An Improved Approximation Algorithm for Minimum Size 2-Edge Connected Spanning Subgraphs

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Abstract. We give a $\frac{17}{12}$-approximation algorithm for the following NP-hard problem:

> Given a simple undirected graph, find a 2-edge connected spanning subgraph that has the minimum number of edges.

The best previous approximation guarantee was $\frac{3}{2}$. If the well known TSP $\frac{4}{3}$ conjecture holds, then there is a $\frac{4}{3}$-approximation algorithm. Thus our main result gets half-way to this target.

1 Introduction

Given a simple undirected graph, consider the problem of finding a 2-edge connected spanning subgraph that has the minimum number of edges. The problem is NP-hard, since the Hamiltonian cycle problem reduces to it. A number of recent papers have focused on approximation algorithms [1] for this and other related problems, [2]. We use the abbreviation 2-ECSS for 2-edge connected spanning subgraph.

Here is an easy 2-approximation algorithm for the problem:

> Take an ear decomposition of the given graph (see Section 2 for definitions), and discard all 1-ears (ears that consist of one edge). Then the resulting graph is 2-edge connected and has at most $2n-3$ edges, while the optimal subgraph has $\geq n$ edges, where n is the number of nodes.

[1] An *α-approximation algorithm* for a combinatorial optimization problem runs in polynomial time and delivers a solution whose value is always within the factor α of the optimum value. The quantity α is called the *approximation guarantee* of the algorithm.

R. E. Bixby, E. A. Boyd, and R. Z. Ríos-Mercado (Eds.): IPCO VI
LNCS 1412, pp. 126–136, 1998.

Khuller & Vishkin [8] were the first to improve on the approximation guarantee of 2. They gave a simple and elegant algorithm based on depth-first search that achieves an approximation guarantee of 1.5. In an extended abstract, Garg, Santosh & Singla [6] claimed to have a 1.25-approximation algorithm for the problem. No proof of this claim is available; on the other hand, there is no evidence indicating that achieving an approximation guarantee of 1.25 in polynomial time is impossible.

We improve Khuller & Vishkin's $\frac{18}{12}$-approximation guarantee to $\frac{17}{12}$. If the well known TSP $\frac{4}{3}$ conjecture holds, then there is a $\frac{4}{3}$-approximation algorithm, see Section 5. Thus our main result gets half-way to this target.

Let $G = (V, E)$ be the given simple undirected graph, and let n and m denote $|V|$ and $|E|$. Assume that G is 2-edge connected.

Our method is based on a matching-theory result of András Frank, namely, there is a good characterization for the minimum number of even-length ears over all possible ear decompositions of a graph, and moreover, an ear decomposition achieving this minimum can be computed efficiently, [4]. Recall that the 2-approximation heuristic starts with an arbitrary ear decomposition of G. Instead, if we start with an ear decomposition that maximizes the number of 1-ears, and if we discard all the 1-ears, then we will obtain the optimal solution. In fact, we start with an ear decomposition that maximizes the number of odd-length ears. Now, discarding all the 1-ears gives an approximation guarantee of 1.5 (see Proposition 8 below). To do better, we repeatedly apply "ear-splicing" steps to the starting ear decomposition to obtain a final ear decomposition such that the number of odd-length ears is the same, and moreover, the internal nodes of distinct 3-ears are nonadjacent. We employ two lower bounds to show that discarding all the 1-ears from the final ear decomposition gives an approximation guarantee of $\frac{17}{12}$. The first lower bound is the "component lower bound" due to Garg et al [6, Lemma 4.1], see Proposition 4 below. The second lower bound comes from the minimum number of even-length ears in an ear decomposition of G, see Proposition 7 below.

After developing some preliminaries in Sections 2 and 3, we present our heuristic in Section 4. Section 5 shows that the well known $\frac{4}{3}$ conjecture for the metric TSP implies that there is a $\frac{4}{3}$-approximation algorithm for a minimum-size 2-ECSS, see Theorem 18. Almost all of the results in Section 5 are well known, but we include the details to make the paper self-contained. Section 6 has two examples showing that our analysis of the heuristic is tight. Section 6 also compares the two lower bounds with the optimal value.

