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COVERING DIRECTED AND ODD CUTS

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Analogous pairs of theorems are investigated concerning coverings of directed and odd cuts. One such pair of results is the Lucchesi–Younger theorem on directed cuts and Seymour's theorem on odd cuts. Here we strengthen these results (incidently providing a simple proof of Seymour's theorem). For example, the minimum cardinality of a T-join in a graph $G=(V, E)$ is proved to equal the maximum of $\sum q_T(V_i)/2$ over all partitions of V where $q_T(X)$ is the number of T-odd components of $V-X$. Moreover, if G is bipartite, there is an optimal partition arising from a partition of the two parts. Secondly some orientation problems of undirected graphs are discussed. The results also emphasize the analogy between strong connectivity and parity conditions.

Key words: Odd Cut, Directed Cut, Matching, Chinese Postman, Orientation.

1. Introduction

In this paper we investigate analogous pairs of theorems concerning directed cuts of digraphs and odd cuts (or T-cuts) of undirected graphs. For example, the Lucchesi–Younger theorem asserts that the minimum cardinality of a covering of directed cuts (i.e. a set of edges meeting all directed cuts) is equal to the maximum number of pairwise disjoint directed cuts. Seymour's theorem states the same for odd cuts of bipartite graphs. Here we shall strengthen these results.

For a fixed directed cut $F=\delta(X)$, a minimax formula is proved for $c'(X)$, the minimum of $|C \cap F|$ over all coverings C of directed cuts. Making use of the supermodularity of c' we prove a variant of the Lucchesi–Younger theorem in which the optimal packing of directed cuts forms a chain.

An analogous minimax formula is proved for the minimum of $|C \cap F|$ over all T-joins C (from which, incidently, a simple proof for Seymour's theorem follows). Here the main result is: the minimum cardinality of a T-join in a bipartite graph $G=(V_1, V_2; E)$ is equal to the maximum of $\sum q_T(X_i)$ over all partitions of V_1 where $q_T(X)$ denotes the number of T-odd components of $G-X$.

This theorem easily implies Seymour's theorem as well as the Berge–Tutte formula. It also provides a formula for the minimum number of edges in a graph G such that G becomes Eulerian by doubling these edges (i.e. for the minimum cardinality of a Chinese postman tour). This minimum is equal to the maximum of $\sum q_0(V_i)/2$ over all partitions of V where $q_0(X)$ is the number of components in $G-X$ of odd degree.

We also deal with several problems concerning orientations of undirected graphs. Frank and Gyárfás [4] gave good characterizations for the existence of strongly connected orientations with certain side-conditions. Here analogous results are proved when strong connectivity is replaced by certain parity restrictions.

We shall use the terms graph and digraph for undirected and directed graph, respectively. By an edge we mean an undirected edge while directed edges are called arrows. Throughout we work with connected graphs and weakly connected digraphs. A *mixed* graph may contain both directed and undirected edges.

For a mixed graph $G=(V,E)$ the *coboundary* $\delta(X)$ of $X\subseteq V$ is the set of edges and arrows entering X, $S(X)$ is the set of edges and arrows incident with at least one node of X, $E(X)$ is the set of edges and arrows induced by X. We denote by $d(X,Y)$ the number of edges and arrows with one end node in $X-Y$ and the other one in $Y-X$ and we set $d(X)=d(X,V-X)$. The number of (weak) components of $V-X$ is denoted by $c(X)$.

Let $G=(V,E)$ be a digraph. The *in-degree* $\rho(X)$ is the number of arrows entering X. A *kernel* X is a subset of nodes with no arrows leaving X. The coboundary $\delta(X)(\neq\emptyset)$ of a kernel X is called a *directed cut* (or dicut) determined by X. A *covering* C (of directed cuts) is a subset of arrows meeting all the directed cuts.

Let $G=(V,E)$ be a graph and $T\subseteq V$ a subset of even cardinality. A set $X\subseteq V$ is *T-odd* if $|X\cap T|$ is odd. $q_T(X)$ denotes the number of T-odd components in $G-X$. In particular $q_V(X)$ is the number of components in $G-X$ of odd cardinality. For a T-odd set X the coboundary $\delta(X)$ is called an *odd cut* or *T-cut*. (We shall prefer the latter.) A subset C of edges is called a *T-join* if the number of edges in C incident with v is odd exactly when $v\in T$. Obviously a T-join and a T-cut have an odd number of edges in common.

An *orientation* of undirected graph G is a digraph arising from G by orienting its edges.

