in V(P)$,

* Received by the editors August 20, 1990; accepted for publication (in revised form) June 21, 1995. This research was partly supported by the Alexander von Humboldt Foundation and project TEMPRA, Région Rhône-Alpes.

http://www.siam.org/journals/sicomp/26-2/18670.html

† CNRS, LEIBNIZ-IMAG, 46 Avenue Félix Viallet, 38031 Grenoble cedex 1, France (andras.sebo@imag.fr).

$P(a,b)$ denotes a simple subpath of P joining a and b. (If P is simple, $P(a,b)$ is uniquely determined.) For $X \subseteq V(G)$, $\delta(X)$ will denote the set of edges with exactly one endpoint in X. We will also use the notation $E^- := \{e \in E(G) : w(e) < 0\}$, $E^+ := \{e \in E(G) : w(e) > 0\}$.

A graph G with a weighting $w : E(G) \to \mathbb{R}$ is called *conservative* if $w(C) \geq 0$ for every circuit $C \subseteq E(G)$. Conservative graphs are characterized as follows.

(1.1) (G,w) *is conservative if and only if for all* $x, y \in V(G)$, $\lambda_w(x,y) = \min\{w(P) :$ P *is a simple* (x,y) *path*$\}$.

In particular, for connected graphs, $\lambda_w(x,y)$ is finite for all $x, y \in V(G)$.

Recall that $\lambda_w(x,y)$ can be computed via matching techniques in various well-known ways (see Edmonds [1965a] or Lawler [1976]); one of these will be explained in section 3. On the other hand *it cannot be reduced to the well-known shortest-path algorithms* because the two directed edges corresponding to an undirected edge of negative weight constitute a negative directed cycle; moreover, subpaths of shortest paths are not necessarily shortest and distances do not satisfy the triangle inequality. Thus the notion of potentials and the related theory are also different in the undirected case.

The behavior of undirected potentials is determined by the following theorem, as will be explained in section 2:

If (G,w) is conservative and $x_0 \in V(G)$, we call each set $V^i = V^i(\lambda) := \{x \in V(G) : \lambda_w(x_0,x) \leq i\}$ $(i = 0, \pm 1, \pm 2, \ldots)$ a *level set* of (G, w, x_0); we denote by G^i the graph induced by V^i, and we call it the *level graph*. If the weights are ± 1, then edges go between neighboring levels; that is, we have the following.

(1.2) *If* $w : E(G) \to \{-1, 1\}$ *is conservative, then for all* $xy \in E(G)$, $|\lambda(x_0,x) - \lambda(x_0,y)| \leq 1$. *If, in addition,* G *is bipartite, then this inequality is satis ed with equality for every edge.*

Indeed, we can assume without loss of generality that $\lambda(x_0,x) \geq \lambda(x_0,y)$. Let P be a simple shortest (x_0,y) path. If $xy \in P$, then $xy \in E^-$ follows, and $\lambda(x_0,x) \leq w(P \setminus \{xy\}) = w(P) + 1$; if $xy \notin P$, then $xy \in E^+$, and $\lambda(x_0,x) \leq w(P \cup \{xy\}) = w(P) + 1$. Thus $\lambda(x_0,y) \leq \lambda(x_0,x) \leq \lambda(x_0,y) + 1$. If G is bipartite, then in addition $\lambda(x_0,x) \neq \lambda(x_0,y)$, whence $\lambda(x_0,x) = \lambda(x_0,y) + 1$, and (1.2) is proved.

For an arbitrary conservative weighting, the inequality $|\lambda(x_0,x) - \lambda(x_0,y)| \leq |w(xy)|$ can be checked in the same way (and also follows easily from (1.2); see the proof of Lemma 2.1(a)).

THEOREM 1.1. *Let* G *be a bipartite graph and* $w : E(G) \to \{-1, 1\}$ *such that* (G,w) *is conservative. Furthermore, let* $x_0 \in V(G)$ *be arbitrary and* D *be the vertex set of a component of* G^i $(i \in \{0, \pm 1, \pm 2, \ldots\}$, $V^i \neq \emptyset)$. *Then* $|\delta(D) \cap E^-| = 1$ *provided that* $x_0 \notin D$ *and* $|\delta(D) \cap E^-| = 0$ *provided that* $x_0 \in D$.

If $\mathcal{D}$ denotes the family of sets occurring as the vertex set of a component of a G^i (as a D in the theorem), where G is bipartite, then because of (1.2), $\{\delta(D) : D \in \mathcal{D}\}$ partitions $E(G)$; the theorem states that the $D \in \mathcal{D}$ with $x_0 \notin D$ partition E^- into singletons.

Theorem 1 contains Seymour's minimax theorem on T-joins and T-cut packings, the Berge–Tutte theorem, and the Gallai–Edmonds structure theorem. It actually implies the generalization of this structure theorem to T-joins and weighted matchings (see Sebő [1990] and section 3 below).

Figure 1 illustrates the components of the level sets of a ± 1-weighted conservative graph. The thick edges are those of weight -1.

Proof of Theorem 1.1. Let (G,w) be conservative. We prove the theorem by

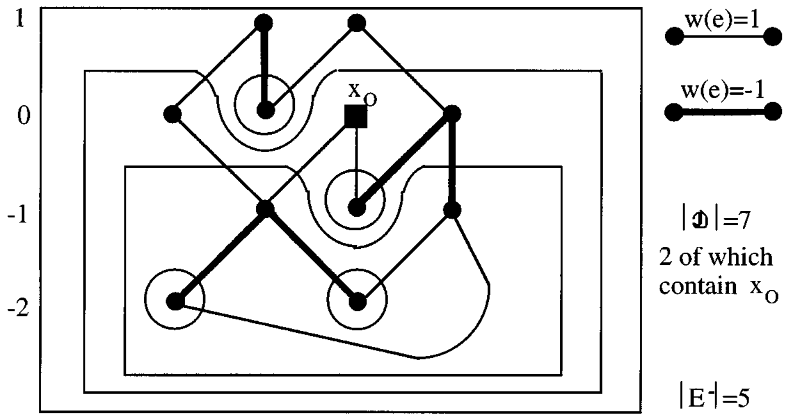

FIG. 1.

induction on $|V(G)|$. Let b be a vertex that satisfies

$$\lambda_w(x_0, b) = m := \min\{\lambda_w(x_0, x) : x \in V(G)\}$$

and let P be a simple (x_0, b) path, with $w(P) = m$. (P exists because of (1.1).) In addition, we assume that b is chosen among all possible choices so that $|P|$ is minimum. (This last assumption is not really essential; the claims that we will prove for b are true without it. However, it will be useful for technical simplicity.)

Let us first check the statement for $\{b\}$: since G is bipartite, V^m does not induce edges, whence $\{b\}$ is one of its components. The statement to check is the following.

CLAIM 1.

$$|\delta(b) \cap E^-| = 1 \quad \textit{provided } b \neq x_0,$$

$$|\delta(b) \cap E^-| = 0 \quad \textit{provided } b = x_0;$$

Moreover, if $b \neq x_0$, then P contains the unique negative edge adjacent to b.

If $b = x_0$, then because of $m = \lambda_w(x_0, b) = 0$, there cannot be a negative edge adjacent to $x_0 = b$. Now suppose that $b \neq x_0$.

Since P is simple, it has exactly one edge adjacent to b. That edge is negative because otherwise, if we delete it from P, we get a path which is shorter than $w(P) = m$. Suppose indirectly that there exists a negative edge $e \in \delta(b) \setminus P$: $P \cup e$ is also a path and $w(P \cup e) < w(P) = m$, and this contradiction proves Claim 1.

Let G^* be the graph that we obtain after contracting $\delta(b)$ or, equivalently, after identifying the vertices adjacent to b and deleting b. (By the choice of b and because of (1.2), all neighbors of b are at level $m - 1$.) We consider that the vertices and edges of G^* are the same as those of $G - b$; w^* is defined as the restriction of w to $E(G - b) = E(G^*)$. (Parallel edges can be replaced by one edge whose weight is the minimum of the weights.)

CLAIM 2. *Suppose $b \neq x_0$. Then (G^*, w^*) is conservative, and for every $x \in V(G^*)$, $x \neq b$, $\lambda_{G^*,w^*}(x_0, x) = \lambda_{G,w}(x_0, x)$.*

A circuit or (x_0, x) path ($x \in V(G)$, $x \neq b$) K^* of G^* is a circuit or (x_0, x) path of G or can be made into one by adding two edges of $\delta(b)$. Denote this corresponding circuit or path of G by K. Since $|\delta(b) \cap E^-| = 1$, $w(K^*) = w(K) - 2$ or $w(K^*) = w(K)$.

If K is not a zero-weight circuit or shortest (x_0, x) path, then since G is bipartite, $w(K) \geq 2$ or $w(K) \geq \lambda_w(x_0, x)+2$, respectively, and $w(K^*) \geq 0$ or $w(K^*) \geq \lambda_w(x_0, x)$ follow.

Therefore, let $w(K) = 0$ or $w(K) = \lambda_w(x_0, x)$, respectively. Claim 2 clearly follows now from the following: either $K \cap \delta(b) = \emptyset$ or $K \cap \delta(b) = \{e_1, e_2\}$, $e_1 \in E^-$, $e_2 \in E^+$.

Suppose indirectly that $K \cap \delta(b) = \{e_1, e_2\}$, $e_1, e_2 \in E^+$.

Case 1. $K \cap P = \emptyset$. If K is a zero-weight circuit, then $P' := (P \cup K) \setminus e_1$ is a path, with $w(P') = w(P) + w(K) - 1 = w(P) + 0 - 1 < m$, a contradiction. Therefore, suppose that K is a w-shortest (x_0, x) path. By (1.1), we can suppose that K is simple. Since the only edge of $K(x_0, b)$ adjacent to b is positive, by Claim 1, $K(x_0, b)$ is not a w-shortest path, that is, $w(K(x_0, b)) > w(P(x_0, b))$. Then, however, $w(P(x_0, b) \cup K(b, x)) = w(P(x_0, b)) + w(K(b, x)) < w(K(x_0, b)) + w(K(b, x)) = \lambda_w(x_0, x)$, a contradiction because $P(x_0, b) \cup K(b, x)$ contains an (x_0, x) path.

