On Combinatorial Properties of Binary Spaces

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Abstract. A *binary clutter* is the family of inclusionwise minimal supports of vectors of affine spaces over GF(2). Binary clutters generalize various objects studied in Combinatorial Optimization, such as paths, Chinese Postman Tours, multiflows and one-sided circuits on surfaces. The present work establishes connections among three matroids associated with binary clutters, and between any of them and the binary clutter. These connections are then used to compare well-known classes of binary clutters; to provide polynomial algorithms which either confirm the membership in subclasses, or provide a forbidden clutter-minor; to reformulate and generalize a celebrated conjecture of Seymour on ideal binary clutters in terms of multiflows in matroids, and to exhibit new cases of its validity.

1 Introduction

A *clutter* is a family of subsets of a finite ground set S, none of which contains any other. We will also suppose that every $e \in S$ is contained in at least one set of the family. A clutter $\mathcal{A}$ is *ideal* (has the max-flow-min-cut property) if its *blocking polyhedron*, that is the polyhedron $\{x \in \mathbb{R}^n_+ : x(A) \geq 1 \text{ for all } A \in \mathcal{A}\}$, has only integer vertices. Clearly, a clutter is ideal precisely when the set of vertices of its blocking polyhedron are exactly the characteristic vectors of its blocker. The *blocking clutter* or *blocker* of the clutter $\mathcal{A} \subseteq 2^S$, denoted by $b(\mathcal{A})$, is defined to be the family of minimal elements of $\{B \subseteq S : |B \cap A| \geq 1 \text{ for all } A \in \mathcal{A}\}$. $b(b(\mathcal{A})) = \mathcal{A}$ [4].

When it causes no confusion, we will speak interchangeably about a subset of S and its incidence vector considered as a member of $GF(2)^S$; similarly, a family of subsets and a 0-1 matrix are interchangeable, as well as the mod 2 sum of vectors and their "symmetric difference". Both operations will be simply denoted by a '+' sign. The linear independence, rank, span, orthogonality etc. is understood over $\mathrm{GF}(2)^S$. This abuse of notation corresponds well to the purposes of the paper: we will be studying the *combinatorial* properties of *affine subspaces*.

A *binary* clutter is the family of (inclusionwise) minimal supports of elements of affine subspaces (shifts of linear subspaces) of vector spaces over GF(2). Equivalently, given $A_1, A_2, \ldots, A_k \subseteq 2^S$, linearly independent with ($k \leq |S|$), a *binary* clutter is the set of all minimal supports of elements obtained by summing an odd number of vectors from $\{A_1, A_2, \ldots, A_k\}$. For results on binary spaces and binary clutters see in [21], [23], [7], [8]. It is easy to see that the blocker of a binary clutter $\mathcal{A}$ is $b(\mathcal{A}) = \{B \subseteq S : |B \cap A| \equiv 1 \bmod 2 \text{ for all } A \in \mathcal{A}, B \text{ minimal}\}$. It follows that $b(\mathcal{A})$ is also a binary clutter (see [21]; also see Section 2 below). The result of *deleting* or *contracting* $e \in S$ in a binary clutter $\mathcal{H}$ is denoted by $\mathcal{H}\backslash e$, and $\mathcal{H}/e$ respectively, and defined by $\mathcal{H}\backslash e := \{A \in \mathcal{H} : e \notin A\}$, and $\mathcal{H}/e :=$ the minimal elements of $\{A - \{e\} : A \in S(\mathcal{H})\}$. $\mathcal{H}\backslash X/Y$ denotes the the result of deleting the elements of X and then contracting those of Y and is called a *minor*. The order in which the deletions and contractions are effectuated does not affect the resulting minor. It is easy to see that $b(\mathcal{H}\backslash e) = b(\mathcal{H})/e$, $b(\mathcal{H}/e) = b(\mathcal{H})\backslash e$.

As with matroids, a class of clutters is *minor-closed* if all minors of every clutter in the class are also in the class. It is easy to see that the class of ideal binary clutters is minor-closed. The *core* of a minimal non-ideal clutter $\mathcal{H}$ is the family of its minimal cardinality elements.

For us, *matroid* will mean "binary matroid" — one representable over $GF(2)$ — and will be a pair $M = (S, \mathcal{C})$, where $\mathcal{C}$ is the set of its circuits. The linear space generated by $\mathcal{C}$ is called the *cycle-space* of M. The linear space orthogonal to $\mathcal{C}$ is called *cocycle space*, its elements are the *cocycles*, its (inclusionwise) minimal elements are the cocircuits. The matroid $M^* = (S, \mathcal{C}^*)$, where $\mathcal{C}^*$ is the set of cocircuits of M is the *dual matroid* of M If the columns of a matrix represent a binary matroid, then the rows generate the cocycle space of the matroid. The ground-set S of a matroid M or of a clutter $\mathcal{H}$ will be referred to as $S(M)$ or $S(\mathcal{H})$. The rank function of the matroid is denoted by $r := r(M) := r(S)$.