A family $\mathcal{F}$ of subsets is called *half-disjoint* if every element of the ground set is in at most two members of $\mathcal{F}$. For a vector $u\in\mathbb{R}^V$ set $u(X)=\sum(u(x): x\in X)$.

2. Orientations I

Our starting point is a result of Frank and Gyárfás [4] concerning orientations of graphs which satisfy lower and upper bound restrictions on the in-degree. Given a graph $G=(V,E)$, let l and u be two integral functions on V such that $0\leqslant l\leqslant u$.

Theorem 1. a. *There is an orientation of G with $\rho(v)\leqslant u(v)$ for $v\in V$ iff $|E(X)|\leqslant u(X)$ for $X\subseteq V$.*

b. *There is an orientation of G with* $\rho(v) \geq l(v)$ *for* $v \in V$ *iff* $|S(X)| \geq l(X)$ *for* $X \subseteq V$.

c. *There is an orientation of G with* $l(v) \leq \rho(v) \leq u(v)$ *for* $v \in V$ *iff there is one with* $\rho(v) \geq l(v)$ *for* $v \in V$ *and there is one with* $\rho(v) \leq u(v)$ *for* $v \in V$.

An earlier orientation result is due to H. Robbins [10].

Theorem 2. *A graph has a strongly connected orientation iff there is no isthmus.*

A slight strengthening of this theorem is

Theorem 2a [2]. *In a mixed graph G there is an orientation of the undirected edges which results in a strongly connected digraph iff there is no isthmus and G contains no directed cuts.*

Theorem 2a can be interpreted so that any partial orientation of an isthmus-free graph can be continued in order to get a strongly connected orientation provided that there is either an undirected edge or an already oriented edge leaving X for every subset X $(\neq \emptyset, V)$.

The following characterization for the existence of a strongly connected orientation satisfying lower and upper bound conditions on the in-degrees was presented in [4].

Theorem 3. *Suppose G is an undirected graph with no isthmus.* a. *There is a strongly connected orientation of G with* $\rho(v) \leq u(v)$ *for* $v \in V$ *iff* $|E(X)| + c(X) \leq u(X)$ *for* $X \subseteq V$.

b. *There is a strongly connected orientation of G with* $\rho(v) \geq l(v)$ *for* $v \in V$ *iff* $|S(X)| - c(X) \geq l(X)$ *for* $X \subseteq V$.

c. *There is a strongly connected orientation of G with* $l(v) \leq \rho(v) \leq u(v)$ *for* $v \in V$ *iff there is one with* $\rho(v) \leq u(v)$ *for* $v \in V$ *and there is one with* $l(v) \leq \rho(v)$ *for* $v \in V$.

In this paper we shall prove some analogous theorems in which the property of strong connectivity is replaced by a certain parity condition—namely, given a subset T of nodes, find an orientation for which $\rho(v)$ is odd exactly if $v \in T$ and ρ satisfies lower and upper bound conditions. Without loss of generality one can suppose that $l(v)$ and $u(v)$ are odd exactly when $v \in T$. Thus we assume that $l(v) \equiv u(v) \pmod 2$ for $v \in V$. Call an orientation *u-congruent* if $\rho(v) \equiv u(v) \pmod 2$ for $v \in V$. Obviously u-congruent and l-congruent mean the same. The following simple observation is analogous to Robbins' theorem.

Theorem 4. *A connected graph* $G = (V, E)$ *has a u-congruent orientation iff*

$$|E(V)| \equiv u(V) \pmod 2. \tag{1}$$

Proof. Orient the edges so that the number of nodes v with $\rho(v) \not\equiv u(v) \pmod 2$ is as small as possible. If there are no such nodes, we are done. Otherwise, there are at least two, say u and w. Choose a path from u to w (in the undirected graph), and reverse the orientation on the edges of this path. This increases the number of nodes v with $\rho(v) \equiv u(v) \pmod 2$. $\square$

Next let us deal with u-congruent orientations satisfying an upper bound restriction for the indegrees. Before the reader thinks this problem too artificial we remark that the 1-factor problem in a graph can be formulated in this way, as follows. Given a graph $H=(V, E)$, place a new node on every edge. Let $u(v)=1$ for original nodes and $u(v)=2$ for new nodes. There is a 1-to-1 correspondence between the 1-factors of H and the u-congruent orientations of the enlarged graph G satisfying $\rho \leqslant u$. Namely, an edge ab of H is in the 1-factor if and only if the in-degree of the middle node of ab is 0. Actually, Tutte's 1-factor theorem easily follows from the following theorem which can be considered as a counterpart to Theorem 3a. On the other hand, in the proof we rely on Tutte's theorem.