Case 2. $K \cap P \neq \emptyset$. Then $b \neq x_0$ (otherwise $P = \emptyset$), and by the last part of Claim 1, the edge of P adjacent to b is negative, so it is different from the edges of K adjacent to b. Thus $V(K) \cap V(P)$ cannot consist of b only. Walking on P from b towards x_0, let $a \neq b$ be the first vertex of K we meet. $K(a, b) \cap P(a, b) = \emptyset$. $K' := (K \setminus K(a, b)) \cup P(a, b)$ is also a circuit (if K is a circuit) or an (x_0, x) path (if K is so), whence $w(K(a, b)) \leq w(P(a, b))$; otherwise, K' would be shorter than K. Moreover, $w(P(a, b)) \leq 0$ because otherwise $w(P(x_0, a)) = w(P(x_0, b)) - w(P(a, b)) < m$. On the other hand, by the conservativeness of (G, w), $w(K(a, b) \cup P(a, b)) \geq 0$, whence we have equality throughout; in particular, $w(K(a, b)) = w(P(a, b)) = 0$ and $w(P(x_0, a)) = m$. $|P(x_0, a)| < |P(x_0, b)|$, contradicting the choice of b. Claim 2 is proved.

Theorem 1.1 now follows by induction in a straightforward way:

- If b can be chosen to be different from x_0, then the level sets of G^* are the same as those of G by Claim 2, except that $\{b\}$ is no longer a level set and its neighbors are identified.
- If $b = x_0$ is the only possible choice for b, then all distances are positive from x_0. In particular, there is no negative edge adjacent to x_0, and in contracting $\delta(x_0)$, all distances from x_0 decrease exactly by one; again, the level sets of G^* are the same as those of G, except that $\{x_0\}$ is no longer more a level set and its neighbors are identified.

In both cases, the components of the level graphs of G that are different from $\{b\}$ are exactly the components of the level graphs of G^*. We apply Claim 1 to $\{b\}$ and apply the induction hypothesis to G^*, and the theorem follows. ∎

In the remaining part of this section, we explore some applications of our results. For the moment, we restrict ourselves to the ± 1-weighted special case.

If $T \subseteq V(G)$, then $F \subseteq E(G)$ is called a *T-join* if T is the set of odd-degree vertices of F. $X \subseteq V(G)$ is said to be T-odd if $|X \cap T|$ is odd. $C \subseteq E(G)$ is a T-cut if $C = \delta(X)$ for some $X \subseteq V(G)$ and X is T-odd. It is an easy exercise to show that a T-join F and a T-cut C have an odd and, in particular, nonempty intersection, and it follows that the minimum cardinality of a T-join is at least as much as the maximum cardinality of a family of pairwise-disjoint T-cuts.

To give a first, typical example of how Theorem 1 relates T-joins and T-cuts, let us show how Theorem 1.1 implies Seymour's [1981] following well-known theorem.

(1.3) *If G is bipartite, then the minimum cardinality of a T-join is equal to the maximum cardinality of a family of pairwise-disjoint T-cuts.*

It is an easy exercise to show that a T-join F has minimum cardinality if and only if, upon defining the weight of the edges in F to be -1 and the weight of the other edges to be 1, we get a conservative weighting (a remark of Guan [1962]). Apply Theorem 1.1 to this conservative weighting and to an arbitrary $x_0 \in V(G)$. If D is a component of a level graph for which $x_0 \notin D$, then $|\delta(D) \cap F| = 1$; it follows that $\delta(D)$ is a T-cut. According to (1.2) applied to the bipartite graph G, every edge leaves some level graph, whence by Theorem 1.1, the number of T-cuts $\delta(D)$, $x_0 \notin D$, is $|F|$, and these are pairwise disjoint. We have thus found a family of pairwise-disjoint T-cuts which has the same cardinality as the minimum T-join, and Seymour's theorem is proved. In fact, the constructed set of disjoint T-cuts has the particular form of the packings presented by the following theorem of Frank, Sebő, and Tardos [1984].

(1.4) *Given a bipartite graph with a* ± 1 *conservative weighting, both classes of the bipartition can be partitioned into classes* $X_1, \ldots, X_k$ *so that* $\delta(C)$*, where* C *is a component of* $G - X_i$ $(i = 1, \ldots, k)$*, contains at most one negative edge.*

Indeed, according to Theorem 1.1, the vertices x, $\lambda_w(x_0, x) = i$, in the components of G^i with i odd, $i = \pm 1, \pm 3, \ldots$ (or of G^i with i even), constitute the classes of a partition which has the claimed property.

(1.3) and (1.4) easily imply half-integer minimax theorems valid for arbitrary graphs. (The half-integer version of (1.3) is a result of Lovász [1975].)

The generalization of theorems on T-joins or conservative weightings to the weighted case is straighforward via subdivision of edges. For instance, given a weight function $w : E(G) \to \mathbb{N}$, the minimum weight of a T-join is at least as much as the maximum cardinality of a w-packing of T-cuts, where a *w-packing* is a multiset which covers edge $e \in E(G)$ at most $w(e)$ times; Seymour's theorem ((1.3)) states that there is equality here for bipartite weightings. (For the sharpening series of integer minimax theorems that have been developed (including weighted variants) and the blocking pair of T-join and T-cut polyhedra, see Lovász and Plummer [1986b]; for a survey of more recent results, see, for instance, Frank [1990] or other recent publications on T-joins mentioned in the reference list. We will state some of these in section 3.) The weighting $w: E(G) \to \mathbb{Z}$ is called *Eulerian* if $w(C)$ is even for every cut C, and it is *bipartite* if it is even for every circuit C. A graph is Eulerian or bipartite if the identically 1 function on its edges is Eulerian or bipartite.

Let us also note that the distances *do not depend on the choice of the minimum T-join F*. (For an easy exercise, see, for example, Sebő [1990].)

The weighted generalization of (1.4) is somewhat artificial. On the other hand, Theorem 1.1 itself can be straightforwardly generalized to the weighted case, and this can be used for various purposes, as we will show in the following sections.

We finish this introduction by showing how Theorem 1.1 already applies to unweighted multiflows, that is, edge-disjoint path problems. We assume that the reader is familiar with the (easy) relation between some notions in planar graphs and those in the planar dual: the dual of the dual of a graph G is equal to G; Eulerian and bipartite graphs or weight functions correspond to each other; disjoint unions of circuits correspond to sets of the form $\delta(X)$ $(X \subseteq V(G))$; etc.

If G is planar, in the dual graph, Theorem 1.1 has the following more apparent meaning.

We are given an Eulerian graph (dual of "bipartite") G embedded in the plane and $R \subseteq E(G)$ so that the *cut condition*

$$|C \cap R| \leq |C \setminus R| \quad \text{for every cut } C$$

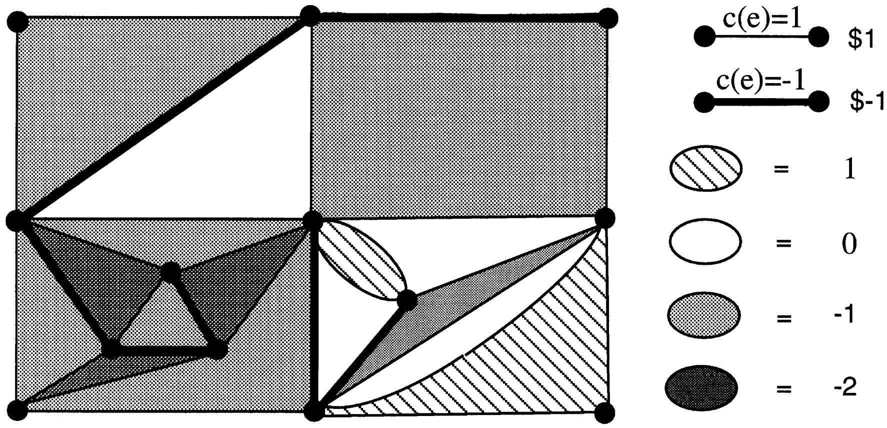

FIG. 2.

is satisfied for (G, R). (This condition is equivalent to the conservativeness of the dual graph. Set weights -1 on (duals of) edges in R.) Assign to every face φ of G the following *magic numbers*:

$$\lambda(\varphi) := \min\{|P \setminus R| - |P \cap R| : P \text{ dual path from the infinite face to } \varphi\}.$$

A "dual" path is a path of the dual graph and can be imagined as going from face to face, crossing edges. $\lambda(\varphi)$ can be interpreted as the minimum cost of reaching face φ if we must pay 1\$ for crossing an edge in $|E(G) \setminus R|$ and -1\$ for crossing an edge of R. (Negative costs correspond to incomes.)

This definition of $\lambda(\varphi)$ corresponds to a natural choice of x_0 in Theorem 1.1. Let x_0 be the vertex corresponding to the infinite face of the dual graph. Let us apply Theorem 1.1 to the dual of G with this choice of x_0.

The union of any set of faces of a planar graph can be partitioned into (topologically) connected regions, where each region is the union of faces corresponding to the components of the dual graph. Consider the regions determined by the faces φ with $\lambda(\varphi) \leq i$ (Figure 2). Among these, those whose territory is bounded (equivalently, which do not contain the infinite face) will be called *patches* (see Figure 2).

THEOREM 1.2. *Let G be an Eulerian graph embedded in the plane, and let $R \subseteq E(G)$ be such that (G, R) satis es the cut condition. Then the boundaries of patches contain exactly one edge of R each, they are pairwise disjoint, and every $e \in R$ is contained in one of them.*

Proof. Apply Theorem 1.1 to the dual of G, where the edge weights are -1 for the duals of edges in R and 1 otherwise; note that the dual of G is bipartite; define x_0 to be the vertex of the dual graph corresponding to the infinite face of G. Clearly, there is a one-to-one correspondence between the patches of G and those components of the level graphs of the dual graph which do not contain x_0. Because of (1.2), the boundaries of these patches are disjoint; the rest of the statement can be extrapolated from Theorem 1.1. ☐

In other words, if G is a planar Eulerian graph and $R \subseteq E(G)$ is such that (G, R) satisfies the cut condition, then Theorem 1.2 provides a uniquely determined integer "multiflow" (see the definition below), which will be called a *patch flow*. Similarly, the packing of odd cuts defined after Theorem 1.1 will be called a *patch packing* for (G, T, x_0).

The reader may find it amusing to translate the proof of Theorem 1.1 to give a direct proof of Theorem 1.2 and, in fact, of the following sharpening.

The necessity of the cut condition for the existence of paths in $G - R$ between the endpoints of the edges in R is trivial. The sufficiency is just the "planar dual" of (1.3). That is, we have shown a constructive proof (see the corresponding polynomial algorithm in section 3) of the following theorem of Seymour [1981].