For more on the basic notions and simple facts related to binary clutters and matroids, in the introductions of [21], [23], [24], [25], [7], [8]. We will focus next on some concepts, used throughout the paper, which we wish to clarify and consolidate.

Three Matroids Associated with each Binary Clutter. Let $\mathcal{H}$ be a binary clutter. We define the *down matroid* of $\mathcal{H}$ to be $M_0(\mathcal{H}) := (S, \mathcal{C}_0)$, where $\mathcal{C}_0$ consists of the minimal non-empty subsets of S that can be written as the sum of an even number of elements of $\mathcal{H}$. The clutter $\mathcal{H}$ is called a *lift* of $M_0(\mathcal{H})$. The *up matroid* of $\mathcal{H}$ is defined by $M_1(\mathcal{H}) := (S, \mathcal{C}_1)$, where $\mathcal{C}_1$ is the set of minimal non-empty elements of the linear space generated by $\mathcal{H}$.

(1.1) *If $\emptyset \neq \mathcal{H} \neq \{\emptyset\}$, then $\mathcal{C}_0$ generates a subspace of corank $= r(C_1) - r(C_0) = 1$ of $\mathcal{C}_1$, and $\mathcal{H} = \mathcal{C}_1 \backslash \mathcal{C}_0$.*

We deleted for this volume the proofs of the three simple claims of this section

One gets M_1 from M_0 by "undeleting" and contracting an element, and the matroid one gets in the intermediate step is uniquely determined:

(1.2) *If $\emptyset \neq \mathcal{H} \neq \{\emptyset\}$, there exists a uniquely determined matroid M_2, and $t \in S(M_2)$ such that $M_2(\mathcal{A}) \setminus t = M_0(\mathcal{A})$, $M_2(\mathcal{A}) / t = M_1(\mathcal{A})$.*

M_2 will be called the *port matroid* of $\mathcal{H}$. Clearly, $M_2(\mathcal{H}) = (S \cup t, \mathcal{C}_2)$, $(t \notin S)$ is connected: since we assumed that every $e \in S$ is contained in some $E \in \mathcal{H}$, and since $E \cup t \in \mathcal{C}_2$ contains both e and t, every $e \in S$ is in the same component of $M_2(\mathcal{H})$ as t. Given our assumption that $\emptyset \neq \mathcal{H} \neq \{\emptyset\}$, M_0, M_1, M_2 are of course uniquely determined by $\mathcal{H}$. Conversely, the pair (M_0, M_1) or the pair (M_2, t), $t \in S(M_2)$ uniquely determines $\mathcal{H}$. We will refer to any $A \in \mathcal{A}$ as a *t-join*, of M_0, or a *t-port* of M_2. A lift of a matroid is in fact the same as a t-join for some t. A set $B \in b(\mathcal{A})$ will be called a *t-cut* of M_0. In Section 2 we justify this terminology.

(1.3) *Let $\mathcal{A} \subseteq 2^S$ be a binary clutter, and let $\mathcal{B}$ be its blocker. For $e \in S$, if $\mathcal{A} \backslash e \neq \emptyset$ and $\mathcal{A}/e \neq \{\emptyset\}$, then $M_i(\mathcal{A} \backslash e) = M_i(\mathcal{A}) \backslash e$ and $M_i(\mathcal{A}/e) = M_i(\mathcal{A})/e$, $(i = 0, 1, 2)$. Moreover, the matroids $M_i(\mathcal{A})$, $M_i(\mathcal{B})$ $(i = 0, 1, 2)$ relate as follows:*

(i) $M_2(\mathcal{A}) \setminus t = M_0(\mathcal{A}), M_2(\mathcal{A}) / t = M_1(\mathcal{A})$.
(ii) $M_2(\mathcal{B}) = M_2^*(\mathcal{A})$.
(iii) $M_0(\mathcal{B}) = M_1^*(\mathcal{A})$.

We make the convention that the down, up and port matroids of the clutter $\mathcal{A} = \emptyset$ is an arbitrary triple of matroids $M_0 = (S, \mathcal{C}_0)$, $M_1 = (S, \mathcal{C}_1)$, $M_2 = (S \cup \{t\}, \mathcal{C}_2)$, where t is a coloop in M_2 and $M_1 = M_2/t$. Accordingly, the down, up and port matroid of $\mathcal{B} = \{\emptyset\}(= b(\mathcal{A}))$ is any triple where t is a loop in M_2 and $M_0 = M_2\backslash t$.

It follows from (ii) that $B \in b(\mathcal{A})$ is also a t-port of $M_2^*(\mathcal{A})$, and from (iii) that $A \in \mathcal{A}$ is a t-cut of $M_1^*(\mathcal{A})$, etc, (see Section 2 for an account of similar remarks), and furthermore that, given our convension above, that the claims of (1.3) hold for the blocking pair of clutters $\emptyset$ and $\{\emptyset\}$ as well.