Theorem 5a. *There exists a u-congruent orientation of G for which $\rho(v) \leqslant u(v)$ if and only if*

$$|E(X)| + \mathrm{o}(X) \leqslant u(X) \quad \text{for } X \subseteq V \tag{2}$$

where $\mathrm{o}(X)$ denotes the number of components C (called u-odd) of $G-X$ for which $u(C) \not\equiv |S(C)| \pmod 2$.

Proof. (Necessity.) In any u-congruent orientation, any u-odd set is left by an odd number of arrows. Thus at least $\mathrm{o}(X)$ arrows enter X and we have

$$|E(X)| + \mathrm{o}(X) \leqslant \sum(\rho(v)\colon v \in X) \leqslant u(X).$$

(Sufficiency.) We deduce sufficiency from Tutte's 1-factor theorem by means of an elementary construction. First place a new node v^e on every edge e, then blow up every original node v to a complete graph of $u(v)$ nodes. More exactly, define $G'=(V', E')$ as follows.

$$V' = \{v_i\colon v \in V \text{ and } 1 \leqslant i \leqslant u(v)\} \cup \{v^e\colon e \in E\},$$

$$E' = \{v_i v_j\colon \text{for } v \in V \text{ and } 1 \leqslant i < j \leqslant u(v)\}$$

$$\cup \{(v^e v_i\colon e \in E, v \in V, e \text{ incident with } v\}.$$

Claim. *G has a u-congruent orientation for which $\rho(v) \leqslant u(v)$ $(v \in V)$ iff G' has a 1-factor.*

Proof. If G' has a 1-factor F and an edge of F incident to v^e is $v^e v_i$, orient e in G toward v. The orientation of G defined this way clearly satisfies the upper bound requirement. Furthermore, an even number of nodes is paired by F in the set of nodes in G' corresponding to a node $v \in V$ therefore the in-degree of v in the orientation is congruent to $u(v) \pmod 2$.

Conversely, the same correspondence shows that a u-congruent orientation with $\rho(v)\leq u(v)$ $(v\in V)$ determines a 1-factor of G'. $\diamond$

By Tutte's theorem, a graph has a 1-factor iff for every subset $S\subseteq V'$, the number $q_{V'}(S)$ of components of $G'-S$ with an odd number of nodes is at most $|S|$. Let S be a minimal subset of V' violating the Tutte condition.

Claim. *For every $v\in V$, S either contains all the copies in G' corresponding to v or none of them.*

Proof. To the contrary, let $v\in V$ and $1\leq i,j\leq u(v)$ be such that $v_i\in S$, $v_j\notin S$. Set $S_1=S-v_i$. Since v_i and v_j are adjacent and the neighbors of v_i and v_j are the same, the components of $G'-S_1$ are the components of $G'-S$ with the only exception that the component containing v_j will also contain v_i. Thus $q_{V'}(S_1)\geq q_{V'}(S)-1>|S|-1=|S_1|$ contradicting the minimal choice of S. $\diamond$

Claim. *S does not contain any new node v^e.*

Proof. For otherwise, set $S_1=S-v^e$. Since every node of G was replaced by a complete graph, v^e can be connected with at most two components, say C_1, C_2, of $G'-S$. If both of them are odd, $C_1\cup C_2\cup\{v^e\}$ is odd, as well, whence we have $q_{V'}(S_1)\geq q_{V'}(S)-1>|S|-1=|S_1|$ contradicting the minimal choice of S. $\diamond$

Using these claims one can easily check that $X=\{v\colon v_i\in S\}$ violates (2). $\square$

An analogous result for l-congruent orientations with lower bounds simply follows from the preceding theorem.

Theorem 5b. *A graph $G=(V,E)$ has an l-congruent orientation for which $\rho(v)\geq l(v)$ $(v\in V)$ iff $|S(X)|-\mathrm{o}(X)\geq l(X)$ for $X\subseteq V$ where $\mathrm{o}(X)$ denotes the number of l-odd components of $G-X$.*

Somewhat surprisingly, the counterpart of Theorem 3c is not true. The graph following in Fig. 1 has a u-congruent orientation with upper bound u and an l-congruent orientation with lower bound l but no u-congruent orientation exists satisfying the upper bound and the lower bound condition at the same time. (At the nodes the first number denotes the lower bound l and the second number denotes the upper bound u.) However we have the following more complicated characterization. Suppose again that $u(v)\equiv l(v)$ (mod 2) $(v\in V)$.