A Useful Assumption

For our heuristic to work, it is essential that the given graph be 2-node connected. Hence, in Section 4 of the paper where our heuristic is presented, we will assume that the given graph G is 2-node connected. Otherwise, if G is not 2-node connected, we compute the blocks (i.e., the maximal 2-node connected subgraphs) of G, and apply the algorithm separately to each block. We compute a 2-ECSS for each block, and output the union of the edge sets as the edge set of a 2-ECSS of G. The resulting graph has no cut edges since the subgraph found for each block has no cut edge, and moreover, the approximation guarantee for G is at most the maximum of the approximation guarantees for the blocks.

2 Preliminaries

Except in Section 5, all graphs are simple, that is, there are no loops nor multi-edges. A closed path means a cycle, and an open path means that all the nodes are distinct.

An *ear decomposition* of the graph G is a partition of the edge set into open or closed paths, $P_0 + P_1 + \ldots + P_k$, such that P_0 is the trivial path with one node, and each P_i $(1 \leq i \leq k)$ is a path that has both end nodes in $V_{i-1} = V(P_0) \cup V(P_1) \cup \ldots \cup V(P_{i-1})$ but has no internal nodes in V_{i-1}. A (closed or open) *ear* means one of the (closed or open) paths $P_0, P_1, \ldots, P_k$ in the ear decomposition, and for a nonnegative integer ℓ, an ℓ-*ear* means an ear that has ℓ edges. An ℓ-ear is called *even* if ℓ is an even number, otherwise, the ℓ-ear is called *odd*. (The ear P_0 is always even.) An *open* ear decomposition $P_0 + P_1 + \ldots + P_k$ is one such that all the ears $P_2, \ldots, P_k$ are open.

Proposition 1 (Whitney [12]).

(i) A graph is 2-edge connected if and only if it has an ear decomposition.
(ii) A graph is 2-node connected if and only if it has an open ear decomposition.

An *odd* ear decomposition is one such that every ear (except the trivial path P_0) has an odd number of edges. A graph is called *factor-critical* if for every node $v \in V(G)$, there is a perfect matching in $G - v$. The next result gives another characterization of factor-critical graphs.

Theorem 2 (Lovász [9], Theorem 5.5.1 in [10]). *A graph is factor-critical if and only if it has an odd ear decomposition.*

It follows that a factor-critical graph is necessarily 2-edge connected. An *open odd* ear decomposition $P_0 + P_1 + \ldots + P_k$ is an odd ear decomposition such that all the ears $P_2, \ldots, P_k$ are open.

Theorem 3 (Lovász & Plummer, Theorem 5.5.2 in [10]). *A 2-node connected factor-critical graph has an open odd ear decomposition.*

Let $\varepsilon(G)$ denote the minimum number of edges in a 2-ECSS of G. For a graph H, let $c(H)$ denote the number of (connected) components of H. Garg et al [6, Lemma 4.1] use the following lower bound on $\varepsilon(G)$.

Proposition 4. *Let $G = (V, E)$ be a 2-edge connected graph, and let S be a nonempty set of nodes such that the deletion of S results in a graph with $c = c(G - S) \geq 2$ components. Then $\varepsilon(G) \geq |V| + c - |S|$.*

Proof. Focus on an arbitrary component D of $G-S$ and note that it contributes $\geq |V(D)|+1$ edges to an optimal 2-ECSS, because every node in D contributes ≥ 2 edges, and at least two of these edges have exactly one end node in D. Summing over all components of $G-S$ gives the result. □

For a set of nodes $S \subseteq V$ of a graph $G=(V,E)$, $\delta(S)$ denotes the set of edges that have one end node in S and one end node in $V-S$. For the singleton node set $\{v\}$, we use the notation $\delta(v)$. For a vector $x: E \rightarrow \mathbb{R}$, $x(\delta(S))$ denotes $\sum_{e \in \delta(S)} x_e$.

