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TOTAL DUAL INTEGRALITY IMPLIES LOCAL STRONG UNIMODULARITY

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We prove that any totally dual integral description of a full-dimensional polyhedron is locally strongly unimodular in every vertex.

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1. Introduction

Total dual integrality and local strong unimodularity have been used in different papers to investigate the integrality of polyhedra (see e.g. Edmonds and Giles [3], Hoffman and Oppenheim [7]). We prove here that the latter notion contains the former one.

We refer to Schrijver [9] for the terminology and basic facts about polyhedra [9, Part III], and for the elements of linear diophantine equations and lattices [9, Part II].

Let $\mathbb{Z}$ and $\mathbb{Q}$ denote the sets of integers and rationals respectively, and let A be an integral $m \times n$ matrix and $b \in \mathbb{Q}^m$.

The system of inequalities $Ax \leq b$ is called *totally dual integral* (TDI) if for each $w \in \mathbb{Z}^n$ for which $\min\{yb \mid yA = w, y \geq 0\}$ exists, there is a $y \in \mathbb{Z}^m$ attaining the minimum. (In general, total dual integrality is also defined for non-integral matrices A, but here we are only interested in integral matrices.)

If x_0 is an element of $P := \{x \in \mathbb{Q}^n \mid Ax \leq b\}$, denote by $A(x_0)$ the matrix consisting of those rows a_i of A for which $a_i x_0 = b_i$. The system of inequalities $Ax \leq b$ is called *locally strongly unimodular* (LSU) in vertex x_0 of P if $A(x_0)$ has an $n \times n$ submatrix with determinant ± 1.

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Hoffman [5] proved for pointed polyhedra and Edmonds and Giles [3] proved in general that if $Ax \leq b$ is TDI and $b \in \mathbb{Z}^n$, then P is integral, i.e. each face of P contains an integral point. (Fulkerson [4] proved this earlier for 0–1 matrices. Edmonds and Giles' results motivated the study of lattices that led to fundamental results in integer programming, see [9].) In the present note we sharpen this result by proving the following theorem. For the sake of simplicity we state the theorem for vertices only (for the general case see the remark below).

Theorem 1. *Let $Ax \leq b$ be a totally dual integral system of inequalities such that $P := \{x \in \mathbb{Q}^n \mid Ax \leq b\}$ is full-dimensional. Then $Ax \leq b$ is locally strongly unimodular in every vertex of P.*

The proof of Theorem 1 will be postponed to section 3. As a consequence, theorems of Hoffman and Oppenheim, stating that some TDI descriptions of matching and b-matching polyhedra are LSU, follow, and are even sharpened: Theorem 1 states that every, in particular the minimal, TDI description of a full-dimensional polyhedron is LSU in every vertex. (Full-dimensional polyhedra have a unique minimal TDI description, see Schrijver [8] or [9, Theorem 22.6]. Cook and Pulleyblank [2] call such a minimal TDI description the Schrijver system of the polyhedron.) New results follow from Theorem 1 by applying it to other TDI systems: e.g. to any description of the clique or the stable set polytope of a perfect graph, since the facets make up already a TDI system. Moreover the Schrijver systems for (maybe capacitated) b-matching polyhedra and t-join polyhedra (Cook and Pulleyblank [2], Sebő [10] respectively) are LSU.

Throughout this text $r(V)$ and $r(M)$ denote the linear rank of the set $V \subset \mathbb{Q}^n$ and of the matrix M respectively.

Remarks. For the sake of simplicity and in view of the applications cited above we have chosen to define local strong unimodularity only with respect to a vertex of P and we restricted ourselves in Theorem 1 to the case that P is full-dimensional. Below we explain how to generalize the notion of local strong unimodularity to be able to obtain more general results. It will turn out that the restrictions are not really essential and are more of a technical nature.

Let $A^+x \leq b^+$ denote the subsystem of $Ax \leq b$ consisting of those inequalities $a_ix \leq b_i$ for which there exists an $x_0 \in P$ satisfying $Ax_0 \leq b$ and $a_ix_0 < b_i$. Let $A^= x \leq b^=$ denote the remaining inequalities in $Ax \leq b$ (i.e., the so called implicit equalities).