Theorem 6. *A graph $G=(V,E)$ has a u-congruent orientation for which $l(v)\leq\rho(v)\leq u(v)$ for $v\in V$ iff*

$$u(A)-l(B)\geq p(A,B)+|E(A)|-|S(B)| \tag{3}$$

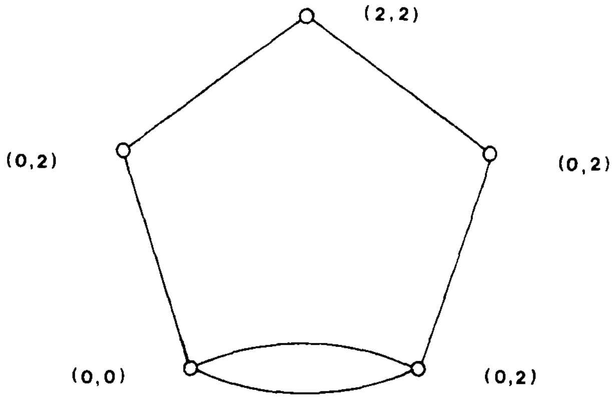

Fig. 1.

for disjoint subsets A, $B \subseteq V$, where $p(A, B)$ denotes the number of those components C of $V-(A \cup B)$ for which $|E(C)|+d(C, A) \not\equiv l(C) \pmod 2$.

The graph in the figure does not satisfy (3): let A consist of the two lower nodes and B of the upper most node.

Proof. The proof is similar to that of Theorem 5a so we outline only the sufficiency. Place a new node v^e on every edge e and blow up every original node v to a graph consisting of $l(v)$ isolated nodes and a complete graph of $u(v)-l(v)$ nodes. One can see that G has a u-congruent orientation ρ for which $l(v) \leq \rho(v) \leq u(v)$ $(v \in V)$ iff G' has a 1-factor.

If there is no 1-factor of G', there exists a subset $S \subset V'$ violating the Tutte condition. Let S be minimal. A simple argument shows that, for every $v \in V$, S either contains all the nodes in G' corresponding to v or none of them. Set $A=\{v \in V\colon v_i \in S\}$ and $B=\{v \in V$: there is no edge $e \in E$ incident with v such that $v^e \not\in S\}$. The proof is completed by showing that $A \cap B = \emptyset$ and A, B violate condition (3). □

3. Orientations II

Let $G=(V, E)$ be an isthmus-free graph and X $(\neq \emptyset, V)$ a specified subset of nodes. In this section we describe a min–max formula for the minimum number $c'(X)$ of arrows entering X in strongly connected orientations of G.

First of all observe that the components of X and $V-X$ can be shrunken into singletons without changing $c'(X)$. This is so because shrinking does not destroy the strong connectivity, on the other hand Theoreom 2.2a shows that a strongly connected orientation of the shrunken graph can be extended to a strongly connected orientation of G. Thus we can suppose that G is a bipartite graph with parts V_1,

V_2 where V_1 denotes the set formed by shrinking the components of X. Recall the definition of $c(X)$. The next lemma occurs in [3, 7].

Lemma 1. $c(A)+c(B) \leq c(A \cup B)+c(A \cap B)+d(A, B)$.

Theorem 2. *If $G=(V_1, V_2; E)$ is a bipartite graph, the minimum number of arrows entering V_1 among all strongly connected orientations of G is equal to*

$$\max(\textstyle\sum c(X_i): \{X_i\} \text{ partitions } V_1,\ X_i \neq \emptyset).$$

Proof. (max $\leq$ min.) For a strongly connected orientation ρ, $\rho(X) \geq c(X)$ $(X \neq \emptyset)$ therefore $\rho(V_1)=\sum \rho(X_i) \geq \sum c(X_i)$.

(max = min.) We describe an algorithm that, given a strongly connected orientation with in-degree function ρ, either finds a better orientation ρ' (i.e. $\rho'(V_1)<\rho(V_1)$) or else finds a partition $\{X_i\}$ of V_1 for which $\rho(V_1)=\sum c(X_i)$.

Call a set *tight* (with respect to ρ) if $\rho(X)=c(X)$. (Note that V is always tight.)