Further Preliminaries. Let us state now some further preliminaries, among them an important open problem about binary clutters. $\mathcal{F}_7$ will denote the clutter consisting of the lines of Fano plane. F_7 will denote the Fano matroid, that is the matroid whose cycle-space is generated by the lines of the Fano plane. AG(2,3) is the matroid represented by the eight 4-dimensional vectors having 1 as last coordinate. $\mathcal{K}_5$ denotes the set of odd circuits of the complete graph K_5. The cycle matroids of K_5 and $K_{3,3}$ will also be denoted, respectively, by K_5 and $K_{3,3}$. R_{12} is the "3-sum" of $K_{3,3}$ and its dual $K_{3,3}^*$, see [25]. S_8 is also defined in [25]. $M_0(\mathcal{F}_7) = F_7^*$, $M_1(\mathcal{F}_7) = F_7$, $M_2(\mathcal{F}_7) = AG(2,3)$. $M_0(\mathcal{K}_5) = R_{10}$. ($R_{10}$ is defined to be the matroid represented by the matrix whose columns are all ten 0-1 vectors of length five having three 1's, see [24]). $M_1(\mathcal{K}_5) = K_5$.

Seymour's Conjecture: *A binary clutter is ideal if and only if it contains none of* $\mathcal{K}_5$, $b(\mathcal{K}_5)$ *or* $\mathcal{F}_7$ *as minors.*

Compare the three excluded minors of this conjecture to the infinite set of minimal non-ideal clutters, see [3], [15].

We will use the notation $\mathcal{S} := \{\mathcal{K}_5, b(\mathcal{K}_5), \mathcal{F}_7\}$ throughout the paper. Seymour has stated several variants of his conjecture ([23] page 200, [25] (9.2), (11.2)) whose equivalence can be easily understood using the equivalence of binary clutters to ports and their relation to multiflows (see Section 2). Gerards [8] surveys a wide range of multiflow theorems which are special cases of Seymour's conjecture.

In Section 2 we present the most well-known particular classes of binary clutters. The basis of the classification is the down up and port matroid. We next axiomatize binary clutters based on these.

In Section 3 we solve the recognition problem for path, t-join, t-cut, one-sided path, odd circuit and signing clutters, and some sub- and superclasses of these. *All of these classes can be defined either in terms of their down up or port matroids.*

Defining a refinement of "ideal clutters", we arrive at a property which is easier to handle with the well-known sum operations:

We *decompose binary clutters in three different ways applying Seymour's* 1-, 2- *and* 3-*sums to the down-, up- and port matroids*[3]. Excluded clutter-minor characterizations follow for the subclasses, generalizing Seymour's characterization [22] of path clutters of graphs. Polynomial recognition algorithms [4] — which either confirm membership in the subclasses, or provide a forbidden clutter-minor — will be reduced to matroid recognition algorithms.

[3] The 3-sum applied to the up matroid is not the same as the 3-sum applied to the down matroid M of the blocker: it correspods to a "dual 3-sum" on M.

[4] We suppose that a clutter is given as the set of minimum supports of equations $Mx \equiv t$, that is, as a t-port of some binary matroid M ($t \in S(M)$) *whose representation is known.* If more generally, the binary clutter is given with a "containment oracle", a beautiful recent result of Coullard and Hellerstein [2] reconstructs a representation of the port matroid of the clutter (see Section 3).

Section 4 is devoted to ideal binary clutters. A further tool is introduced: metrics, which provide conditions for multiflow problems. With the help of general conditions for matroid flow feasibility ([17], [19], [14]) we extend the connection between the Cut Condition and ideal clutters (largely exploited in [25], [7], [8]), refine Seymour's conjecture, and prove it for a new set of cases.

2 Representations of Binary Clutters

Sections 2 and 3 wish to provide basic facts about binary clutters in a similar way as introductions to matroid theory do: first several equivalent definitions are presented by analogy with various notions of graph theory, then subclasses are defined and their interrelations are studied. As for matroids, it will be comfortable to pick-up always the most suitable definition, and to be able to switch easily between them. Proofs of the stated claims in these sections are not difficult, and are, for the most part, equivalent to well-known facts (see [12], [21], [23], [7]).