(1.5) *Let G be a planar Eulerian graph and $R \subseteq E(G)$. Then there exist edge-disjoint paths in $G - R$ between the endpoints of the edges of R if and only if the cut condition is satisfied for (G, R).*

We conclude this section by defining multiflows precisely and listing a sequence of known results about them. Let G be a graph and $c: E(G) \to \mathbb{R}$. c will mean capacity on the positive edges and demand on the negative edges. We define the *demand* of $e \in E(G)$, $c(e) < 0$, to be $-c(e)$. If $c(e) < 0$, e is called a demand edge.

Given a weight function in a graph, let us denote the set of negative edges by E^- and the set of nonnegative edges by E^+. A *multiflow* for (G, c) is a set of circuits $\mathcal{C}$ with "multiplicities" $f: \mathcal{C} \to \mathbb{R}$ such that

$$|C \cap E^-| = 1 \quad \text{for all } C \in \mathcal{C},$$

$$\sum_{e \in C} f(C) \leq c(e) \quad \text{if } e \in E^+, \quad \text{and}$$

$$\sum_{e \in C} f(C) = -c(e) \quad \text{if } e \in E^-.$$

If $f(e)$ is (half-) integer for every $e \in E(G)$, we say that the flow is (*half-*) *integer*. If in this definition we replace "circuit" by "cut," we say that $(\mathcal{C}, f)$ is a *dual multiflow*. The planar special case of dual multiflows is the panar multiflow problem, but dual multiflows have the advantage that the theorems concerning them are valid for arbitrary graphs.

The (*integer*) *multiflow problem* is the problem whose instances are (G, c) pairs, where G is a graph and $c: E(G) \to \mathbb{R}$, and the question is to decide the existence of an (integer) multiflow. In the *planar multiflow problem*, we consider only instances where G is planar.

Statements about capacitated multiflow problems can be reduced to uncapacitated ones by replacing e by $|c(e)|$ parallel copies of e. For instance, the cut condition becomes

$$c(\delta(X)) \geq 0 \quad \text{for every } X \subseteq V(G).$$

The results of Edmonds and Johnson [1973], Lovász [1975], Barahona [1980], and Korach [1982] proved that in a planar graph, a half-integer flow or a violating cut can be found in polynomial time. The best complexity was attained by Barahona [1989]. The closest predecessor to our approach is in the work of Matsumoto, Nishizeki, and Saito [1986]. They decreased c along a face for all possible choices of faces and checked the cut condition for each choice. The proof of Theorem 1.1 indicates explicitly that the face to be chosen is φ with $\lambda(\varphi)$ minimum; moreover, Theorem 1.1 foresees the entire multiflow, the same one which would be the result of alternatively determining φ and deleting its boundary.

Various results about integral (dual) multiflows have been obtained by Seymour [1977, 1981], Korach and Penn [1992], Frank [1990], Sebő [1987a, b], and, more recently, Frank and Szigeti [1995] and Ageev, Kostochka, and Szigeti [1995]. These integer multiflows can also be obtained with the help of the "magic numbers"—some of them with considerable additional work, but also with the algorithmic advantages that this represents (see section 3).

The planar multiflow problem in general was proved by Middendorf and Pfeiffer [1989] to be NP-complete.

In section 2, we will extrapolate the weighted generalization of the results in this section, and in section 3, we will apply the results that we obtain.

2. Potentials. Potentials in directed graphs are defined in the following way:

Let G be a directed graph, $w: E(G) \to \mathbb{R}$, and $x_0 \in V(G)$. (G, w) is said to be *conservative* if $w(C) \geq 0$ for every directed circuit C. $\pi : V(G) \to \mathbb{R}$ is called a *potential* (centered at x_0) if

$$\pi(x_0) = 0 \quad \text{and}$$

$$\pi(y) - \pi(x) \leq w(x, y) \quad \text{for every directed edge } xy.$$

The role of potentials is apparent from, for instance, the following well-known proposition.

(2.1)

(a) (G, w) *is conservative if and only if there exists a potential.*

(b) *If* π *is a potential centered at* x_0, *then for all* $x \in V(G)$,

$$\lambda_w(x_0, x) \geq \pi(x).$$

(c) *If* (G, w) *is conservative, the function de ned by* $\pi(x) := \lambda_w(x_0, x)$ *is a potential centered at* x_0.

(b) and the *if* part of (a) are trivial; (c) is also easy and proves the *only if* part of (a). In other words, potentials centered at x_0 give an apparent proof of conservativeness and a lower bound for the distances from x_0. (c) claims the a priori surprising fact that $\max\{\pi(x) : \pi \text{ is a potential centered at } x_0\}$ is attained by one and the same potential for every $x \in V(G)$, that is, by the distance function λ, $\lambda(x) := \lambda_w(x_0, x)$ $(x \in V(G))$.

Now let G be undirected. Suppose first that G is bipartite, $w: E(G) \to \{-1, 1\}$, and $x_0 \in V(G)$. $\pi: V(G) \to \mathbb{Z}$ will be called a *potential* centered at x_0 if

(i) $\pi(x_0) = 0$ and

(ii) $|\pi(y) - \pi(x)| = 1$ for every $xy \in E(G)$.

If D is a component of the graph $G^i = G^i(\pi)$ induced by the level set $V^i = V^i(\pi)$ $:= \{x \in V(G) : \pi(x) \leq i\}$ $(i = 0, \pm 1, \pm 2, \ldots)$, then

(iii)

$$|\delta(D) \cap E^-| = 1 \quad \text{provided that } x_0 \notin D,$$

$$|\delta(D) \cap E^-| = 0 \quad \text{provided that } x_0 \in D.$$

Theorem 1.1 can now be restated in the following way.

(2.2) *If* G *is* ± 1*-weighted and bipartite, then* (2.1) *holds.*

The *if* part of (2.1a) and all of (2.1b) can be proved in a straightforward way. (For details, see Sebő [1990].) To prove (2.1c), which also implies the *only if* part of (2.1a), we have to check (i), (ii), and (iii): (i) is trivial, (ii) is easy (we have already proved it; see (1.2)), and (iii) is Theorem 1.1.

We have arrived at the main purpose of this section: we will generalize potentials for arbitrary undirected graphs with arbitrary weights. Of course, we have to satisfy two constraints: it should be easy to check whether a given function is a potential (like the inequality for directed graphs or (i), (ii), and (iii) for bipartite ±1-weighted graphs); (2.1) should be true.

For the sake of simplicity, we first do this work only for integer weights, and then we will observe that the theorems hold for arbitrary real weights almost without change and that the extension is straightforward.

If w is not bipartite, we associate with the pair (G, w) a bipartite graph $\hat{G}$ and a weight function $\hat{w}: E(\hat{G}) \to \{-1, 1\}$ in the following way:

Contract the zero-weight edges of G and replace each edge $e \in E(G), w(e) \neq 0$, by $2|w(e)|$ edges in series. (Divide e into $2|w(e)|$ edges by $2|w(e)| - 1$ new points.) We think of $V(G)$ as a subset of $V(\hat{G})$. If $\hat{e} \in E(\hat{G})$ is an element of the subdivision of $e \in E(G)$, let $\hat{w}(\hat{e}) = -1$ if $w(e) < 0$ and let $\hat{w}(\hat{e}) = +1$ if $w(e) > 0$.

Clearly, the natural correspondence between paths of G and paths of $\hat{G}$ simply doubles the weights. Thus

$$\forall x, y \in V(G), \quad \lambda_{\hat{G},\hat{w}}(x, y) = 2\lambda_{G,w}(x, y).$$

This shows that we shall have an easy task. We know that the potentials in $(\hat{G}, \hat{w})$ are the functions for which (i), (ii), and (iii) hold. On the other hand (e.g., for applications), we need theorems that consider only (G, w) directly; potentials should be defined in terms of (G, w). We will face no obstacles in obtaining this direct definition because it turns out that a potential $\hat{\pi} : V(\hat{G}) \to \mathbb{Z}$ is *already determined by its restriction to* $V(G)$.

LEMMA 2.1. *Suppose* $\hat{\pi} : V(\hat{G}) \to \mathbb{Z}$ *is a potential in* $(\hat{G}, \hat{w})$*. Then* $\hat{\pi}$ *is even on* $V(G)$*, and if* $\pi : V(G) \to \mathbb{Z}$ *denotes the restriction of* $\hat{\pi}/2$ *to* $V(G)$*, then for every* $ab \in E(G)$*,*

(a) $|\pi(a) - \pi(b)| \leq |w(ab)|$.

Denote the path of $\hat{G}$*, replacing* $ab \in E(G)$ *by* $Q = \{p_0,\ p_1, \ldots, p_{t-1},\ p_t\}$ $(p_0 = a,\ p_t = b,\ t = 2|w(ab)|)$.

(b) π *uniquely determines* $\hat{\pi}$*, namely, if* $w(ab) \neq 0$*, then in* $(\hat{G}, \hat{w})$*,*

$$\hat{\pi}(p_i) := \begin{cases} 2\pi(a) + i & \text{provided that } 0 \leq i \leq i_0, \\ 2\pi(b) + |2w(ab)| - i & \text{provided that } i_0 \leq i \leq t, \end{cases}$$

where $i_0 = \pi(b) - \pi(a) + |w(ab)|$.

(c) $\max_{0 \leq i \leq t}\ \hat{\pi}(p_i) = \hat{\pi}(p_{i_0}) = \pi(a) + \pi(b) + |w(ab)|$.

Remark. Lemma 2.1 gives the value of $\hat{\pi}(p_i)\,(0 \leq i \leq t)$; it even gives two definitions for $\hat{\pi}(p_{i_0})$, but both of them define the value $\hat{\pi}(p_{i_0}) = \hat{\pi}(a) + \hat{\pi}(b) + |\hat{w}(ab)|$.

Lemma 2.1 is illustrated in Figure 3.

Proof of Lemma 2.1. First, we prove that $\hat{\pi}$ is even on $V(G)$. Indeed, it follows from (ii) that $\hat{\pi}$ has different parity on the two endpoints of every edge, and hence its parity is fixed on each class of the bipartition of $\hat{G}$. Since one of these classes contains $V(G)$, and since $\hat{\pi}(x_0) = 0$, we get that $\hat{\pi}$ is even on $V(G)$.

We now prove (a). If $w(ab) = 0$, then $\hat{\pi}(a) = \hat{\pi}(b)$, and (a) is trivial.