Lemma 3. *If A, B are intersecting tight sets, $A \cap B$, $A \cup B$ are also tight.*

Proof. Using Lemma 1 we get

$$\rho(A)+\rho(B)=c(A)+c(B) \leq c(A \cap B)+c(A \cup B)+d(A, B) \leq \rho(A \cap B) + \rho(A \cup B)+d(A, B)=\rho(A)+\rho(B)$$

from which we have $c(A \cap B)=\rho(A \cap B)$ and $c(A \cup B)=\rho(A \cup B)$. $\diamond$

An easy consequence is

Lemma 4. *If a family of tight sets forms a connected hypergraph the union is tight. The intersection $P(v)$ of all tight sets containing a node v is tight.* $\diamond$

Let ρ be a strongly connected orientation of G. There may be two cases.

Case 1: For every $x \in V_1$, $P(x) \subseteq V_1$. Let $X_1, X_2, \ldots, X_n$ denote the components of the hypergraph $\{P(x): x \in V_1\}$. Then $\{X_i\}$ forms a partition of V_1 and, by Lemma 4, each X_i is tight. Thus $\rho(V_1)=\sum \rho(X_i)=c(X_i)$, i.e. for the given orientation we have the required equality.

Case 2: There are nodes $x \in V_1$ and $y \in V_2$ such that $y \in P(x)$. Let P be a directed path from y to x. Reverse the orientation of the arrows of P. We get another orientation for which $\rho'(V_1)=\rho(V_1)-1$. Furthermore the new orientation is still strongly connected, for otherwise there would have been a set A with $\rho(A)=1$, $x \in A$, $y \notin A$ contradicting the assumption that $y \in P(x)$. $\square$

Corollary 5. *The minimum over all strongly connected orientations of a graph $G=$*

(V, E) of the number of arrows entering a specified subset X is equal to $\max(\sum c(X_i)$: *$\{X_i\}$ partitions X and no edges lead between distinct X_i's, $X_i \neq \emptyset$).*

The next corollary concerns only c and not the orientation but we do not see any direct proof.

Corollary 6. *In a bipartite graph $G=(V_1, V_2; E)$,* $\max(\sum c(X_i)$: *$X_i \neq \emptyset$, $\{X_i\}$ partitions V_1*$) = \max(\sum c(Y_j)$: *$Y_j \neq \emptyset$, $\{Y_j\}$ partitions V_2).*

We conclude this section by showing that Lemma 1 is also true for c'.

Theorem 7. $c'(A)+c'(B) \leq c'(A \cap B)+c'(A \cup B)+d(A, B)$.

Proof. It suffices to prove the theorem when $d(A, B)=0$. If $d(A, B)>0$, place a new node on the middle of every edge between $A-B$ and $B-A$. The new nodes do not belong to A and B. Obviously this transformation increases $c'(A \cup B)$ by $d(A, B)$ but it does not affect $c'(A)$, $c'(B)$, and $c'(A \cap B)$. Furthermore there is no edge between $A-B$ and $B-A$ any more. Thus the theorem follows from the special case when $d(A, B)=0$ and henceforth we assume this.

The next lemma is due to Dunstan and Lovász [2, 7].

Lemma 8. *Let $\mathcal{F}$ be a family of subsets with $\emptyset \notin \mathcal{F}$ and let f be a function on $\mathcal{F}$ such that X, $Y \in \mathcal{F}$, $X \cap Y \neq \emptyset$ imply that $X \cap Y$, $X \cup Y \in \mathcal{F}$ and $f(X)+f(Y) \leq f(X \cap Y)+f(X \cup Y)$. Then the family*

$$\mathcal{F}' = \{\textstyle\bigcup F_i: \text{ and } F_i \in \mathcal{F},\ F_i\text{'s are disjoint}\} \cup \{\emptyset\}$$

is closed under taking union and intersection. Moreover, let f' be the function on $\mathcal{F}'$ defined by

$$f'(F) = \max(\textstyle\sum f(F_i): \{F_i\} \text{ partitions } F,\ F_i \in \mathcal{F}) \quad \text{if } F \in \mathcal{F}'-\{\emptyset\}, \qquad f'(\emptyset)=0.$$

Then, for X, $Y \in \mathcal{F}'$,

$$f'(X)+f'(Y) \leq f'(X \cup Y)+f'(X \cap Y). \qquad \diamond$$

Let us orient the edges of G so that no arrow enters A and B and set $\mathcal{F}=\{X: X \neq \emptyset, \rho(X)=0\}$. By Corollary 5, it is obvious for $X \in \mathcal{F}$ that $c'(X)=\max(\sum c(X_i)$: $\{X_i\}$ partitions X, $X_i \in \mathcal{F})$. By Lemma 1, Lemma 8 applies to $\mathcal{F}$ with $f=c$ and we are done. $\square$

4. Directed cuts, T-cuts

A fundamental theorem on directed cuts is due to Lucchesi and Younger [8].