Let x_0 be an arbitrary element of $P := \{x \in \mathbb{Q}^n \mid Ax \leq b\}$ and denote by $A^+(x_0)$ the matrix consisting of those rows a_i of A^+ for which $a_ix_0 = b_i$. Let r be the rank of $A^+(x_0)$. We say that $Ax \leq b$ is LSU′ in x_0 if $r = 0$ (i.e., there exists no row a_i of A^+ with $a_ix_0 = b_i$) or there exists an $r \times n$ submatrix M of $A^+(x_0)$ such that the g.c.d. of all $r \times r$ subdeterminants of M is 1. (Since the *faces* of P can be defined as sets of the form $\{x \in P \mid A^+(x) = A^+(x_0)\}$, LSU′ concerns faces of polyhedra.) LSU′ does not depend on implicit equalities and if P is full-dimensional and x_0 is a vertex of it, then P is LSU′ in x_0 iff it is LSU in x_0. If P is not full-dimensional and $Ax \leqslant b$ is LSU′ in x_0, then by choosing the description of the affine hull appropriately (i.e., assuming that $A^{=}$ is a basis of the lattice generated by $A^{=}$) it is also LSU in x_0. Thus LSU′-ness is essentially the extension of LSU-ness to arbitrary faces of an arbitrary polyhedron.

The following can be proved similarly to the proof of Theorem 1 in Section 3.

If $Ax \leqslant b$ is a TDI system, then $Ax \leqslant b$ is LSU′ in every $x \in P$.

In fact essentially this is the result which is proved in Section 3.

2. Hilbert bases

Giles and Pulleyblank [5], and Schrijver [8] (cf. [9, Theorem 22.5]) make clear that total dual integrality is closely related to a more algebraic notion, the notion of Hilbert bases.

If $V = \{v_1, \ldots, v_k\} \subset \mathbb{Z}^n$, then the *conic hull* of V is the set

$$\operatorname{cone}(V) := \left\{ \sum_{i=1}^{k} \lambda_i v_i \mid \lambda_i \in \mathbb{Q},\ \lambda_i \geqslant 0\ (i = 1, \ldots, k) \right\}.$$

We call cone (V) *pointed* if $\sum_{i=1}^{k} \lambda_i v_i = 0$, $\lambda_i \geqslant 0$ $(i = 1, \ldots, k)$ implies that $\lambda_1 = \cdots = \lambda_k = 0$. $H = \{h_1, \ldots, h_k\} \subset \mathbb{Z}^n$ is called a Hilbert basis, if for every $z \in \operatorname{cone}(H) \cap \mathbb{Z}^n$ there exist non-negative integers $\alpha_1, \ldots, \alpha_k$ such that $z = \sum_{i=1}^{k} \alpha_i h_i$. The following statement has been used in Giles and Pulleyblank [5], and Schrijver [8] (cf. Schrijver [9, Theorem 22.8], it can be proved easily).

(1) *$Ax \leqslant b$ is TDI if and only if for every face F of $\{x \in \mathbb{Q}^n \mid Ax \leqslant b\}$ the set of all rows a_i of A for which $x \in F$ implies $a_i x = b_i$, forms a Hilbert basis.*

This explains that any statement for Hilbert bases has an implication for TDI systems. For example the simple property of a Hilbert basis H that every integral vector in its linear hull is an integer combination of elements of H (cf. [1, Theorem 2.4b]) implies that if $Ax \leqslant b$ is TDI then it is locally unimodular in every vertex ($Ax \leqslant b$ is *locally unimodular* in a vertex x_0 if the g.c.d. of the $n \times n$ subdeterminants of $A(x_0)$ is 1; Hoffman and Oppenheim [7]). Local strong unimodularity will follow from the following stronger but still easy statement:

(2) *If $H \subset \mathbb{Z}^n$ is a Hilbert basis, $r = r(H)$, and* cone(H) *is pointed, then there exist vectors $h_1, \ldots, h_r$ in H such that $\{h_1, \ldots, h_r\}$ is linearly independent and is a Hilbert basis.*

We postpone the proof of (2) until Section 3. We finish this section with a trivial statement characterizing linearly independent Hilbert bases.

(3) *The following statements about* $\{h_1, \ldots, h_r\} \subset \mathbb{Z}^n$ *are equivalent*:

(i) $\{h_1, \ldots, h_r\}$ *is linearly independent, and it is a Hilbert basis.*

(ii) If $\sum_{i=1}^r \lambda_i h_i$ *is integral,* $\lambda_i \in \mathbb{Q}$, $0 \le \lambda_1 < 1$ $(i = 1, \ldots, r)$, *then* $\lambda_1 = \cdots = \lambda_r = 0$.

(iii) *The g.c.d. of the* $r \times r$ *subdeterminants of the matrix formed by* $h_1, \ldots, h_r$ *is* 1.