Suppose that $w(ab) \neq 0$. According to (ii), $|\hat{\pi}(p_i) - \hat{\pi}(p_{i-1})| = 1$, whence

$$2|\pi(a) - \pi(b)| = |\hat{\pi}(a) - \hat{\pi}(b)| = \left|\sum_{i=1}^{t} \hat{\pi}(p_i) - \hat{\pi}(p_{i-1})\right| \leq e \sum_{i=1}^{t} |\hat{\pi}(p_i) - \hat{\pi}(p_{i-1})|$$

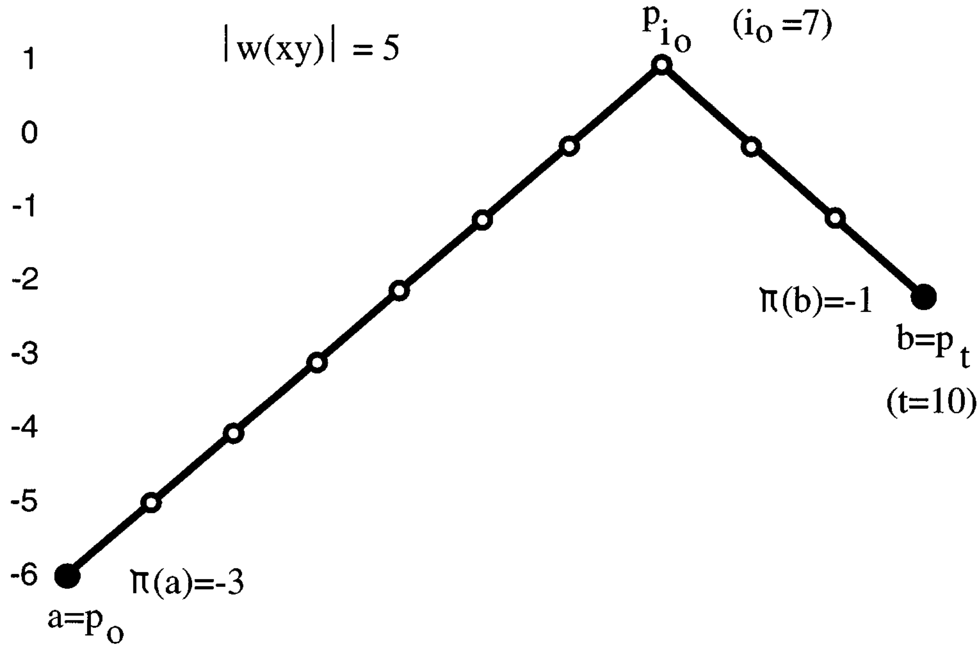

FIG. 3.

$$= t = |2w(ab)|,$$

and (a) is proved.

If $\hat{\pi}(p_i) = \hat{\pi}(p_{i-1}) - 1$, then we call the edge $p_{i-1}p_i$ *descending*, and if $\hat{\pi}(p_i) = \hat{\pi}(p_{i-1}) + 1$, then we call it *ascending*. By (ii), every edge is either descending or ascending. If $p_{i-1}p_i$ is descending, $i < t$, then $p_i p_{i+1}$ cannot be ascending; otherwise, $\{p_i\} \in D$, which contradicts (iii) because $\delta(p_i)$ consists of two positive or two negative edges. From this it follows that there exists an $i_0 \in \mathbb{N}$ such that $p_{i-1}p_i$ is ascending if $1 \le i \le i_0$ and descending if $i_0 < i \le t$. Consequently, with this i_0, $\hat{\pi}(p_i) = \hat{\pi}(p_0) + i$ if $0 \le i \le i_0$ and $\hat{\pi}(p_i) = \hat{\pi}(p_t) + (t - i)$ if $i_0 \le i \le t$. Hence $\hat{\pi}(p_0) + i_0 = \hat{\pi}(p_{i_0}) = \hat{\pi}(p_t) + (t - i_0)$, that is, $i_0 = (\hat{\pi}(b) - \hat{\pi}(a) + 2|w(ab)|)/2$. Thus we have proved (b). (c) is the immediate consequence of (b). □

We call a function $\pi : V(G) \to \mathbb{Z}$ a *potential centered at* $x_0 \in V(G)$ if the function 2π is the restriction to $V(G)$ of a potential in $\hat{G}$ centered at $x_0 \in V(\hat{G})$. Given G, w, and a potential π, we will keep the notation $\hat{G}$, $\hat{w}$, $\hat{\pi}$; these are uniquely determined (see Lemma 2.1(b)). The properties of $\hat{\pi}$ can be "translated" into properties of π:

Let $\pi : V(G) \to \mathbb{Z}$ and define $G_w^i = G_w^i(\pi)$ by $V(G_w^i) := \{x \in V(G) : \pi(x) \le i\}$ and $E(G_w^i) := \{xy \in E(G) : (\pi(x) + \pi(y) + |w(xy)|)/2 \le i\}$ ($i = 0, \pm 1/2, \pm 1, \pm 3/2, \ldots$). We also introduce the notation

$$\mathcal{D} := \{D \subseteq V(G)\text{: } D \text{ is a component of } G_w^i \text{ for some } i\}.$$

LEMMA 2.2. *Let G be an arbitrary undirected graph, $w : E(G) \to \mathbb{Z}$, $x_0 \in V(G)$. $\pi : V(G) \to \mathbb{Z}$ is a potential centered at x_0 if and only if the following hold:*

(î) $\pi(x_0) = 0$.

(îî) $|\pi(y) - \pi(x)| \le |w(x, y)|$ *for all* $xy \in E(G)$.

(îîî) *If D is a component of $G_w^i(\pi)$, then*

- *all negative edges induced by D in G are in $E(G_w^i)$ and*
- (iii) *holds for D.*

Proof. To prove the *only if* part, suppose that $\pi : V(G) \to \mathbb{Z}$ is a potential centered at x_0, that is, 2π is the restriction to $V(G)$ of a potential $\hat{\pi}$ of $(\hat{G}, \hat{w})$ centered at x_0. We must prove that (î), (îî), and (îîî) hold for π provided that (i), (ii), and (iii) hold for $\hat{\pi}$. (î) is obvious and (îî) is just Lemma 2.1(a).

To prove (îîî), let $\hat{\imath} \in \mathbb{Z}$ and note the following: the level graph $\hat{G}^{\hat{\imath}}(\hat{\pi})$ of $\hat{G}$ contains exactly those vertices $v \in V(G)$ for which $\hat{\pi}(v) = 2\pi(v) \leq \hat{\imath}$; by Lemma 2.1(c), it entirely contains the subdivision of exactly those edges $ab \in E(G)$ for which $\pi(a) + \pi(b) + |w(ab)| \leq \hat{\imath}$. That is, the components of $\hat{G}^{\hat{\imath}}$ are determined by those vertices $v \in V(G)$ for which $\pi(v) \leq \hat{\imath}/2$ and those edges $ab \in E(G)$ for which $(\pi(a) + \pi(b) + w(ab))/2 \leq \hat{\imath}/2$ ($\hat{\imath} = 0, \pm 1, \pm 2, \ldots$). Defining $i := \hat{\imath}/2$, we see that these vertices and edges are exactly those of $E(G_w^i(\pi))$. Thus there is a one-to-one correspondence between the components of $\hat{G}^{\hat{\imath}}(\hat{\pi})$ and those of $G_w^i(\pi)$. If D is a component of the latter, we will denote the corresponding component of the former by $\hat{D}$.

(îîî) now follows easily. If a negative edge ab induced by D is not in $E(G_w^i)$, then $(\pi(a) + \pi(b) + |w(ab)|)/2 > i$, and it follows from Lemma 2.1 that there are two negative edges in $\delta(\hat{D})$. (Follow on Figure 3: since $p_{i_0} \notin \hat{D}$ but $a, b \in \hat{D}$, $\delta(\hat{D})$ contains one negative edge from each of the paths (a, p_{i_0}) and (p_{i_0}, b).) Therefore, since by assumption $\hat{\pi}$ satisfies (ii), such an edge ab does not exist, and the first part of (îîî) is proved. Furthermore, it now follows that the negative edges of $\hat{D}$ are in one-to-one correspondence with those entering D, so the second part of (îîî) holds as well.

To prove the *if* part, suppose that (î), (îî), and (îîî) hold for π. Define $\hat{\pi}$ to be equal to 2π on $V(\hat{G}) \cap V(G)$, and extend it to the entire $V(\hat{G})$ in the unique way dictated by Lemma 2.1 (b). By definition, $\hat{\pi}$ satisfies (i) and (ii). If D is as in (îîî), then by the same correspondence as in the proof of the *only if* part, D correponds to a component $\hat{D}$ of $\hat{G}^{\hat{\imath}}(\hat{\pi})$, and because of the first part of (îîî), there is a one-to-one correspondence between the negative edges of $\delta(\hat{D})$ and those of $\delta(D)$. Now the second part of (îîî) implies that (iii) holds for $\hat{D}$. $\square$

The following theorem is a straightforward reformulation of Theorem 1.1 (that is, of (2.2)) for the weighted case.

THEOREM 2.1 *Let G be an arbitrary undirected graph, and let $w : E(G) \to \mathbb{Z}$ such that (G, w) is conservative; furthermore, $x_0 \in V(G)$. Then $\lambda : V(G) \to \mathbb{Z}$, where $\lambda(x) := \lambda_{G,w}(x_0, x)$ $(x \in V(G))$ is a potential centered at x_0. Furthermore,* (2.1) *holds.*

Proof. Apply Theorem 1.1 and then (2.2) to the conservative bipartite $(\hat{G}, \hat{w})$. Then apply Lemma 2.2 to obtain the result for (G, w). $\square$

The reader may find it to be a useful exercise to give a direct proof of the trivial *if* part of (2.1a) and all of (2.1b), where the definition of potentials is given in (î), (îî), and (îîî).

The main point of Theorem 2.1 is that (îîî) holds for the distances from x_0. Since this statement will be often used in what follows, let us restate it separately in a slightly different form for later convenience.

(2.3) *Let G be a undirected graph, $w : E(G) \to \mathbb{Z}$ such that (G, w) is conservative, and $x_0 \in V(G), \lambda(x) := \lambda_{G,w}(x_0, x)$, $G_w^i := G_w^i(\lambda)$.*

(a) *If D is a component of G^i_w,*

$$|\delta(D) \cap E^-| = 1 \quad \textit{provided that } x_0 \notin D,$$

$$|\delta(D) \cap E^-| = 0 \quad \textit{provided that } x_0 \in D.$$

(b) *Let $y(D) := \max\{i : D$ is a component of $G^i\} - \min\{i : D$ is a component of $G^i\}$. Then for all $e \in E^-$, $\sum_{D\in\mathcal{D},e\in\delta(D)} y(D) = |w(e)|$, and for all $e \in E(G)$, $\sum_{D\in\mathcal{D},e\in\delta(D)} y(D) \le |w(e)|$.*

(2.3) shows how the magic numbers $\lambda(x_0, x)$ generate a special dual flow (packing of T-cuts). This will be the basis of the applications in the following section.