The Most Well-Known Classes of Binary Clutters. All of these originate in graph theory. Let $G = (V, E)$ be an undirected graph. For $r, s \in V$, the collection of (r, s) paths and its blocker, the collection of minimal (r, s) cuts, are binary clutters, special cases of the following. Let $t : V \mapsto \{0, 1\}$. A subset A of E is a *t-join* if $\deg_{G(A)}(v) \equiv t(v) \, mod \, 2$. If $X \subseteq V(G)$ and $t(X)$ is odd, the cut $\{xy \in E(G) : x \in X, y \in V(G) \backslash X\}$ is called a t-cut. The collection of minimal t-joins of G and that of minimal t-cuts of G is a blocking pair of binary clutters. We say that $\mathcal{H}$ is a *join* (*cut*) clutter if there exist a graph G and $t : V \mapsto \{0, 1\}$ such that the elements of $\mathcal{H}$ are the minimal t-joins (t-cuts) of G. When $\sum_{v \in V} t(v) = 2$, we say that $\mathcal{H}$ is *path* clutter; its blocker a *one-cut* clutter.

The collection of all odd circuits of an undirected graph and its blocker are binary clutters. More generally, a *signed* graph is (G, R), where $G = (V, E)$ is an undirected graph, and $R \subseteq E(G)$. We define the pair of binary clutters

$$\mathcal{A}(G, R) : = \{A : |A \cap R| \text{ is odd, } A \text{ a circuit of } G\}.$$
$$\mathcal{B}(G, R) : = \{B = R + Q : Q \text{ is a cocycle of } G, \ B \text{ minimal non-empty}\}.$$

$\mathcal{A}(G, R)$ is called an *odd circuit* clutter, and $\mathcal{B}(G, R)$, the blocker of $\mathcal{A}(G, R)$, a *signing* clutter. It is easy to see that $\mathcal{A}(G, R)$ and $\mathcal{B}(G, R)$ form a blocking pair of clutters. Signed graphs and the related binary clutters are studied by Gerards in [7], [8], who gives a topological meaning to particular cases of binary clutters [8].

For the topological definitions, see for instance [8]. If a graph is embedded in a non-orientable surface of genus k the *one-sided circuits* of the graph form a binary clutter. If, for a binary clutter $\mathcal{H}$, there exists a graph G and an embedding in a non-orientable surface of genus k such that $\mathcal{H}$ is the set of one-sided circuits of G, then $\mathcal{H}$ will be called a k-clutter. If $\mathcal{H}$ is a k-clutter and $R \in b(\mathcal{H})$, then $\mathcal{H} = \mathcal{A}(G, R)$, so k-clutters are signed graph clutters for every k.

If a graph G is embedded in a compact surface, then a cycle of G is called 0-*homologic* if it is the symmetric difference of face-bounding circuits of G. Two cycles C_1 and C_2 are called *homologic* if $C_1 + C_2$ is 0-homologic. Clearly, the homology relation of cycles is an equivalence relation, and the minimal cycles of fixed non-0 homology type form a binary clutter. If G is embedded in a non-orientable surface of this clutter will be called a homology clutter. If the genus of the surface is k and the fixed homology type is *orienting* (that is "goes through all the cross-caps"), we say that it is a k-homology clutter. We know already that k-homology clutters are all signing clutters. It is easy to prove that $\mathcal{H}$ *is a k-clutter if and only if* $b(\mathcal{H})$ *is a k-homology clutter.* (Indeed, with an arbitrary $R \in b(\mathcal{H})$, as noticed above $\mathcal{H} = \mathcal{A}(G,R)$, $b(\mathcal{H}) = \mathcal{B}(G,R)$. Let $\bar{G}$ be the surface dual of G. R is a cycle of $\bar{G}$, because every face of G (and the faces generate the cuts of $\bar{G}$) is in the down-space of $\mathcal{H}$, whence its intersection with every $B \in b(\mathcal{H})$ is even. On the other hand the above definition of $\mathcal{B}(G,R)$ shows that $\mathcal{B}(G,R)$ is exactly the set of minimal cycles homologic to R in $\bar{G}$, and the claim is proved.) Since the projective planar graphs contain only one non-0 homology type of circuits 1-clutters are the same as 1-homology clutters. Note that every odd circuit (signing) clutter is a k-clutter (k-homology clutter) for some k. 1- and 2-clutters are ideal according to results of Lins [13] and Schrijver [18]. Seymour's conjecture is open for k-clutters, $k \geq 3$.

A. Matroid Ports. Matroid ports were introduced by Seymour [20]. Let $M = (S, \mathcal{C})$ be a binary matroid, $t \in S$. P is called a t-port of M, if $P = C \backslash t$, $C \in \mathcal{C}$, $t \in C$. Clearly, t-ports generalize (r,s)-paths in graphs: the (r,s)-paths of G are exactly the rs-ports of $G \cup rs$. The set of t-ports of M will be denoted by $\mathcal{C}(M,t)$.