Theorem 1. *In a digraph the minimum cardinality of a covering of directed cuts is equal to the maximum number of pairwise disjoint directed cuts.*

For a covering C and a dicut F by definition we have $|C \cap F| \geq 1$. We are going to determine the minimum of $|C \cap F|$ over all coverings C.

An answer can be read out from Theorem 1, namely, the minimum in question is equal to the maximum number of pairwise disjoint directed cuts in F. The main content of Theorem 2 is that the maximum is determined by certain disjoint kernels.

Theorem 2. *In a digraph, given a specified directed cut* $F = \delta(X)$, $\min(|C \cap F|$: C *covering*$) = \max(\sum c(X_i)$: $\bigcup X_i = X$, X_i*'s are disjoint kernels*$)$.

Proof. We can suppose that G does not contain isthmuses.

Lemma. *If C is a covering minimal with respect to inclusion, reorienting the elements of C results in a strongly connected digraph.*

Proof. Delete the orientation of the elements of C. We get a mixed graph to which Theorem 2.2a applies. In the resulting strongly connected reorientation of G each element of C must have direction reversed. For C is minimal and so for each $e \in C$ there is a directed cut T of G such that $T \cap C = \{e\}$ for each $e \in C$ and in the strongly connected reorientation of G, T is no longer a directed cut. $\diamond$

This lemma and Corollary 3.5 proves the theorem. $\square$

Notice that in Theorem 2, if G does not contain isthmuses, the minimum is exactly $c'(X)$. Thus we can use the notation $c'(X)$ for the minimum in Theorem 2 when G may contain isthmuses. With the help of c', the Lucchesi–Younger theorem can be formulated in a more compact form.

Theorem 3. *In a digraph the minimum cardinality of a covering of directed cuts is equal to* $\max(\sum c'(X_i)$: X_i*'s are kernels,* $X_1 \subset X_2 \subset \cdots \subset X_k$ *and no arrow enters more than one* $X_i)$.

Proof. Theorem 1 provides a covering C and a family $\mathscr{F}$ of disjoint directed cuts $\delta(X_i)$ such that $|C| = |\mathscr{F}| \leq \sum c'(X_i) \leq |C|$, i.e. $|C| = \sum c'(X_i)$. Making use of Theorem 3.7 we can apply a well-known trick [11] to 'uncross' $\{X_i\}$. Namely, let us choose $\{X_i\}$ in such a way that $|C| = \sum c'(X_i)$, $\delta(X_i)$ are pairwise disjoint, and $\sum |X_i|^2$ is maximal. We claim for each i, j that $X_i \subset X_j$ or $X_j \subset X_i$. If this were not true for X_1 and X_2, say, in the directed cut family replace X_1 and X_2 by $X_1 \cap X_2$ and $X_1 \cup X_2$. Applying Theorem 3.7 we get $|C| \geq \sum c'(X_i') \geq \sum c'(X_i) = |C|$, i.e. $\sum c'(X_i') = |C|$ and $\sum |X_i'|^2 > \sum |X_i|^2$, a contradiction. $\square$

Next, let us consider a T-cut theorem of Lovász analogous to Theorem 1.

Theorem 4 [5]. *In a graph the minimum cardinality of a T-join is equal to the one-half of the maximum number of half-disjoint T-cuts.*

Unfortunately the analogy is not quite perfect. K_4 shows that the minimum cardinality of a T-join may be strictly bigger than the maximum number of pairwise disjoint T-cuts. As we shall see, however, this nicer min-max form is true for bipartite graphs.

As in the directed case, set up the following problem. Given a fixed coboundary $F = \delta(X)$, determine $\min|C \cap F|$ over all T-joins C. Parallel to Theorem 2 we have

Theorem 5. $\min|C \cap F|$ *over T-joins C is equal to* $\max \sum q_T(X_i)$ *over all partitions $\{X_i\}$ of X such that no edge connects distinct X_i's.*

Like in the digraph case, first we reduce the problem to bipartite graphs, as follows.

Form a bipartite graph $G' = (V_1, V_2; E)$ by shrinking the components of X and $V - X$ into singletons. Let T' consist of those new nodes which came from a T-odd component. After the shrinking, a T-join of G transforms to a T'-join of G' and conversely, since a graph has a T-join iff its components are T-even, a T-join of G' can be extended in V_1 and V_2 to a T-join of G. Hence the minimum value for G' is the same as the minimum value for G, and therefore what we need is to find in G' a T'-join of minimum cardinality.