In the rest of the paper, G^i will denote the graph defined in (2.3) (given that (G, w) is conservative and $x_0 \in V(G)$). The family $\mathcal{D}$ can be split into the union of $\mathcal{D}'$ and $\mathcal{D}''$, where $\mathcal{D}'$ is the family of all components of G^i with i as an integer and $\mathcal{D}''$ is the family of all components of G^i with i as a noninteger (but, of course, as a half-integer). Each element of $\mathcal{D}''$ is partitioned by some elements of $\mathcal{D}'$ because $V(G^i) = V(G^{i+1/2})$, and $E(G^i) \subseteq E(G^{i+1/2})$. The following remark will be useful in the construction of integer (dual) flows or packings.

(2.4) *Let G be an undirected graph, and let $w : E(G) \to \{-1, 1\}$ so that (G, w) is conservative. Then $\{\delta(D) : D \in \mathcal{D}'', x_0 \notin D\}$ is a set of disjoint cuts each of which contains one negative edge, and $xy \in E(G)$ is in none of these cuts if and only if $\lambda(x) = \lambda(y)$.*

Indeed, let $xy \in E(G)$, and suppose without loss of generality that $\lambda(x) \le \lambda(y)$. Then either $\lambda(x) = \lambda(y)$ or $\lambda(x) = \lambda(y) - 1$. In the latter case, xy is in $\delta(D)$, where D is the component of $G^{\lambda(x)+1/2}$ containing x; xy is in none of the other sets of the form $\delta(X)$, $X \in \mathcal{D}''$. In the former case, $(\lambda(x)+\lambda(y)+w(xy))/2 = \lambda(x)+1/2$, that is, $xy \in E(G^{\lambda(x)+1/2}$. It follows that x and y are in the same component of $G^{\lambda(x)+1/2})$, whence none of the sets $\delta(D)$ $(D \in \mathcal{D}'')$ contains it, and (2.4) is proved.

The dual flow defined by

$$\mathcal{V} := \{\delta(D) : D \in \mathcal{D},\, x_0 \notin D\}$$

with multiplicities $y(D)$ will be called the *patch dual flow* of (G, w, x_0). It is, in fact, a dual multiflow. A patch dual flow becomes a multiflow in the dual of a planar graph (and if we dualize so that x_0 is the infinite face, this is the uniquely determined patch flow); applying it to the conservative graph corresponding to a minimum T-join, we get a *patch packing* of T-cuts (see section 1 for the unweighted case), which is a maximum packing of T-cuts.

Note that the results can be *generalized to arbitrary* $w\colon E(G) \to \mathbb{R}$, where the potentials can be defined by (î), (îî), and (îîî). *The distances still form a potential*; *furthermore,* (2.1) *still holds.*

Indeed, let $\varepsilon \le (1/2n^2)\mu$, where μ is the minimum difference between *different* path lengths. The function $w_\varepsilon(e) := \lceil w(e)/\varepsilon \rceil$ $w_\varepsilon\colon E(G) \to \mathbb{N}$ has the property that for two paths P and Q (not necessarily between the same pair of vertices), $w(P) < w(Q)$ implies $w_\varepsilon(P) < w_\varepsilon(Q)$, whence the w_ε-shortest paths are also w-shortest paths.

Since $w(e)/\varepsilon \le w_\varepsilon \le 1 + w(e)/\varepsilon$, we have $w(e) \le \varepsilon w_\varepsilon \le \varepsilon + w(e)$. Thus, choosing an arbitrary series $\varepsilon_n \to 0$, $\varepsilon_n w_{\varepsilon_n} \to w$ holds (uniformly on the paths). It follows that, applying (2.1) to w_{ε_n}, the same follows for $\varepsilon_n w_{\varepsilon_n}$. Since, as we noticed, the w_{ε_n}-shortest paths are also w-shortest, the distances according to the weight function $\varepsilon_n w_{\varepsilon_n}$ converge to λ_w. Now since in (î), (îî), and (îîî) we only have linear functions of convergent series, these linear functions also converge to the same linear function of the limits, and (2.1) follows for arbitrary real weights in a straightforward way.

Patch flows and patch packings (without using this term) have been applied to prove integer-path or cut-packing theorems. See, for instance, Sebő [1990], Frank and Szigeti [1995], or Ageev, Kostochka, and Szigeti [1995].

3. Algorithms and applications. It is well known that an algorithm for minimum-weight T-joins has been deduced from weighted matching algorithms in various ways (see Edmonds [1965b] and Lawler [1976]). The same sources reduce the shortest-undirected-path problem of conservative graphs to matching problems with similar gadget-type reductions. However, these and other "Waterloo folklore solutions" do not give a satisfactory answer to the dual of the minimum-weight T-join problem (though it is often needed in applications; see below). That is why there were later several attempts at the solution of the primal and dual problem at the same time; see Barahona [1980, 1989], Edmonds and Johnson [1973], and Korach [1982]. Barahona [1989] gave a clear presentation; Barahona and Cunningham [1989] showed an elegant way to provide an integer dual solution in the bipartite case.

Let n stand for $|V(G)|$ and m stand for $|E(G)|$ in the rest of the paper. In this section, we will show that a dual flow or a maximum w-packing of T-cuts can be found by solving n independent matching problems (which can also be carried out in parallel). Furthermore, we always get a half-integer solution, and if for all circuits $w(C)$ is even, then we automatically get an integer solution. Then we collect applications of weighted and unweighted potentials, the complexity of which is determined by our algorithm.

In the following, we shall need a subroutine that determines a minimum-weight T-join. For this we shall use Edmonds and Johnson's [1973] reduction to matchings, which we shall describe now. It is the following obvious fact that makes possible the use of this method.

LEMMA 3.1. *Suppose that* (G,w) *is conservative,* $a,b \in V(G)$, *and* $T := \{x \in V(G) : d_{E^-}(x)$ *is odd*$\}$. *If* F *is a* $|w|$*-minimum* $T \triangle \{a,b\}$*-join, then in* $F \triangle E^-$, *any* (a,b)*-path is* w*-shortest.*

Remark. Since $F \triangle E^-$ is an $\{a,b\}$-join, an (a,b)-path contained in it is trivial to find. Thus Lemma 3.1 reduces the shortest-path problem to finding a minimum T-join for nonnegative weights. Since, moreover, the zero-weight edges can be contracted, in the following, we can concentrate on T-joins in graphs with positive weights.

We also remark that the converse of Lemma 3.1 is also true (but irrelevant here): if P is a w-shortest (a,b)-path, then $P \triangle E^-$ is a $|w|$-minimum $T \triangle \{a,b\}$-join.

Proof of Lemma 3.1. Clearly, $F \triangle E^-$ is an $\{a,b\}$-join, and

$$\begin{aligned} w(F \triangle E^-) &= w(F \setminus E^-) + w(E^- \setminus F) \\ &= |w|(F \setminus E^-) + |w|(F \cap E^-) - |w|(E^- \cap F) + w(E^- \setminus F) \\ &= |w|(F) + w(E^-). \end{aligned}$$

Thus $|w|(F)$ and $w(F \triangle E^-)$ differ only in a constant independent of F. F is a $|w|$-minimum $T \triangle \{a,b\}$-join $\Leftrightarrow$ $F \triangle E^-$ is a w-minimum (a,b)-path $\Leftrightarrow$ in $F \triangle E^-$, an arbitrary circuit has weight 0. $\square$

Now let $w : E(G) \to \mathbb{N}$, and on the edges of the complete graph on H with $V(H) := T$, define the weighting $c(x,y) := \min\{w(P) : P \subseteq E(G),\ P \text{ is an } (x,y) \text{ path}\}$

> 0. ($c(x, y)$ can, for instance, be computed by Dijkstra's algorithm; see, for example, Lawler [1976].) The following lemma of Edmonds and Johnson [1973] relates the w-minimum T-joins of G to the c-minimum matchings of H.

LEMMA 3.2. *Let G be a connected graph, $T \subseteq V(G)$, where $|T|$ is even, and $w : E(G) \to \mathbb{N}$, and from these let us define H and c in the above way. Furthermore, let $k := |V(H)|/2$. If $\{x_i y_i : i = 1, \ldots, k\}$ is a c-minimum perfect matching in H and P_i is a shortest (x_i, y_i)-path $(i = 1, \ldots, k)$, then $P_i \cap P_j = \emptyset$ $(i \neq j)$ and $\bigcup_{i=1}^{k} P_i$ is a w-minimum T-join in G.*

Remark. Applying both Lemmas 3.1 and 3.2, the shortest paths of conservative graphs can be determined using any matching algorithm (see Algorithm 1 below).

Proof of Lemma 3.2. Let $M := \{x_i y_i : i = 1, \ldots, |V(H)|/2\}$ be a c-minimum perfect matching and P_i be a w-minimum (x_i, y_i) path $(i = 1, \ldots, k)$. Thus $P_i \cap P_j = \emptyset$ since if $P_i \cap P_j \neq \emptyset$, then $w(P_i \triangle P_j) < w(P_i) + w(P_j)$, and $P_i \triangle P_j$ contains two (edge-) disjoint paths between two disjoint pairs of points in $\{x_1, y_1, x_2, y_2\}$, which contradicts the minimality of M. □

The converse of this lemma is also easy. Given nonnegative weights, a minimum-weight T-join (which is a forest) is easy to split into edge-disjoint paths between pairs in $V(H)$, and no matter how we carry this out, the pairs will create a c-minimum matching of H.

We now have the means at our disposal to describe the algorithm based on Theorem 2.1 (section 2).

ALGORITHM 1.

Input: graph G, $x_0 \in V(G)$, and $w : E(G) \to \mathbb{Z}$.

Output: either a negative circuit in (G, w) or a feasible w-packing which is furthermore the uniquely existing patch dual flow belonging to (G, w, x_0).

0. Contract the zero-weight edges and with the help of Lemma 3.2 above, determine a $|w|$-minimum T-join F, $T := \{x \in V(G) \ : \ d_{E^-}(x) \text{ is odd}\}$.

• If $|w|(F) < |w|(E^-)$, then a negative circuit can easily be found in $F \triangle E_-$. STOP.

• If $|w|(F) = |w|(E_-)$, then GOTO 1.