3 Frank's Theorem and a New Lower Bound for ε

For a 2-edge connected graph G, let $\varphi(G)$ (or φ) denote the minimum number of even ears of length ≥ 2, over all possible ear decompositions. For example: $\varphi(G)=0$ if G is a factor-critical graph (e.g., G is an odd clique $K_{2\ell+1}$ or an odd cycle $C_{2\ell+1}$), $\varphi(G)=1$ if G is an even clique $K_{2\ell}$ or an even cycle $C_{2\ell}$, and $\varphi(G)=\ell-1$ if G is the complete bipartite graph $K_{2,\ell}$ ($\ell \geq 2$). The proof of the next result appears in [4], see Theorem 4.5 and Section 2 of [4].

Theorem 5 (A. Frank [4]). *Let $G=(V,E)$ be a 2-edge connected graph. An ear decomposition $P_0+P_1+\ldots+P_k$ of G having $\varphi(G)$ even ears of length ≥ 2 can be computed in time $O(|V|\cdot|E|)$.*

Proposition 6. *Let G be a 2-node connected graph. An* open *ear decomposition $P_0+P_1+\ldots+P_k$ of G having $\varphi(G)$ even ears of length ≥ 2 can be computed in time $O(|V|\cdot|E|)$.*

Proof. Start with an ear decomposition having $\varphi(G)$ even ears of length ≥ 2 (the ears may be open or closed). Subdivide one edge in each even ear of length ≥ 2 by adding one new node and one new edge. The resulting ear decomposition is odd. Hence, the resulting graph G' is factor critical, and also, G' is 2-node connected since G is 2-node connected. Apply Theorem 3 to construct an open odd ear decomposition of G'. Finally, in the resulting ear decomposition, "undo" the $\varphi(G)$ edge subdivisions to obtain the desired ear decomposition $P_0+P_1+\ldots+P_k$ of G. □

Frank's theorem gives the following lower bound on the minimum number of edges in a 2-ECSS.

Proposition 7. *Let $G=(V,E)$ be a 2-edge connected graph. Then $\varepsilon(G) \geq |V|+\varphi(G)-1$.*

Proof. Consider an arbitrary 2-ECSS of G. If this 2-ECSS has an ear decomposition with fewer than $\varphi(G)+1$ even ears, then we could add the edges of G not in the 2-ECSS as 1-ears to get an ear decomposition of G with fewer than $\varphi(G)+1$ even ears. Thus, every ear decomposition of the 2-ECSS has $\geq \varphi(G)+1$ even ears. Let $P_0+P_1+\ldots+P_k$ be an ear decomposition of the 2-ECSS, where $k \geq \varphi(G)$. It is easily seen that the number of edges in the 2-ECSS is $k+|V|-1 \geq \varphi(G)+|V|-1$. The result follows. □

The next result is not useful for our main result, but we include it for completeness.

Proposition 8. *Let $G = (V, E)$ be a 2-edge connected graph. Let $G' = (V, E')$ be obtained by discarding all the 1-ears from an ear decomposition $P_0 + P_1 + \ldots + P_k$ of G that has $\varphi(G)$ even ears of length ≥ 2. Then $|E'|/\varepsilon(G) \leq 1.5$.*

Proof. Let t be the number of internal nodes in the odd ears of $P_0 + P_1 + \ldots + P_k$. (Note that the node in P_0 is not counted by t.) Then, the number of edges contributed to E' by the odd ears is $\leq 3t/2$, and the number of edges contributed to E' by the even ears is $\leq \varphi + |V| - t - 1$. By applying Proposition 7 (and the fact that $\varepsilon(G) \geq |V|$) we get, $|E'|/\varepsilon(G) \leq (t/2 + \varphi + |V| - 1)/\max(|V|, \varphi + |V| - 1) \leq (t/2|V|) + (\varphi + |V| - 1)/(\varphi + |V| - 1) \leq 1.5$. □

4 Approximating ε via Frank's Theorem

For a graph H and an ear decomposition $P_0 + P_1 + \ldots + P_k$ of H, we call an ear P_i of length ≥ 2 *pendant* if none of the internal nodes of P_i is an end node of another ear P_j of length ≥ 2. In other words, if we discard all the 1-ears from the ear decomposition, then one of the remaining ears is called pendant if all its internal nodes have degree 2 in the resulting graph.