(Each of (i)⇒(ii), (ii)⇒(iii), (iii)⇒(i) can be proved straightforwardly.)

3. Proof of Theorem 1

First we prove (2). We need the following lemma.

Lemma. *Let H be a Hilbert basis such that* cone(H) *is full dimensional. Moreover let* $vx \ge 0$ *be a facet inducing inequality for* cone(H). *If the components of* v *are relatively prime integers, then there exists an* $h \in H$ *with* $vh = 1$.

Proof. Since g.c.d. $\{v_j \mid j = 1, \ldots, n\} = 1$ there exists an $x_0 \in \mathbb{Z}^n$ with $vx_0 = 1$. First we show that we may assume $x_0 \in \text{cone}(H)$. Suppose $x_0 \notin \text{cone}(H)$. Let $v_i x \ge 0$ $(v_i \in \mathbb{Z}^n$, $i = 1, \ldots, t)$ be the complete list of those facet inducing inequalities for cone(H) for which $v_i x_0 < 0$. Since no two facets can contain one another, there exists for each $i = 1, \ldots, k$ an $x_i \in \text{cone}(H)$ such that $vx_i = 0$ and $v_i x_i > 0$. We may suppose $x_i \in \mathbb{Z}^n (i = 1, \ldots, t)$. Clearly, $x := x_0 + \sum_{i=1}^t (-v_i x_0) x_i$ satisfies all the facet inequalities for cone(H). Moreover $vx = 1$. Hence the assumption $x_0 \in \text{cone}(H)$ was correct.

Since x_0 is integral, we have

$$x_0 = \sum_{i=1}^k \alpha_i h_i \quad \text{with } \alpha_i \ge 0, \alpha_i \in \mathbb{Z} \ \ (i = 1, \ldots, k).$$

Now

$$\sum_{i=1}^k \alpha_i (vh_i) = v\left(\sum_{i=1}^k \alpha_i h_i\right) = vx_0 = 1.$$

Since both α_i and vh_i are non-negative integers for each $i = 1, \ldots k$, there exists an $i \in \{1, \ldots, k\}$ with $vh_i = 1$.

Proof of (2). Let H be a Hilbert basis, and suppose cone(H) is pointed. We use induction on $r := r(H)$. We may assume that cone(H) is full dimensional, i.e. $r = n$. (Indeed, if not take a basis for the lattice of all integral vectors in the linear hull of H. One may use this basis to obtain a unimodular transformation mapping the linear hull of H onto the linear space of vectors having the last $n - r$ components zero.) It follows that there exists a facet inducing inequality $vx \ge 0$, v integral, and a vector, h_r say, in H such that $vh_r = 1$ (by the Lemma). The set $H_v := \{h \in H \mid vh = 0\}$ is a Hilbert basis, $r(H_v) = r - 1$, and cone(H_v) is pointed. By induction there exist vectors $h_1, \ldots, h_{r-1}$ in H_v such that $\{h_1, \ldots, h_{r-1}\}$ is a linearly independent Hilbert basis. Obviously $\{h_1, \ldots, h_r\}$ is linearly independent. It is a Hilbert basis too. Indeed, let w be an integral vector satisfying $w = \sum_{i=1}^r \lambda_i h_i$ for certain $\lambda_i \in \mathbb{Q}$ with $0 \le \lambda_i < 1$ for $i = 1, \ldots, r$. Then $\lambda_r = \sum_{i=1}^r \lambda_i (vh_i) = vw$ is an integer, so $\lambda_r = 0$. Since $\{h_1, \ldots, h_{r-1}\}$ is a linear independent Hilbert basis we get $\lambda_1 = \cdots = \lambda_{r-1} = 0$ ((3) (i)⇒(ii)). Now (3) (ii)⇒(i) yields that $\{h_1, \ldots, h_r\}$ is a Hilbert basis.

Finally we prove Theorem 1.

Proof of Theorem 1. Let $Ax \leqslant b$ be TDI, and let x_0 be a vertex of $P := \{x \in \mathbb{Q}^n \mid Ax \leqslant b\}$. Moreover let P be full-dimensional. Denote the set of rows of $A(x_0)$ by H. By (1) H is a Hilbert basis. Since P is full-dimensional, cone(H) is pointed. Now using (2), (3) (i)$\Rightarrow$(iii), and the fact that $r(H) = r(A(x_0)) = n$ we get that $A(x_0)$ has an $n \times n$ submatrix of determinant 1. So $Ax \leqslant b$ is LSU in x_0.

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