1. With the help of Lemmas 3.1 and 3.2, determine the weight of a w-shortest (x_0, x) path for every $x \in V(G), x \neq x_0$. Let this number be denoted by $\lambda(x)$. GOTO 2.

2. Let the function $\lambda : V(G) \cup E(G) \to \mathbb{Z}$ be the following:

$$\lambda(x) := \begin{cases} \lambda(x) & \text{if } x \in V(G), \\ \dfrac{\lambda(u) + \lambda(v) + w(uv)}{2} & \text{if } x = uv \in E(G). \end{cases}$$

• For the value of $\lambda(x)$ $(x \in V(G))$ and $\lambda(xy)$ $(xy \in E(G))$, define the components of the graph G^i, that is, the set system $\mathcal{D}$. (In the case of bipartite weightings, λ takes only integer values because $\lambda(x)$ and $\lambda(y)$ have the same parity if $w(xy)$ is even and have different parities if it is odd.) The multiplicities $y(D)$ $(D \in \mathcal{D})$ are easily seen to be computable in the following way:

$$y(D) \leftarrow \max_{e \in E^- \cap \delta(D)} \lambda(e) - \min_{e \in E^- \cap E(D)} \lambda(e).$$

(See (2.3b); for a proof, use (îîî).)

$$\mathcal{V} \leftarrow \{\delta(D) : D \in \mathcal{D}, x_0 \notin D\},$$

where we mean the multiplicity of $C \in \mathcal{V}, C = \delta(D)$ $(D \in \mathcal{D})$, to be $y(C) := y(D)$.

Comments.

• The first step of Algorithm 1 associates the execution of a matching algorithm with each point $x \neq x_0$ (see Lemma 3.1) of step 0. The matching subroutine carried out in step 0 can be associated with x_0, so a matching algorithm has been associated with each $x \in V(G)$.

In fact, there is no real assymmetry between steps 0 and 1, in other words, between x_0 and the other vertices of G. The assymetry disappears as soon as we consider a somewhat more general object than graphs (see "towers" in Sebő [1990]).

• For arbitrary real weights, Algorithm 1 works without any change; the only difference is that the function λ defined in step 2 will not be an integer function. We prefer to assume that the weights are integer and often that they are bipartite because then λ is also an integer function and the integrality results that we obtain are included in a natural way.

THEOREM 3.1.

(a) *$\mathcal{V}$ with multiplicities $y(V)$ $(V \in \mathcal{V})$ is a dual flow.*

(b) *$\mathcal{V}$ and $y(V)$ $(V \in \mathcal{V})$ can be determined by rst computing the distances in $(G, |w|)$, then using n parallel running and noncommunicating matching subroutines, and then executing at most $n + m$ subroutines that nd the connected components of a graph. (These can also be executed in parallel and without communication.) The inputs of the subroutines are graphs on at most n points, and in the input of the matching algorithm, every weight is the sum of the weights $w(e)$ of at most n edges $e \in E(G)$.*

We will assume throughout the paper that the complexity of computing the distances (in parallel or nonparallel) in a graph with nonnegative weights does not exceed that of the (parallel or nonparallel) matching algorithms, whence it can be neglected.

Proof of Theorem 3.1. (a) can immediately be seen from (2.3) (that is, Theorem 2.1), and it is an immediate consequence of Lemma 3.1 and 3.2 that Algorithm 1 satisfies the properties described in (b). □

COROLLARY 1. *Suppose that we have a weighted matching algorithm whose running time is $t(n)$ for an input of n points, and suppose that it uses $p(n)$ processors. Then either a dual multiflow or a negative circuit can also be determined in $t(n)$ time and with $np(n)$ processors.*

According to the results of Mulmuley, Vazirani, and Vazirani [1986], a maximum matching can be determined in $O(\log^2 n)$ time with a random parallel algorithm. The same article solves weighted matching problems for particular weights, but we do not know of any general efficient parallel algorithm that solves this problem. However, Mulmuley, Vazirani, and Vazirani note that the general weighted problem is also in RNC^2 if the encoding of the weights is unary. Wein [1991] developed a Las Vegas RNC algorithm for minimum-weight perfect matchings which is logarithmic polynomial if the encoding of the weights is unary. Through Theorem 2.1, we get the same results for dual multiflows and all of their applications. (Wein's [1991] result itself uses Theorem 2.1.)

In Algorithm 1, Theorem 3.1, and Corollary 1, instead of "dual flow," we could of course have written "maximum T-cut packing" for a (G, T) and a weighting $w : E(G) \to \mathbb{Z}_+$. Furthermore, Algorithm 1 always defines *the uniquely existing patch packing.* Thus for every application of patch packings, we have the following corollary:

In planar networks, with the "dualization" described in section 1 and within the time limits in Corollary 1, Algorithm 1 either finds a cut that violates the cut condition or finds a feasible flow. The "dual distances" (see section 1) taken from the infinite face can be determined by Algorithm 1, and in this way our result will be the uniquely existing patch flow. The same holds for the Ising model (see Barahona [1980]). Thus for these problems as well, a structural decomposition is implied. In the case of an Eulerian graph, this will automatically be an integer flow.

Of course, to get the best complexity results, one should substitute into Corollary 1 the best possible complexity results that exploit planarity. Finding a logarithmic polynomial parallel matching algorithm is still an open problem. The best known complexity for matchings in planar graphs is by Lipton and Tarjan [1980]: $O(n^{3/2}\log n)$. It follows that the sequential complexity of finding a patch flow in planar graphs—and of all the planar problems mentioned above—is $O(n^{5/2}\log n)$.

Nishizeki, Matsumoto, and Saito [1986] also solved planar flow problems within this time limit via n matching algorithms; however, these must be successive. In their algorithm, the input of every matching subroutine depends on the output of all subroutines carried out previously, and the output of the whole algorithm depends on the choices made during the running of the algorithm.

Note that the best sequential complexity for the planar Chinese postman problem has thus far been reached by Barahona [1989] with a direct algorithm that has the same complexity as the best planar matching algorithm at present, $O(n^{3/2}\log n)$.

We now mention some applications that require more than a simple use of Theorems 2.1 and 3.1.

3.1. Integer packings in graphs without odd K_4.

(3.1) (Seymour [1977]) *Let G be a graph and suppose that $T \subseteq V(G)$, where $|T|$ is even. If $V(G)$ cannot be partitioned into four T-odd parts so that each induces a connected graph, and if there exists an edge between any two of the four parts, then the maximum cardinality of a family of disjoint T-cuts is equal to the minimum cardinality of a T-join.*

Since the constraint of (3.1) remains true after the usual subdivision (or contraction) of edges, the weighted generalization can be straightforwardly deduced. The condition of (3.1) is not only sufficient but also necessary to have a maximum w-packing of T-cuts that is integer for an arbitrary nonnegative integer weight function w.

(3.1) is an important special case of Seymour's general characterization of binary clutters with the strong max-flow min-cut property. It is closely related to the following result. (Again, for the sake of simplicity, we state only the cardinality case, which implies the general case through the usual subdivision of edges.)

If $\mathcal{P}$ is a partition of $V(G)$ into T-odd parts each of which induces a connected graph, then the set of edges whose endpoints are in different parts of $\mathcal{P}$ is called a *T-border*. The *value* of this T-border is $|\mathcal{P}|/2$. A T-border B is called *bicritical* if, upon contracting all the edges of $E(G) \setminus B$ (that is, shrinking all the classes of $\mathcal{P}$), we get a bicritical graph, that is, a graph that has a perfect matching, and according to the weight function that is -1 on a perfect matching and 1 everywhere else, the distance between any two points is -1. (It is an easy exercise to show that this definition does not depend on the chosen matching.)

(3.2) (Sebő [1988]) *Let G be a graph and $T \subseteq V(G)$, where $|T|$ is even. The minimum cardinality of a T-join of G is equal to the maximum sum of the values of a set of edge-disjoint bicritical T-borders.*

Again, (3.2) implies its own weighted generalization in the usual way. (3.2) implies (3.1) using the following.

(3.3) (Sebő [1988]) *The vertices of a bicritical graph can be partitioned into four classes of odd cardinality so that, upon contracting all edges with both endpoints in the same class, we get* K_4.

The original proof of (3.3) reduces the statement to Seymour's theorem and points out that an elementary proof would generate a simple proof of Seymour's theorem via (3.2). A simple proof of (3.3) was given by Lovász through the ear decomposition of nonbipartite matching-covered graphs and by Gerards [1987] in an elementary way. For an elegant, elementary proof of (3.3), see Frank and Szigeti [1994].

The original proof (see Sebő [1987a]) of (3.2) used Theorem 1.1, and for the sake of the algorithmic consequences, this is what we must follow here. (In Sebő [1988], the same proof was described in a self-contained way by substituting the proof of Theorem 1.1 instead of using it. Frank and Szigeti [1994] replaced Theorem 1.1 by using (1.4), which combined with their proof of (3.3) is the shortest variant.)

Proof of (3.2). It is trivial that the minimum is greater than or equal to the maximum. To prove equality, let F be a minimum T-join and define $w(e) := -1$ if $e \in F$ and $w(e) := 1$ if $e \in E(G) \setminus F$. By Guan's remark (see section 1), (G, w) is conservative. We can suppose without loss of generality that G is connected. Let $x_0 \in V(G)$ be arbitrary. Define $\mathcal{D}$, $\mathcal{D}'$, and $\mathcal{D}''$ as we did immediately after (2.3), and apply (2.3).

As we noticed after (2.3), each $D \in \mathcal{D}''$ is partitioned by some elements $D_1, \ldots, D_k$ of $\mathcal{D}'$. If $x_0 \notin D$, then $x_0 \notin D_1 \in \mathcal{D}, \ldots, x_0 \notin D_k \in \mathcal{D}$, and by (2.3a), $\delta(D_i)$ contains exactly one negative edge for $i = 1, \ldots, k$, and since $\delta(V(G) \setminus D) = \delta(D)$, so does $\delta(V(G) \setminus D)$. $G - D$ may be disconnected, but for one of its components—denote it by D_0—$\delta(D_0)$ contains the unique negative edge of $\delta(D)$. Add each component of $G - D$ except D_0 to one of the D_i's ($i = 1, \ldots, k$) adjacent to it. (Since G is connected, there exists at least one such D_i for every component of $G - D$.) In this way, we get a partition $\mathcal{P}(D) := \{D_0, D'_1, \ldots, D'_k\}$ whose classes are matched by $(k+1)/2$ edges of F, and there is no other edge of F in the set $B(D)$ of edges joining different classes of $\mathcal{P}(D)$. Thus B(D) is a T-border.