Let $G = (V, E)$ be the given graph, and let $\varphi = \varphi(G)$. Recall the assumption from Section 1 that G is 2-node connected. By an *evenmin ear decomposition* of G, we mean an ear decomposition that has $\varphi(G)$ even ears of length ≥ 2. Our method starts with an open evenmin ear decomposition $P_0 + P_1 + \ldots + P_k$ of G, see Proposition 6, i.e., for $2 \leq i \leq k$, every ear P_i has distinct end nodes, and the number of even ears is minimum possible. The method performs a sequence of "ear splicings" to obtain another (evenmin) ear decomposition $Q_0 + Q_1 + \ldots + Q_k$ (the ears Q_i may be either open or closed) such that the following holds:

Property (α)

(0) the number of even ears is the same in $P_0 + P_1 + \ldots + P_k$ and in $Q_0 + Q_1 + \ldots + Q_k$,
(1) every 3-ear Q_i is a pendant ear,
(2) for every pair of 3-ears Q_i and Q_j, there is no edge between an internal node of Q_i and an internal node of Q_j, and
(3) every 3-ear Q_i is open.

Proposition 9. *Let $G = (V, E)$ be a 2-node connected graph with $|V| \geq 4$. Let $P_0 + P_1 + \ldots + P_k$ be an open evenmin ear decomposition of G. There is a linear-time algorithm that given $P_0 + P_1 + \ldots + P_k$, finds an ear decomposition $Q_0 + Q_1 + \ldots + Q_k$ satisfying property (α).*

Proof. The proof is by induction on the number of ears. The result clearly holds for $k = 1$. Suppose that the result holds for $(j - 1)$ ears $P_0 + P_1 + \ldots + P_{j-1}$. Let

$Q'_0 + Q'_1 + \ldots + Q'_{j-1}$ be the corresponding ear decomposition that satisfies property (α). Consider the open ear P_j, $j \geq 2$. Let P_j be an ℓ-ear, $v_1, v_2, \ldots, v_\ell, v_{\ell+1}$. Possibly, $\ell = 1$. (So v_1 and $v_{\ell+1}$ are the end nodes of P_j, and $v_1 \neq v_{\ell+1}$.)

Let T denote the set of internal nodes of the 3-ears of $Q'_0 + Q'_1 + \ldots + Q'_{j-1}$. Suppose P_j is an ear of length $\ell \geq 2$ with exactly one end node, say, v_1 in T. Let $Q'_i = w_1, v_1, w_3, w_4$ be the 3-ear having v_1 as an internal node. We take $Q_0 = Q'_0, \ldots, Q_{i-1} = Q'_{i-1}, Q_i = Q'_{i+1}, \ldots, Q_{j-2} = Q'_{j-1}$. Moreover, we take Q_{j-1} to be the $(\ell+2)$-ear obtained by adding the last two edges of Q'_i to P_j, i.e., $Q_{j-1} = w_4, w_3, v_1, v_2, \ldots, v_\ell, v_{\ell+1}$, and we take Q_j to be the 1-ear consisting of the first edge $w_1 v_1$ of Q'_i. Note that the parities of the lengths of the two spliced ears are preserved, that is, Q_{j-1} is even (odd) if and only if P_j is even (odd), and both Q_j and Q'_i are odd. Hence, the number of even ears is the same in $P_0 + P_1 + \ldots + P_j$ and in $Q_0 + Q_1 + \ldots + Q_j$.

Now, suppose P_j has both end nodes v_1 and $v_{\ell+1}$ in T. If there is one 3-ear Q'_i that has both v_1 and $v_{\ell+1}$ as internal nodes (so $\ell \geq 2$), then we take Q_{j-1} to be the $(\ell + 2)$-ear obtained by adding the first edge and the last edge of Q'_i to P_j, and we take Q_j to be the 1-ear consisting of the middle edge $v_1 v_{\ell+1}$ of Q'_i. Also, we take $Q_0 = Q'_0, \ldots, Q_{i-1} = Q'_{i-1}, Q_i = Q'_{i+1}, \ldots, Q_{j-2} = Q'_{j-1}$. Observe that the number of even ears is the same in $P_0 + P_1 + \ldots + P_j$ and in $Q_0 + Q_1 + \ldots + Q_j$.