Theorem 4 of course gives an answer, the following result of Seymour, however, provides a better one.

Theorem 6 [13]. *In a bipartite graph $G = (V_1, V_2; E)$ the minimum cardinality of a T-join is equal to the maximum number of pairwise disjoint T-cuts.*

From this one can easily deduce Theorem 4 by placing a new node on the middle of each edge and keeping T unchanged.

Here we shall prove another formula, similar to that in Theorem 2, for the minimum cardinality of a T-join in a bipartite graph. Incidently, we get a proof for Theorem 6 which looks much simpler than the earlier proofs.

Theorem 7. *In a bipartite graph $G = (V_1, V_2; E)$ the minimum cardinality of a T-join is equal to*

$$\max \sum q_T(X_i)$$

taken over all partitions $\{X_i\}$ of V_1.

This theorem is considered as the main result of this section. It immediately implies Theorem 5. The nontrivial $\max \geq \min$ part of Theorem 6 also follows easily since the T-odd components of $G-X_i$ define disjoint T-cuts. Two further consequences will be mentioned at the end of this section.

At this point we can observe a difference between T-cuts and directed cuts. For T-cuts the bipartite case (Theorem 6) is stronger than the non-bipartite one (Theorem 2) while Theorem 3 does not seem to imply the Lucchesi–Younger theorem.

Before proving Theorem 7 we point out that the minimal T-join problem is equivalent to finding a negative circuit in a ± 1-weighted graph. To be more precise suppose the edges of G are assigned weights $+1$ or -1.

By the *length* $w(P)$ of a path or circuit P we mean the sum of its edge weights. A circuit is called *negative* if its length is negative. The *distance* $\lambda(u, v)$ of two nodes u, v is the minimum length of a (simple) path between u and v.

Theorem 6′. *In a ± 1-weighted bipartite graph there is no negative circuit if and only if there is a set of disjoint cuts such that every negative edge occurs in one of them and each cut contains only one negative edge.*

Theorem 7′. *In a ± 1-weighted bipartite graph $G=(V_1, V_2; E)$ there is no negative circuit if and only if there is a partition $X_1, X_2, \ldots, X_k$ of V_1 such that no component of $G-X_i$ is entered by more than one negative edge.*

Observe that a T-join F is minimal iff there is no negative circuit with respect to w where $w(e)=-1$ if $e \in F$ and $w(e)=+1$ if $e \in E-F$. Now Theorem 7′ implies Theorem 7 since a set X is T-odd if X is entered by exactly one edge of F. Conversely, for a given ± 1-weighting w set $F=\{e\colon w(e)=-1\}$ and define

$$T=\{v\colon \text{an odd number of negative edges is incident with } v\}.$$

Since no negative circuit exists, F is a minimal T-join; therefore by Theorem 7 there is a partition $\{X_1, \ldots, X_k\}$ of V_1 for which $|F|=\sum qT(X_i)$. This partition will do for Theorem 7′ as well.

The equivalence between Theorems 6 and 6′ is seen similarly.

Proof of Theorem 7′. The if part is straightforward. To see the other direction, the next lemma is crucial.

Lemma. Let $G=(V_1, V_2; E)$ $(|V_1 \cup V_2| \geqslant 3)$ be a simple bipartite graph and w a ± 1-weighting on E such that no negative circuit exists and there is a negative path between every two nodes in the same class V_i. Then G is a tree and w is -1 everywhere.

Proof. Let x_0 be an arbitrary node. Let $m=\min \lambda(x_0, x)$ and choose a path $P_0=(x_0, x_1, \ldots, x_n)$ from x_0 to x_n such that $w(P_0)=m$ and $\lambda(x_0, x_i)>m$ for $0 \leqslant i<n$. Obviously $m<0$ and $w(x_{n-1}x_n)=-1$. $\diamond$

By induction the next claim implies the lemma.

Claim. *$x_{n-1}x_n$ is the only edge incident with x_n.*

Proof. Suppose x_ny is another edge ($y \neq x_{n-1}$). Now $w(x_ny)$ cannot be -1 since then, if y is on P_0, $P_0[yx_n] \cup x_ny$ is a negative circuit, if y is not on P_0, $P_0 + x_ny$ is a path of length $m-1$.