(2.A.1) *The t-ports of a binary matroid form a binary clutter, and every binary clutter is the set of t-ports of some matroid M $(t \in S(M))$. The t-ports of M and those of M^* form a blocking pair of binary clutters.*

(2.A.2) *The matroid M and $t \in S(M)$ so that $\mathcal{A}$ is the set of t-ports of M is uniquely determined, namely $M = M_2(\mathcal{A})$, (and $M^* = M_2(\mathcal{B})$).*

(2.A.3) *Let M be a binary matroid, $t \in M$. If $e \in S(M) \setminus t$, then $\mathcal{C}(M,t) \setminus e = \mathcal{C}(M \setminus e, t)$, and $\mathcal{C}(M,t)/e = \mathcal{C}(M/e,t)$.*

B. t-joins and t-cuts of matroids. Every binary clutter $\mathcal{A}$ can be written as the set of minimal support solutions of the equation $Ax \equiv t \mod 2$, where A is a 0-1 matrix. By analogy to graphs we can call each $A \in \mathcal{A}$ a t-join of the matroid M represented by A; a t-cut B of M is a cocycle of M which is the $mod\, 2$ sum of some rows of A (cocycles of M) an odd number of which is "t-odd". $M_0(\mathcal{A}) = M$, and the new definitions of t-joins and t-cuts are in accordance with those of Section 1. t can also be defined independently of the representation, as a linear function on the cocycle space of M; then a t-cut is a cocycle whose t-value is 1.

(2.B.1) *The set of t-cuts (t-joins) of a binary matroid form a binary clutter, and every binary clutter is the set of t-cuts (t-joins) of some uniquely determined binary matroid M; The t-cuts and t-joins of a matroid M are a blocking pair of binary clutters.*

Let $\mathcal{A}$ be the clutter of t-joins of matroid M, and let $\mathcal{B}$ be its blocker.

(2.B.2) *$\mathcal{B}$ is the clutter of t-cuts of M. $M_0(\mathcal{A}) = M$, the circuits of $M_1(\mathcal{A})$ are the circuits and t-joins of M.*

Suppose that $\mathcal{A}$ is the set of t-joins of M, and $e \in S$.

(2.B.3) *$\mathcal{A} \backslash e$ and $\mathcal{A}/e$ is the set of t-joins* [5] *of $M \backslash e$, M/e respectively.*

[5] Contractions change the cocycle-space, and consequently the basis of cocycles which provides the rows of the matrix representing the matroid, changes. So the representation of the t-column after a contraction of $e \in S(M)$ changes accordingly, similarly to "t-contractions" of graphs ! Still, since for us t is an element of a matroid, and not a vector, we can and will keep the notation t after the contraction.

C. Signed Matroids. Let $M = (E, \mathcal{C})$ be a binary matroid and $R \subseteq S(M)$. Define $\mathcal{A}(M,R) := \{A \in \mathcal{C} : |A \cap R| \text{ is odd}\}$. $\mathcal{B}(M,R) := \{B = R+Q : Q \text{ is a cocycle of } M,$ B minimal non-empty$\}$, see [8]. Clearly, $R \in \mathcal{B}(M,R)$. Following Gerards, let us call (M,R) a signed matroid, and note the following:

(2.C.1) *$\mathcal{A}(M,R)$ is a binary clutter, and every binary clutter $\mathcal{A}$ is equal to $\mathcal{A}(M,R)$ for some uniquely determined binary matroid M, and arbitrary $R \in b(A)$; the same is true for $\mathcal{B}(M,R)$; $\mathcal{A}(M,R)$ and $\mathcal{B}(M,R)$ are the blocker of each other.*

(2.C.2) *If $\mathcal{A}$ is an arbitrary binary clutter, then $\mathcal{A} = \mathcal{A}(M,R)$, $\mathcal{B} = \mathcal{B}(M,R)$ with $M = M_1(\mathcal{A})$, and arbitrary $R \in b(\mathcal{A})$.*

The fact that the sets $\mathcal{A} = \mathcal{A}(M,R)$, $\mathcal{B} = \mathcal{B}(M,R)$, and $e \in S(M)$ do not change if we replace R by any $R' \in \mathcal{B}$, (that is, by $R + C$ where C is an arbitrary cut), is used in the following claim:

(2.C.3) *$\mathcal{A}\backslash e = \mathcal{A}(M\backslash e, R)$; $\mathcal{A}/e = \mathcal{A}(M/e, R')$, where $R' \in \mathcal{B}$, $e \notin R'$, or if $e \in R'$ for all $R' \in \mathcal{B}$, then $\mathcal{A}/e = \{\emptyset\}$.*

3 The Map of Classes

In this section we study the question of deciding whether a given binary clutter is in one of the well-known subclasses introduced in Section 2. We show good characterization theorems — involving excluded minors and decompositions — for membership in the classes; this helps in comparing the classes with each other and with the class of ideal binary clutters.

Defining Clutters with their Matroids. We shall see that each class of binary clutters discussed in Section 2 is minor-closed. This important property will be one of the consequences of the crucial fact that each class *can be defined with its down, up or port matroids*, that is, each class consists of *all* clutters whose down, up or port matroids are in a well-known class of matroids. The following facts are easy:

(3.1) *$\mathcal{H}$ is a path clutter if and only if $M_2(\mathcal{H})$ is graphic; $\mathcal{H}$ is a one-cut clutter if and only if $M_2(\mathcal{H})$ is cographic.*

Path and one-cut clutters are ideal according to the max-flow-min-cut theorem of Ford and Fulkerson (this is a special case of (3.3) below).