If there are two 3-ears Q'_i and Q'_h that contain the end nodes of P_j, then we take Q_{j-2} to be the $(\ell+4)$-ear obtained by adding the last two edges of both Q'_i and Q'_h to P_j, and we take Q_{j-1} (similarly, Q_j) to be the 1-ear consisting of the first edge of Q'_i (similarly, Q'_h). (For ease of description, assume that if a 3-ear has exactly one end node v of P_j as an internal node, then v is the second node of the 3-ear.) Also, assuming $i < h$, we take $Q_0 = Q'_0, \ldots, Q_{i-1} = Q'_{i-1}, Q_i = Q'_{i+1}, \ldots, Q_{h-2} = Q'_{h-1}, Q_{h-1} = Q'_{h+1}, \ldots, Q_{j-3} = Q'_{j-1}$. Again, observe that the number of even ears is the same in $P_0+P_1+\ldots+P_j$ and in $Q_0+Q_1+\ldots+Q_j$.

If the end nodes of P_j are disjoint from T, then the proof is easy (take $Q_j = P_j$). Also, if P_j is a 1-ear with exactly one end node in T, then the proof is easy (take $Q_j = P_j$).

The proof ensures that in the final ear decomposition $Q_0 + Q_1 + \ldots + Q_k$, every 3-ear is pendant and open, and moreover, the internal nodes of distinct 3-ears are nonadjacent. We leave the detailed verification to the reader. Therefore, the ear decomposition $Q_0 + Q_1 + \ldots + Q_k$ satisfies property (α). □

Remark 10. In the induction step, which applies for $j \geq 2$ (but not for $j = 1$), it is essential that the ear P_j is open, though Q'_i (and Q'_h) may be either open or closed. Our main result (Theorem 12) does not use part (3) of property (α).

Our approximation algorithm for a minimum-size 2-ECSS computes the ear decomposition $Q_0 + Q_1 + \ldots + Q_k$ satisfying property (α), starting from an open evenmin ear decomposition $P_0 + P_1 + \ldots + P_k$. (Note that $Q_0 + Q_1 + \ldots + Q_k$ is an evenmin ear decomposition.) Then, the algorithm discards all the edges in 1-ears. Let the resulting graph be $G' = (V, E')$. G' is 2-edge connected by Proposition 1.

Let T denote the set of internal nodes of the 3-ears of $Q_0+Q_1+\ldots+Q_k$, and let $t=|T|$. (Note that the node in Q_0 is not counted by t.) Property (α) implies that in the subgraph of G induced by T, $G[T]$, every (connected) component has exactly two nodes. Consider the approximation guarantee for G', i.e., the quantity $|E'|/\varepsilon(G)$.

Lemma 11. *$\varepsilon(G) \geq 3t/2$.*

Proof. Apply Proposition 4 with $S=V-T$ (so $|S|=n-t$) and $c=c(G-S)=t/2$ to get $\varepsilon(G)\geq n-(n-t)+(t/2)$. □

Theorem 12. *Given a 2-edge connected graph $G=(V,E)$, the above algorithm finds a 2-ECSS $G'=(V,E')$ such that $|E'|/\varepsilon(G)\leq \frac{17}{12}$. The algorithm runs in time $O(|V|\cdot|E|)$.*

Proof. By the previous lemma and Proposition 7,

$$\varepsilon(G) \geq \max(n+\varphi(G)-1,\ 3t/2)\ .$$

We claim that

$$|E'| \leq \frac{t}{4}+\frac{5(n+\varphi(G)-1)}{4}\ .$$

To see this, note that the final ear decomposition $Q_0+Q_1+\ldots+Q_k$ satisfies the following: (i) the number of edges contributed by the 3-ears is $3t/2$; (ii) the number of edges contributed by the odd ears of length ≥ 5 is $\leq 5q/4$, where q is the number of internal nodes in the odd ears of length ≥ 5; and (iii) the number of edges contributed by the even ears of length ≥ 2 is $\leq \varphi(G)+(n-t-q-1)$, since there are $\varphi(G)$ such ears and they have a total of $(n-t-q-1)$ internal nodes. (The node in Q_0 is not an internal node of an ear of length ≥ 1.)