Moreover, $\mathcal{B} := \{B(D) : D \in \mathcal{D}'', x_0 \notin D\}$ is a set of disjoint T-borders whose sum of values is equal to $|F|$. This proves (3.2) without proving that these T-borders are bicritical. If some of the T-borders $B \in \mathcal{B}$ are not bicritical, we will *nd a larger number of disjoint T-borders with the same value.* This will complete the proof of (3.2) because the number of disjoint T-borders is bounded by the number of edges, so it has a maximum, and then all T-borders are bicritical.

Let B^* be the graph that we obtain by contracting $E(G) \setminus B$ (that is, by shrinking the classes of the underlying partition). $F \cap B$ is a perfect matching of B^*, and the weighting w becomes w^*: -1 on F and 1 elsewhere. (B^*, w^*) is conservative. Suppose that B^* is not bicritical; let $x_0^* \neq x^* \in V(B^*)$ and $\lambda_{B^*,w^*}(x_0^*, x^*) \geq 0$.

Apply to (B^*, w^*) the argument that we used for (G, w) to find a set of disjoint $V(B^*)$-borders of maximum sum of values. We get a set $\mathcal{B}^*$ of disjoint T-borders of total value $|B \cap F|$, the same as the value of B. The T-border in $\mathcal{B}^*$ containing the negative edge e adjacent to x_0^* contains only edges adjacent to vertices at distance -1, whence it does not contain the negative edge adjacent to x^*. Thus $|\mathcal{B}^*| \geq 2$, as claimed. ∎

The algorithm that follows from this proof for finding the integer packing of (3.2) is the following.

ALGORITHM 2.

Input: a graph G and $w : E(G) \to \mathbb{N}$.

Output: a w-packing of T-borders (T-cuts if the condition of (3.1) is satisfied).

1. Find a patch dual flow (see Algorithm 1).

2. Find a w-packing of T-borders using the guidelines of the first part of the proof of (3.2).

Comment. Of course, the proof must be applied to the graph where each $e \in E(G)$ is subdivided into a path of $w(e)$ edges, and if the algorithm really does the subdivision, the polynomial bound is lost. However, from the multiplicities of the cuts in the dual flow provided by Algorithm 1, it is straightforward to compute the multiplicities of the T-borders in the packing without actually doing the subdivision.

3. Decompose each T-border B of the constructed packing into disjoint bicritical T-borders using the guidelines of the second part of the proof of (3.2), and replace B with the elements of the decomposition, assigning its multiplicity to each. Continue this procedure with each of the newly constructed T-borders until there is no more proper decomposition, that is, until the distance between any two vertices in all of the T-borders is -1 (according to the weight function defined in the proof of (3.2)), i.e., until they are all bicritical.

Comment. This step is the same in the weighted and unweighted cases. The decomposition of a T-border into bicritical T-borders is an *unweighted* problem.

THEOREM 3.2. *The integer packing of odd cuts in* (3.2) *and* (3.1) *can be found by solving* $O(mn^2)$ *matching algorithms on minors of* G (*graphs that arise with the contraction and deletion of some edges from* G) *and performing some additional steps whose order of magnitude is smaller.*

Proof. To execute steps 1 and 2, the number of matching algorithms to be solved is n. After the execution of step 3, the number of bicritical T-borders that decompose a bicritical T-border is at most the number of edges of the latter. Thus m is an upper bound for the number of times Algorithm 1 must be applied to the subsequent T-borders. On each of these T-borders, we have to execute n^2 matching algorithms to prove that they are bicritical or decompose them. □

It follows that the maximum integral dual solution to the minimization problem on a T-join polyhedron described with a minimal totally dual integral (TDI) description (see Sebő [1988]) can also be computed within the same time limit.

However, note that the bounds in Theorem 3.2 are weaker than those in Corollary 1. We do not see how our parallel complexity estimates could be saved for this case.

3.2. Integer flows—almost. Let (G, w) be ± 1-weighted and conservative. Of course, E^- forms a forest. Korach and Penn [1992] proved that there exist pairwise-disjoint cuts, each of which contains one negative edge, so that every negative edge is in some of the cuts except perhaps at most one edge of each component of E^-; in fact, for one component of E^-, one can require every edge to be in some of the cuts. Equivalently, for arbitrary weights, Korach and Penn proved the following theorem.

(3.4) *If* (G, w) *is conservative, then there exists* $F \subseteq E^-$ *with* $|F \cap C| \leq 1$ *for every connected component* C *of* E^- *such that for the weight function* w', *there exists an integer dual flow, where* $w'(e) := w(e)$ *if* $e \in E \setminus F$ *and* $w'(e) := w(e) - 1$ *if* $e \in F$.

We give the proof of Sebő [1990], which provides an algorithm for finding this dual flow and the multiflow of Corollary 2 below via Theorem 2.1, with the time limits of Theorem 3.1.

Proof of (3.4). The usual subdivision of edges leads to a ± 1-weighted conservative graph, so suppose that (G, w) is already one. Let $x_0 \in V(G)$ be arbitrary and $\lambda(x) := \lambda_w(x_0, x)$ $(x \in V(G))$. Denote the components of E^- by $E_0, \ldots, E_k$.

Claim 3. *For every $x \in V(E_i)$, there exists at most one $y \in V(E_i)$ with $xy \in E_i$ such that $\lambda(y) \geq \lambda(x)$ $(i = 1, \ldots, k)$.*

Indeed, let $xy_1, xy_2 \in E_i$, $y_1 \neq y_2$. A shortest simple (x_0, x) path P contains at most one of these edges—say, it does not contain the edge xy_2. Then, however, $P \cup xy_2$ is an (x_0, y_2) path and $\lambda(y_2) \leq w(P \cup xy_2) = w(P) - 1 = \lambda(x) - 1$.

Claim 4. *The maximum of $\lambda(x)$ on $V(E_i)$ is reached on one vertex or the two endpoints of an edge $(i = 1, \ldots, k)$. For every edge $uv \in E_i$ but this one, $|\lambda(u) - \lambda(v)| = 1$.*

Indeed, if there are two nonadjacent vertices of E_i with $\lambda(x)$ maximum, then these two vertices are joined by a subpath of E_i, which, if it consists of at least two edges, must contain a vertex, contradicting the claim. If $|\lambda(u) - \lambda(v)| = 0$, then in exactly the same way, we get from Claim 3 that $\lambda(u) = \lambda(v)$ are the only maxima of $\lambda(x)$, $x \in V(E_i)$.

Thus each E_i $(i = 1, \ldots, k)$ contains at most one edge xy with $\lambda(x) = \lambda(y)$. According to (2.4), $\{\delta(D) : D \in \mathcal{D}'', x_0 \notin D\}$ is a set of disjoint cuts which contains every negative edge except those with $\lambda(x) = \lambda(y)$, that is, there is at most one exception in each E_i $(i = 1, \ldots, k)$ (see Claim 4). □

The following statement of Korach and Penn [1992] is an immediate corollary.

Corollary 2. *If (G, c) is an instance of the planar multiflow problem and the cut condition is satis ed, then upon decreasing all but one of the demands by* 1 (*that is, increasing all the negative capacities by* 1)*, there exists an integer multiflow*.

Korach and Penn also noticed that one can, in fact, require $F \cap E_0 = \emptyset$ for any given component E_0 of E^-. Indeed, in the above proof as well, the choice $x_0 \in V(E_0)$ makes sure that in Claim 4, $\lambda(x_0) = 0$, and for every other vertex of E_0, $\lambda(x) < 0$. In E_0 there is no edge xy with $\lambda(x) = \lambda(y)$.

Frank and Szigeti [1995] generalized (3.4) in the following way.

(3.5) *Let G be a graph, $w : E(G) \to \mathbb{Z}$, and $E_0, E_1, \ldots, E_k, E_{k+1}, \ldots, E_{k+l}$ be the components of E^-. Suppose that $w(C) \geq s(C)$ holds for every circuit C of G, where $s(C) := |\{i \in \{1, \ldots, k\} : E_i \cap C \neq \emptyset\}|$.*

Then there exists a set $F \subseteq E_{k+1} \cup \cdots \cup E_{k+l}$ such that $|F \cap E_i| \leq 1$ if $i = k+1, \ldots, k+l$, with the property that, upon decreasing the demand of $e \in E_i \cap F$ by 1 $(i = k+1, \ldots, k+l)$, there exists a dual multiflow.

The proof of Frank and Szigeti consists of splitting every vertex $v \in E_1 \cup \cdots \cup E_k$ into two vertices v_1 and v_2, where v_1 is incident to the positive edges and v_2 is incident to the negative edges of $\delta(v)$. v_1 and v_2 are joined by an edge of weight $-1/2$. If the condition of (3.5) is satisfied, then the constructed graph with the constructed weighting is conservative, and applying the proof of (3.4) to this graph with this weight, we can obtain the dual multiflow stated in (3.5).

Frank and Szigeti [1995] also observed the following immediate corollary.

Corollary 3. *If (G, w) is conservative and $w(\delta(X)) + w(\delta(X) \cap E^-) \geq 0$, then there exists an integer dual multiflow.*

Theorem 3.1 then implies the following:

The integer (dual) multiflow in (3.4), (3.5), and Corollary 3 can be found within the same time limits as those in Corollary 1. Of course, the same holds for integer multiflows in planar graphs.

3.3. Packing cuts exactly. Finally, let us comment on the complexity of integer odd cut packings. Let G be a graph and $T \subseteq V(G)$, where the number of vertices of T in each component of G is even. (G, T) is said to have the *Seymour property* if the minimum cardinality of a T-join is equal to the maximum cardinality of a set of disjoint T-cuts. Accordingly, (G, w) $(w : E(G) \to \mathbb{Z})$ has the Seymour property if there exists an integer dual flow. Middendorf and Pfeiffer [1989] proved that it is NP-complete to test the Seymour property, even in planar graphs.