(3.2) *$\mathcal{H}$ is a join clutter if and only if $M_0(\mathcal{H})$ is graphic; $\mathcal{H}$ is a cut clutter if and only if $M_1(\mathcal{H})$ is cographic.*

The following statement is Edmonds and Johnson's result [5].

(3.3) *Join clutters and cut clutters are ideal.*

Signed graph and signing clutters are not necessarily ideal, in fact it is easy to see that $\mathcal{K}_5$ is a 3-clutter. According to Schrijver, [18] 2-clutters are ideal. They can all be defined with their up matroids:

(3.4) *$\mathcal{H}$ is an odd circuit clutter if and only if $M_1(\mathcal{H})$ is graphic;*
$\mathcal{H}$ is a signing clutter if and only if $M_0(\mathcal{H})$ is cographic.

(3.5) *$\mathcal{H}$ is a k-clutter if and only if $M_1(\mathcal{H})$ is graphic and embeddable on a non-orientable surface of genus k; $\mathcal{H}$ is a k-homology clutter if and only if $M_0(\mathcal{H})$ is cographic and the graph representing $M_0^*(\mathcal{H})$ is embeddable to a surface of genus k.*

The minor-closed property as well as the excluded minor characterizations of all these classes of clutters follows from the next four general claims:

The minor of a matroid M, $t \in S(M)$ containing t will be called a *t-minor* of M.

(3.6) *If $\mathcal{H}$ is a binary clutter, then $\mathcal{H}'$ is a minor of $\mathcal{H}$ if and only if $M_2(\mathcal{H}')$ is a t-minor of $M_2(\mathcal{H})$, where the "t" of the two matroids coincide.*

Proof. Immediate from 2.A.3. □

(3.7) *Let $\mathcal{M}$ be a set of binary matroids, and denote by Ω the class of clutters $\mathcal{H}$ for which $M_0(\mathcal{H})$ does not contain any minor in $\mathcal{M}$. Then $\mathcal{H} \in \Omega$ if and only if $\mathcal{H}$ does not contain a lift of some $M \in \mathcal{M}$ as clutter-minor.*

Proof. Immediate from the first part of (1.3) □

(3.8) *Let $\mathcal{M}$ be a set of binary matroids, and denote by Ω the class of clutters for which $M_1(\mathcal{H})$ does not contain any minor in $\mathcal{M}$. Then $\mathcal{H} \in \Omega$ if and only if it does not contain the blocker of a lift of some M^*, where $M \in \mathcal{M}$, as clutter minor.*

Proof. Apply the previous claim to the blocker. □

The following statement is equivalent to Seymour's theorem on rounded classes of matroids [22] to which we give a simple proof [6].

(3.9) *Let $\mathcal{M}$ be a set of binary matroids, and denote by Ω the class of clutters for which $M_2(\mathcal{H})$ does not contain any minor in $\mathcal{M}$. Then $\mathcal{H} \in \Omega$ if and only if it contains neither a port of some $M \in \mathcal{M}$, nor a lift of some $M \in \mathcal{M}$, nor $b(\mathcal{H})$ contains the lift of M^* ($M \in \mathcal{M}$) as a minor.*

Proof. A minor of $M_2(\mathcal{H})$ is either a t-minor, and its relation to clutter minors was settled in (3.6), or it is a minor of either M_0 or M_1, and we can then apply (3.7), or (3.8) respectively. □

Some lifts may contain some ports as minors, and can thus be omitted.

Recognizing Subclasses of Binary Clutters. In the algorithmic results we suppose that binary matroids are given with their representation, and binary clutters are given with the representation of their port matroids (with a particular element), or equivalently as the minimal support solutions of an equation $Mx \equiv t \bmod 2$. If Ω is a class of binary clutters (matroids), then *recognizing Ω* means deciding whether a clutter (matroid) given as input is in the class Ω. If $\mathcal{T}$ is a set of binary clutters (matroids), then *testing for $\mathcal{T}$-minors* means deciding whether a binary clutter (matroid) given as input contains $\mathcal{H} \in \mathcal{T}$ as a clutter-minor (minor), and if yes, finding such a minor. From (3.6)-(3.9) we can easily conclude:

Theorem 3.1 *Let $\mathcal{M}$ be a set of matroids and $i \in \{0, 1, 2\}$. The class of binary clutters $\mathcal{H}$ for which $M_i(\mathcal{H})$ has no minor in $\mathcal{M}$ is minor-closed, and has a finite set $\mathcal{T}$ of excluded minors provided $\mathcal{M}$ is finite. Moreover binary clutters can be tested for $\mathcal{T}$-minors in polynomial time, provided binary matroids can be tested for $\mathcal{M}$-minors in polynomial time.* □