The approximation guarantee follows since

$$\begin{aligned}\frac{|E'|}{\varepsilon(G)} &\leq \frac{t/4+5(n+\varphi(G)-1)/4}{\varepsilon(G)}\\ &\leq \frac{t/4+5(n+\varphi(G)-1)/4}{\max(n+\varphi(G)-1,\ 3t/2)}\\ &\leq \frac{t}{4}\,\frac{2}{3t}+\frac{5(n+\varphi(G)-1)}{4}\,\frac{1}{n+\varphi(G)-1}\\ &=\frac{17}{12}\ .\end{aligned}$$

□

5 Relation to the TSP $\frac{4}{3}$ Conjecture

This section shows that the well known $\frac{4}{3}$ conjecture for the metric TSP (due to Cunningham (1986) and others) implies that there is a $\frac{4}{3}$-approximation algorithm for a minimum-size 2-ECSS, see Theorem 18. Almost all of the results in this section are well known, except possibly Fact 13, see [1,3,5,7,11,13]. The details are included to make the paper self-contained.

In the *metric TSP* (traveling salesman problem), we are given a complete graph $G' = K_n$ and edge costs c' that satisfy the triangle inequality ($c'_{vw} \leq c'_{vu} + c'_{uw}, \forall v, w, u \in V$). The goal is to compute c'_{TSP}, the minimum cost of a Hamiltonian cycle.

Recall our 2-ECSS problem: Given a simple graph $G = (V, E)$, compute $\varepsilon(G)$, the minimum size of a 2-edge connected spanning subgraph. Here is the multiedge (or uncapacitated) version of our problem. Given $G = (V, E)$ as above, compute $\mu(G)$, the minimum size (counting multiplicities) of a 2-edge connected spanning submultigraph $H = (V, F)$, where F is a multiset such that $e \in F \Longrightarrow e \in E$. (To give an analogy, if we take $\varepsilon(G)$ to correspond to the f-factor problem, then $\mu(G)$ corresponds to the f-matching problem.)

Fact 13. *If G is a 2-edge connected graph, then $\mu(G) = \varepsilon(G)$.*

Proof. Let $H = (V, F)$ give the optimal solution for $\mu(G)$. If H uses two copies of an edge vw, then we can replace one of the copies by some other edge of G in the cut given by $H - \{vw, vw\}$. In other words, if S is the node set of one of the two components of $H - \{vw, vw\}$, then we replace one copy of vw by some edge from $\delta_G(S) - \{vw\}$. □

Remark 14. The above is a lucky fact. It *fails* to generalize, both for minimum-cost (rather than minimum-size) 2-ECSS, and for minimum-size k-ECSS, $k \geq 3$.

Given an n-node graph $G = (V, E)$ together with edge costs c (possibly c assigns unit costs), define its *metric completion* G', c' to be the complete graph $K_n = G'$ with c'_{vw} ($\forall\, v, w \in V$) equal to the minimum-cost of a v-w path in G, c.

Fact 15. *Let G be a 2-edge connected graph, and let c assign unit costs to the edges. The minimum cost of the TSP on the metric completion of G, c, satisfies $c'_{TSP} \geq \mu(G) = \varepsilon(G)$.*

Proof. Let T be an optimal solution to the TSP. We replace each edge $vw \in E(T) - E(G)$ by the edges of a minimum-cost v-w path in G, c. The resulting multigraph H is obviously 2-edge connected, and has $c'_{TSP} = c(H) \geq \mu(G)$. □

Here is the *subtour* formulation of the TSP on G', c', where $G' = K_n$. This gives an integer programming formulation, using the subtour elimination constraints. There is one variable x_e for each edge e in G'.

$$
\begin{array}{llll}
c'_{TSP} = \text{minimize} & c' \cdot x & & \\
\text{subject to} & x(\delta(v)) & = 2, & \forall v \in V \\
 & x(\delta(S)) & \geq 2, & \forall S \subset V,\ \emptyset \neq S \neq V \\
 & x & \geq 0, & \\
 & x & \in \mathbb{Z}\ . &
\end{array}
$$

The *subtour LP* (linear program) is obtained by removing the integrality constraints, i.e., the x-variables are nonnegative reals rather than nonnegative integers. Let z_{ST} denote the optimal value of the subtour LP. Note that z_{ST} is computable in polynomial time, e.g., via the Ellipsoid method. In practice, z_{ST} may be computed via the Held-Karp heuristic, which typically runs fast.