Therefore, instead of the Seymour property, our interest turns towards Seymour graphs. G is called a Seymour graph if for arbitrary $w : E(G) \to \{-1, 1\}$ for which the cut condition holds, there exists an integer dual multiflow. Through the subdivision of edges, the weighted generalization is the following: G is a Seymour graph with respect to the weight function $w : E(G) \to \mathbb{N}$ if for an arbitrary $w' : E(G) \to [-|w(e)|, w(e)]$ such that $w'(e) \equiv w(e) \bmod 2$ and the cut condition holds, there exists an integer dual multiflow. (This is the class of weight functions that one gets by independently signing the parts of a subdivided edge. In the following, we will restrict ourselves to unweighted Seymour graphs since the generalization is trivial and somewhat artificial.) It is not known whether the problem of deciding whether a graph G is a Seymour graph is polynomially solvable or coNP-complete. (It is not trivial but true that it is in coNP, as we shall see later.) However, the following somewhat weaker results are known:

As Seymour's result (1.3) implies that bipartite graphs are Seymour graphs, so does (3.1) imply that series-parallel graphs are Seymour graphs. (The latter statement can also be easily proved directly without using (3.1).) Gerards [1992] proved a common generalization of these two results. A sharpening of this that provides a coNP characterization of Seymour graphs was conjectured by Sebő [1991] and proved by Ageev, Kostochka, and Szigeti [1995]. The following simplified version still provides a polynomially checkable obstacle, and it suffices for our purposes.

(3.6) G *is not a Seymour graph if and only if there exists a signing of the edges such that the union of zero-weight circuits* (*in fact, of only two zero-weight circuits*) *is nonbipartite.*

The proof of the *if* part of this conjecture is a straightforward exercise.

The proof of the *only if* part given by Ageev, Kostochka, and Szigeti is based on Theorem 1.1 in such a way that Algorithm 1 can be straightforwardly substituted in it, and the corresponding polynomial bounds follow easily:

Given the graph G and $w : E(G) \to \{-1, 1\}$, there is a polynomial algorithm which either (a) finds an integer dual multiflow in (G, w) or a negative circuit or (b) exhibits a weight function $w' : E(G) \to \{-1, 1\}$ for which (G, w') is conservative but no integer dual multiflow exists. In the latter case, a "good certificate" (a nonbipartite graph and two zero-weight circuits covering all the edges) is provided.

As a consequence, *the unweighted dual multiflow problem is polynomially solvable in Seymour graphs*, and all of the consequences provided by the Corollary 1 also hold. Let us finally state the specialization to planar graphs (after dualization):

Given the graph $G = (V, E)$ and $R \subseteq E$, there is a polynomial algorithm which either (a) finds an integer multiflow in (G, R) or a violated cut or (b) exhibits $R' \subseteq E$ for which the cut condition is satisfied (certificate: a half-integer multiflow) but no integer multiflow exists (certificate: two zero-weight cuts whose union is odd).

As a consequence, *the edge-disjoint paths problem is polynomially solvable in dual Seymour planar graphs.* The main open problem of whether Seymour graphs are in NP and whether they can be recognized in polynomial time remains.

3.4. Weighted and canonical matching and T-join structure. If $V(G)$ is odd, a matching M will be called *perfect* if it leaves exactly one vertex uncovered. The maximum- (or minimum-) weight matching problem can be easily reduced to the problem of finding minimum-weight perfect matchings. If $w : E(G) \to \mathbb{R}$, we will denote by $\tau(G, w)$ the minimum weight of a perfect matching in G.

It is now easy to deduce the consequences of Theorems 2.1 and 3.1 to weighted matchings:

Add a large number N (say, N is the sum of the absolute values of the weights) to the weight of every edge. Add a new vertex x_0 to G, and join it to every vertex with an edge of weight N. Let G' denote this new graph and w' denote the weight function that we defined on its edges. It is easy to see that the w'-minimum $T := V(G')$-joins (or $T := V(G') \setminus \{x_0\}$-joins, depending on which of the two is even) of G' are exactly the w-minimum perfect matchings of G (after deleting the edge adjacent to x_0 if $|V(G)|$ is odd).

Choose a minimum-weight perfect matching of G or G' (depending on whether $|V(G)|$ or $|V(G')|$ is even), and change the sign of the weights of the edges in it.

It is now easy to see that the distance of the vertex $x \in V(G)$ from x_0 is the number $\tau(G - x, w) - \tau(G, w)$. Note that it is independent of the particular w-minimum matching chosen.

From a maximum 2-packing of T-cuts in G', a maximum odd cut packing of G can also be reconstructed. That is, Theorem 1.1 can be adapted to weighted matchings. The "magic" numbers should be defined to be $\tau(G-x, w) - \tau(G, w)$. These numbers will be equal to the distances with the chosen weights; these independently computable numbers determine a packing of odd cuts, which could be called a "patch packing" and which generalizes the Gallai–Edmonds structure of maximum matchings. (See more about the relation of Theorem 1.1 and the Gallai–Edmonds structure theorem in Sebő [1990].)

Algorithm 1 and Lemmas 3.1 and 3.2 can be applied to other problems involving integer packings in a similar way. We mention some results whose proofs involve Theorem 1.1 (or 2.1) so that they can be accompanied by a polynomial algorithm that combines Lemmas 3.1 and 3.2 and Algorithm 1.

A generalization of the Kotzig–Lovász theorem (Sebő [1987a, b]) provides a "canonical partition" (implying new results on integer feasible flows or the computation of the dimension of T-join polyhedra or multiflows). This canonical partition can be computed via the distances, that is, with Lemmas 3.1 and 3.2. Thus the integer packings and flows of Sebő [1990], or perhaps a negative cut or an odd circuit consisting of tight edges, or a set violating the cut condition or some stronger condition (see Sebő [1987b]) can also be found in polynomial time.

Acknowledgments. I am thankful to András Frank for his encouragement and to Zoltán Szigeti and an anonymous referee for a lot of helpful advice.

REFERENCES

A. Ageev, A. Kostochka, and Z. Szigeti [1995], *A characterization of Seymour graphs*, J. Graph Theory, to appear.

F. Barahona [1980], *Application de l'optimisation combinatoire á certains modéles de verres de spin: Complexité et simulations*, thèse de docteur ingénieur, Université Scientifique et Médicale de Grenoble, Institut National Polytechnique de Grenoble, Grenoble, France.

F. Barahona [1989], *Planar multicommodity flows: Max cut and the Chinese postman problem*, in Polyhedral Combinatorics, W. Cook and P. Seymour, eds., DIMACS Series in Discrete Mathematics and Computer Science, Vol. 1, AMS, Providence, RI, pp.189–202.

F. Barahona and W. Cunningham [1989], *On dual integrality in matching problems*, Oper. Res. Lett., 8, pp. 245–249.

J. Edmonds [1965a], *Paths, trees and flowers,* Canad. J. Math., 17, pp. 449–467.

J. Edmonds [1965b], *The Chinese postman problem,* Oper. Res., 13 (supplement 1), p. 373.

J. Edmonds and E. L. Johnson [1973], *Matching, Euler tours and the Chinese postman*, Math. Programming, 5, pp. 88–124.

A. Frank [1990], *Packing paths, circuits and cuts: A survey,* in Paths, Flows and VLSI-Layout, B. Korte, L. Lovász, H. Prömel, and A. Schrijver, eds., Springer-Verlag, Berlin, Heidelberg, pp. 47–100.

A. Frank, A. Sebő, and É. Tardos [1984], *Covering directed and odd cuts,* Math. Programming Stud., 22, pp. 99–112.

A. Frank and Z. Szigeti [1994], *On packing T-cuts*, J. Combin. Theory Ser. B, 61, pp. 263–271.

A. Frank and Z. Szigeti [1995], *A note on packing paths in planar graphs*, Math. Programming, 70, pp. 201–209.

A. Gerards [1987], private communication.

A. Gerards [1992], *On shortest T-joins and packing T-cuts*, J. Combin. Theory Ser. B, 55, pp. 73–82.

M. Guan [1962], *Graphic programming using odd or even points*, Chinese J. Math., 1, pp. 273–277.

E. Korach [1982], *Packing of T-cuts, and other aspects of dual integrality,* Ph.D. thesis, Waterloo University, Waterloo, ON, Canada.

E. Korach and M. Penn [1992], *Tight integral duality gap in the Chinese postman problem,* Math. Programming, 55, pp. 183–191.

E. Lawler [1976], *Combinatorial Optimization, Networks and Matroids,* Holt, Rinehart, and Winston, New York.

R. J. Lipton and R. E. Tarjan [1980], *Applications of a planar separator theorem,* SIAM J. Comput., 9, pp. 615–627.

L. Lovász [1975], 2*-matchings and* 2*-covers of hypergraphs,* Acta Math. Acad. Sci. Hungar., 26, pp. 433–444.

L. Lovász and M. Plummer [1986a], *On bicritical graphs*, in Infinte and Finite Sets, A. Hajnal, R. Rado, V. T. Sós, et al., eds., Colloq. Math. Soc. János Bolyai, North–Holland, Amsterdam, pp. 1051–1079.

L. Lovász and M. Plummer [1986b], *Matching Theory*, Akadémiai Kiadó, Budapest.

K. Matsumoto, T. Nishizeki, and N. Saito [1986], *Plane multicommodity flows, maximum matchings and negative cycles*, SIAM J. Comput., 15, pp. 495–510.

M. Middendorf and F. Pfeiffer [1989], *On the complexity of the disjoint path problem*, in Polyhedral Combinatorics, W. Cook and P. Seymour, eds., DIMACS Series in Discrete Mathematics and Computer Science, Vol. 1, AMS, Providence, RI, pp. 171–178.

K. Mulmuley, U. V. Vazirani, and V. V. Vazirani [1987], *A parallel algorithm for matching,* Combinatorica, 7, pp. 105–113.

A. Schrijver [1986], *Theory of Linear and Integer Programming*, John Wiley, Chichester, UK.

A. Sebő [1987a], *The factors of graphs: Structures and algorithms,* Ph.D. thesis, Eötvös Loránd University, Budapest.

A. Sebő [1987b], *Dual integrality and multicommodity flows,* in Infinite and Finite Sets, A. Hajnal and V. T. Sós, eds., Colloq. Math. Soc. János Bolyai, North–Holland, Amsterdam, pp. 453–469.

A. Sebő [1988], *The Schrijver system of odd join polyhedra*, Combinatorica, 8, pp. 103–116.

A. Sebő [1990], *Undirected distances and the postman-structure of graphs,* J. Combin. Theory Ser. B, 49, pp. 10–39.

A. Sebő [1991], *On two multiflow problems*, lecture, Graph Minors Meeting, Seattle.

P. D. Seymour [1977], *The matroids with the max-flow min-cut property*, J. Combin. Theory Ser. B, 23, pp. 189–222.

P. D. Seymour [1981], *On odd cuts and plane multicommodity flows*, Proc. London. Math. Soc., 42, pp. 178–192.

J. Wein [1991], *Las Vegas RNC algorithm for weighted matchings*, Inform. Process. Lett., 40, pp. 161–167.