Corollary 3.2 *The following classes of clutters can be recognized in polynomial time: path clutters, one-cut clutters; join clutters, cut clutters; odd circuit clutters, signing clutters; k-clutter for fixed k, and k-homology clutters for fixed k. All of these classes of clutters are minor-closed and have a finite number of excluded clutter*

[6] The easiness of (3.9) can be due to the fact that $M_2(\mathcal{A})$ is always connected for $\mathcal{A} := \mathcal{H}$ as well as for its clutter minors, which makes the main difficulty of [22] disappear.

minors which can be tested in polynomial time. If k is part of the input it is NP-complete to decide whether $\mathcal{H}$ is a k-clutter; as well as to decide whether it is a k-homology clutter.

The proof consists of piecing together known algorithms via the down, up and port matroids. (See the "general framework of an algorithm" below.) The statements about k-clutters and k-homology clutters follow in the same way from the finite number of excluded minors and the polynomial solvability of the "graph genus problem" for fixed k (see [16]). For the NP-completeness of this problem, if k is part of the input, (see [28]).

The characterization of path clutters has already been known from Seymour [22].

If now the clutter $\mathcal{H} \subseteq 2^S$ is given with a *containment oracle*, which, for any $X \subseteq S$ as input tells whether there exists $E \in \mathcal{H}$ such that $X \supseteq E$, yes or no, then the above arguments fail to work. However, according to the recent breakthrough of Coullard and Hellerstein [2] there exists a polynomial algorithm which *given a containment oracle of a port clutter $\mathcal{H}$ of a connected binary matroid M, a matrix representing the matroid M can be constructed in polynomial time.* Since $M_2(\mathcal{H})$ is always connected, a representation of $M_2(\mathcal{H})$, and consequently of $M_0(\mathcal{H})$ and $M_1(\mathcal{H})$ can be found in polynomial time ! Using this result, the algorithmic remarks of Theorem 3 1 and Corollary 3.2 remain valid for this more general computational model.

Let us sketch a general framework of an algorithm which works for every class of binary clutters studied in this paper:

1. If the clutter is given with a containment oracle then use Coullard and Hellerstein's algorithm in order to reconstruct a representation of the port matroid of the clutter.
2. Use Bixby and Cunningham's algorithm [1] to decompose the up down or port matroid.
3. Use known testing membership algorithms for the bricks of the decomposition ([27] or [16] for the above classes of clutters).

Note that while the recognition of the class of ideal clutters is open — and closely related to Seymour's conjecture — Hartvigsen and Wagner [11] have developed a polynomial algorithm recognizing the "strong max-flow-min-cut property".

At the end of this section and in Section 4 we will define sum operations which will it make possible to increase the known classes of ideal clutters.

Compositions of Clutter Classes and Containments. We do not have enough space here to include our diagrams and charts about containment relations between subclasses of clutters. Let us mention, however, some containment relations which are not difficult to establish from (3.1)-(3.5):

The class of k-clutters is contained in the class of $k+1$-clutters, the union of these over all k is the class of odd circuit clutters. The class of k-homology clutters is contained in the class of $k+1$-homology clutters, and their union over all k is the class of signing clutters. Join clutters and cut clutters can also be structured in the same way as odd circuit and signing clutters: according to the genus of the underlying graph.

The intersection of the class of k-clutters and k-homology clutters consists of the 1-clutters; these are also 1-homology clutters. The intersection of odd circuit and cut clutters is the class of cut clutters of planar graphs.

The intersections of unrelated clutters are often small. Surprisingly though, the intersection of the class of join and of the class of cut clutters is quite rich and it led us to develop the three kinds of sum operations — one related to each of the down up and port matroids.

A clutter $\mathcal{H}$ will be called a *join-and-cut* clutter, if it can be represented as the family of t-joins of a graph G, and also as the family of t'-cuts of a graph G'. A binary clutter is join-and-cut if and only if its down matroid is graphic, and its up matroid is cographic.

It came to our knowledge that Gerards, Lovász, Schrijver, Seymour, Shih, Truemper [10] contains a systematic study of binary spaces contained in one another with codimension one, including a characterization of cographic spaces containing a graphic subspace of codimension one. (Another claim of this type occurs in the proof of Theorem 4.6.) The proof we sketch below illustrates the use of one of the sum operations, the one based on the decomposition of M_2:

Lemma 3.3 *Let $\mathcal{H} \subseteq 2^V$ be a binary clutter, and let $M = (E, \mathcal{C})$ be an arbitrary binary matroid, $V \cap E = \{e\}$, where e is non-series and non-parallel to $t \in S(M_2)$. Then the set of t-ports of the 1-sum, or of the 2-sum of $M_2(\mathcal{H})$ and M with marker e is a join-and-cut clutter if and only if $\mathcal{H}$ is a join-and-cut clutter and M is the circuit matroid of a planar graph.*