Theorem 16 (Wolsey [13]). *If c' is a metric, then $c'_{TSP} \leq \frac{3}{2} z_{ST}$.*

TSP $\frac{4}{3}$ Conjecture. If c' is a metric, then $c'_{TSP} \leq \frac{4}{3} z_{ST}$.

To derive the lower bound $z_{ST} \leq \varepsilon(G)$, we need a result of Goemans & Bertsimas on the subtour LP, [7, Theorem 1]. In fact, a special case of this result that appeared earlier in [11, Theorem 8] suffices for us.

Proposition 17 (Parsimonious property [7]). *Consider the TSP on $G' = (V, E'), c'$, where $G' = K_{|V|}$. Assume that the edge costs c' form a metric, i.e., c' satisfies the triangle inequality. Then the optimal value of the subtour LP remains the same even if the constraints $\{x(\delta(v)) = 2, \forall v \in V\}$ are omitted.*

Note that this result does not apply to the subtour integer program given above.

Let z_{2CUT} denote the optimal value of the LP obtained from the subtour LP by removing the constraints $x(\delta(v)) = 2$ for all nodes $v \in V$. The above result states that if c' is a metric, then $z_{ST} = z_{2CUT}$. Moreover, for a 2-edge connected graph G and unit edge costs $c = \mathbb{1}$, we have $z_{2CUT} \leq \mu(G) = \varepsilon(G)$, since $\mu(G)$ is the optimal value of the integer program whose LP relaxation has optimal value z_{2CUT}. (Here, z_{2CUT} is the optimal value of the LP on the metric completion of G, c.) Then, by the parsimonious property, we have $z_{ST} = z_{2CUT} \leq \varepsilon(G)$. The main result in this section follows.

Theorem 18. *Suppose that the TSP $\frac{4}{3}$ conjecture holds. Then*

$$z_{ST} \leq \varepsilon(G) \leq c'_{TSP} \leq \frac{4}{3} z_{ST} \quad .$$

A $\frac{4}{3}$-approximation of the minimum-size 2-ECSS is obtained by computing $\frac{4}{3} z_{ST}$ on the metric completion of G, c, where $c = \mathbb{1}$.

The Minimum-Cost 2-ECSS Problem

Consider the weighted version of the problem, where each edge e has a nonnegative cost c_e and the goal is to find a 2-ECSS (V, E') of the given graph $G = (V, E)$ such that the cost $c(E') = \sum_{e \in E'} c_e$ is minimum. Khuller & Vishkin [8] pointed out that a 2-approximation guarantee can be obtained via the weighted matroid intersection algorithm. When the edge costs satisfy the triangle inequality (i.e., when c is a metric), Frederickson and Ja'Ja' [5] gave a 1.5-approximation algorithm, and this is still the best approximation guarantee known. In fact, they proved that the TSP tour found by the Christofides heuristic achieves an approximation guarantee of 1.5. Simpler proofs of this result based on Theorem 16 were found later by Cunningham (see [11, Theorem 8]) and by Goemans & Bertsimas [7, Theorem 4].

Consider the minimum-cost 2-ECSS problem on a 2-edge connected graph $G = (V, E)$ with nonnegative edge costs c. Let the minimum cost of a simple 2-ECSS and of a multiedge 2-ECSS be denoted by c_ε and c_μ, respectively. Clearly, $c_\varepsilon \geq c_\mu$. Even for the case of arbitrary nonnegative costs c, we know of no example where $\frac{c_\mu}{z_{ST}} > \frac{7}{6}$. There is an example G, c with $\frac{c_\mu}{z_{ST}} \geq \frac{7}{6}$. Take two copies of K_3, call them C_1, C_2, and add three disjoint length-2 paths P_1, P_2, P_3 between C_1 and C_2 such that each node of $C_1 \cup C_2$ has degree 3 in the resulting graph G. In other words, G is obtained from the triangular prism $\overline{C_6}$ by subdividing once each of the 3 "matching edges". Assign a cost of 2 to each edge in $C_1 \cup C_2$, and assign a cost of 1 to the remaining edges. Then $c_\varepsilon = c_\mu = 14$, as can be seen by taking 2 edges from each of C_1, C_2, and all 6 edges of $P_1 \cup P_2 \cup P_3$. Moreover, $z_{ST} \leq 12$, as can be seen by taking $x_e = 1/2$ for each of the 6 edges e in $C_1 \cup C_2$, and taking $x_e = 1$ for the remaining 6 edges e in $P_1 \cup P_2 \cup P_3$.