By hypothesis there is a negative path P between x_{n-1} and y. By parity, $w(P) \leqslant -2$. P goes through x_n, for otherwise P and $x_{n-1}x_n$, x_ny would form a negative circuit. Moreover P traverses the edge $x_{n-1}x_n$. For otherwise the length of segment $P[x_{n-1}x_n]$ is at least $+1$ therefore the length of $P[x_ny]$ is at most -3. Thus $P[x_ny]$ and x_ny would form a negative circuit.

We also see that $w(P[x_n, y]) \geqslant -1$ and $w(P[x_n, y]) \leqslant -1$, i.e. $C = P[x_n, y] + x_ny$ is a circuit of length 0 and x_{n-1} is not on C.

We claim that x_n is the only common node of C and P_0. If not, let x be the next one on P_0 starting at x_n. By the choice of P_0, $w(P[x_n, x]) < 0$. Hence the length of both arcs between x and x_n on C is positive contradicting that $w(C) = 0$.

Now P_0 and $P[x_ny]$ have one node in common, namely x_n, therefore $P_0 \cup P[x_n, y]$ is a simple path from x_0 to y the length of which is $m-1$, a contradiction. $\diamond$

Turning back to the proof of Theorem 7′, we can suppose that G is simple and $|V_1 \cup V_2| \geqslant 3$. There are two cases.

Case 1: There are two nodes x, y in the same class V_i with no negative path between them.

Identify x and y into a single new node z. The resulting bipartite graph has no negative circuit. By induction there is a partition with the desired property. After splitting up z the same partition of V_1 satisfies the requirements.

Case 2: There is a negative path between any two nodes in the same class V_i.

Apply the lemma. The required partition consists of the singletons of V_1. $\square$

Next we exhibit a consequence of Theorem 7.

Theorem 8. *In a graph $G = (V, E)$ the minimum cardinality of a T-join is equal to* $\frac{1}{2} \max \sum q_T(V_i)$ *over all partitions $\{V_i\}$ of V.*

Proof. Place a new node on every edge, keep T the same and apply Theorem 7 to the resulting bipartite graph. $\square$

If T consists of the nodes of odd degree, Theorem 8 provides

Corollary. *The minimum number of edges in a graph G such that G becomes Eulerian by doubling these edges, is equal to* $\frac{1}{2} \max \sum q_0(V_i)$ *over all partitions of V where $q_0(X)$ denotes the number of components C in $V - X$ for which $d(C)$ is odd.*

It is interesting that the same function q_0 plays a central role in Mader's famous A-path theorem [9]: Given a subset $A = \{v_1, \ldots, v_k\}$ of nodes in $G = (V, E)$, the maximum number of pairwise edge-disjoint paths joining distinct nodes of S is equal to $\frac{1}{2}\min\sum(d(V_i)-q_0(\cup V_i))$ over all families of disjoint subsets V_i with $V_i \cap A=\{v_i\}$.

Formulating Theorem 8 in terms of negative circuits we get

Theorem 8′. *In a ± 1-weighted graph $G=(V, E)$ there is no negative circuit iff there is a partition $\{V_i\}$ of V such that each component of $V-V_i$ is entered by at most one negative edge, and, for each i, no negative edge has both ends in V_i.*

This easily implies

Berge–Tutte formula. *In a graph $G=(V, E)$ the maximum cardinality of a matching is $\frac{1}{2}\min(-q_V(X)+|V-X|)$.*

Proof. To prove the nontrivial inequality max ≥ min, we have to show a matching F and a subset $X \subseteq V$ such that each component of $V-X$ contains at most one node not covered by F and for every node v in X there is an odd component C_v of $V-X$ (with $C_v \neq C_u$ if $u \neq v$) and an edge in F connecting v and C_v.

Let F be a maximal matching. Adjoin a new node r to the graph and lead new edges from r to the nodes exposed by F. Define $w(e)=-1$ if $e \in F$ or e is a new edge and $w(e)=+1$ if $e \in E-F$. Since F is maximal this graph does not contain a negative circuit.

Consider the partition provided by Theorem 8′. Suppose V_1 contains r. It is easy to see that F and $X = V_1 - r$ satisfies the requirements. □

Remark. Originally this paper was written by A. Frank and É. Tardos. Their proof for Theorem 7′ made use of the deep result of Seymour (Theorem 6) and was rather complicated. Before the paper was finished, A. Sebő discovered that Theorem 7′ can be proved directly and this approach is much better since it provides a simple proof for Seymour's theorem as well. Therefore we have chosen to include his proof. He also established some fundamental structural properties of T-joins. See [12].

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