6 Conclusions

Our analysis of the heuristic is (asymptotically) tight. We give two example graphs. Each is an n-node Hamiltonian graph $G = (V, E)$, where the heuristic (in the worst case) finds a 2-ECSS $G' = (V, E')$ with $17n/12 - \Theta(1)$ edges. The first example graph, G, is constructed by "joining" many copies of the following graph H: H consists of a 5-edge path $u_0, u_1, u_2, u_3, u_4, u_5$, and 4 disjoint edges $v_1w_1, v_2w_2, v_3w_3, v_4w_4$. We take q copies of H and identify the node u_0 in all copies, and identify the node u_5 in all copies. Then we add all possible edges u_iv_j, and all possible edges u_iw_j, i.e., we add the edge set of a complete bipartite graph on all the u-nodes and all the v-nodes, and we add the edge set of another complete bipartite graph on all the u-nodes and all the w-nodes. Finally, we add 3 more nodes u'_1, u'_2, u'_3 and 5 more edges to obtain a 5-edge cycle $u_0, u'_1, u'_2, u'_3, u_5, u_0$. Clearly, $\varepsilon(G) = n = 12q + 5$. If the heuristic starts with the closed 5-ear $u_0,u'_1,u'_2,u'_3,u_5,u_0$, and then finds the 5-ears u_0,u_1,u_2,u_3,u_4,u_5 in all the copies of H, and finally finds the 3-ears $u_0v_jw_ju_5$ $(1 \leq j \leq 4)$ in all the copies of H, then we have $|E'| = 17q + 5$.

Here is the second example graph, $G = (V, E)$. The number of nodes is $n = 3 \times 5^q$, and $V = \{0, 1, 2, ..., 3 \times 5^q - 1\}$. The "first node" 0 will also be denoted 3×5^q. The edge set E consists of (the edge set of) a Hamiltonian cycle together with (the edge sets of) "shortcut cycles" of lengths $n/3, n/(3 \times 5), n/(3 \times 5^2), \ldots, 5$. In detail, $E = \{i(i+1) \mid \forall 0 \leq i \leq q-1\} \cup \{(3 \times 5^j \times i)(3 \times 5^j \times (i+1)) \mid \forall 0 \leq j \leq q-1, 0 \leq i \leq 5^{q-j}-1\}$. Note that $|E| = 3 \times 5^q + 5^q + 5^{q-1} + ... + 5 = (17 \times 5^q - 5)/4$. In the worst case, the heuristic initially finds 5-ears, and finally finds 3-ears, and so the 2-ECSS (V, E') found by the heuristic has all the edges of G. Hence, we have $|E'|/\varepsilon(G) = |E|/n = 17/12 - 1/(12 \times 5^{q-1})$.

How do the lower bounds in Proposition 4 (call it L_c) and in Proposition 7 (call it L_φ) compare with ε? Let n denote the number of nodes in the graph. There is a 2-node connected graph such that $\varepsilon/L_\varphi \geq 1.5-\Theta(1)/n$, i.e., the upper bound of Proposition 8 is tight. There is another 2-edge connected (but not 2-node connected) graph such that $\varepsilon/L_c \geq 1.5-\Theta(1)/n$ and $\varepsilon/L_\varphi \geq 1.5-\Theta(1)/n$. Among 2-node connected graphs, we have a graph with $\varepsilon/L_c \geq 4/3-\Theta(1)/n$, but we do not know whether there exist graphs that give higher ratios. There is a 2-node connected graph such that $\varepsilon/\max(L_c, L_\varphi) \geq 5/4-\Theta(1)/n$, but we do not know whether there exist graphs that give higher ratios.

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