Proof. Let $\bar{M}$ be the resulting 1- or 2-sum. The condition is necessary, because $\bar{M} \setminus t$ is graphic, and since t is not parallel to e, $\bar{M} \setminus t$ contains both $M_2 \setminus t$ and M as a minors. whence both $M_2 \setminus t$ and M are graphic; applying the same to the dual, since $\bar{M}/t$ is cographic and t is non-series to e, both M_2/t and M are cographic. The only if part is proved. Conversely, if $\mathcal{H}$ is join-and-cut and M is planar, then, since the 2-sum of two graphic matroids and of two cographic matroids is graphic and cographic respectively, we get the right claim about $M_2 \setminus t$, and M_2/t. □

Theorem 3.4 *Let G be a graph, $t : V(G) \mapsto \{0, 1\}$ and let $\mathcal{H}$ be the clutter of t-joins of G. Then $\mathcal{H}$ is a join-and-cut clutter if and only if one of the following (self-dual) conditions holds.*

(i) G is one of K_4 or $K_{3,3}$ with $t(v) = 1$ everywhere, or it is $K_{2,3}$ and $t(v) = 0$ on a vertex of degree 3, and otherwise 1.

(ii) $\sum_{v \in V(G)} t(v) = 2$, and the graph we get after identifying the two vertices v with $t(v) = 1$ is planar, or G is a planar graph and has a face F such that $t(v) = 0$ if v is not on F.

(iii) G is the 1- or 2-sum of the graphs G_1 and G_2 and $t(v) = 0$ if $v \in V(G_2)$, moreover the t-join clutter of G_1 is join-and-cut and G_2 is planar.

Proof. (Sketch) The if part is easy to check: in (i) $M_2(\mathcal{H})$ is F_7^*, R_{10} and F_7 in order, $M_2(b(\mathcal{H})) = M_2^*(\mathcal{H})$ is then F_7, R_{10} and F_7^* respectively, whence $\mathcal{H}$ in these three cases is a t'-cut in G', where the list of the (G', t') is the list of the (G, t) in reverse order. In (ii) we suppose G is a planar graph where F is a face. $\{v : t(v) = 1\}$ splits up F into an even number of paths, let us number these in the order defined by F; let the set of neighbors of F along the paths which got an even number be A, and those which are neighbouring F along a path that got an odd number be B. Define now G' by unshrinking F in the dual G^* of G so that if we shrink the arising two vertices $\{a, b\} \subseteq V(G')$ we get $F \in V(G^*)$, and join a to $A \subseteq V(G^*)$ and b to $B \subseteq V(G^*)$. It is easy to see that the t-cuts of G correspond exactly to the t'-joins of G', where t' is 1 on a and b and 0 elsewhere. The converse of this construction shows what is (G', t') in the second case listed in (ii).

For case (iii) the if part of the statement follows from the if part of Lemma 3.3. In order to prove the only if part suppose that $\mathcal{H}$ can be represented as the $t'-cut$ clutter of a graph G'. We suppose that (iii) does not hold and prove that one of (i) or (ii) holds.

Claim 1 If (ii) does not hold, $M_2(\mathcal{H})$ has no 1- or 2-separation.

Claim 2 M_2 has no $AG(2,3)$, S_8 or R_{12} minor.

Claim 3 Either (ii) holds for M_2, or M_2 is regular without R_{10} minor.

Claim 4 If (i) does not hold for M_2, then (ii) holds. □

Theorem 3.4 does not provide new classes of ideal clutters: join clutters and cut clutters (see (3.3)) are already ideal.

Let us realize that minor-closed conditions on M_0 and M_1 are more general than on M_2. While for join-and-cut clutters the suitable decomposition was that of M_2, in the next section, where the goal is to compose as large a class of (ideal) clutters as possible, the decomposition of M_0 and M_1 is more appropriate.

4 Ideal Clutters, Multiflows and Metrics

Let $M = (E, \mathcal{C})$ be a matroid. $m : E \mapsto \mathbb{R}_+ \cup \{\infty\}$ is called a *metric*, if for every circuit $C \in \mathcal{C}$ and every $e \in C$: $m(e) \leq m(C \setminus e)$. $d : E \mapsto \mathbb{N}$ is called a *distance function* if for some $F \subseteq E$ it is defined in the following way: for $e \in F$, $d(e) := 1$, and for $e \notin F$ $d(e) := \min\{|C\setminus e| : C \in \mathcal{C}, \{e\} = C\setminus F\}$. It is easy to see that a distance function is a metric and that it is finite if and only if $S(M)\setminus F$ does not contain a cut. This metric will be denoted by $[M, F]$. For instance an $[M(K_5), E(K_{2,3})]$ metric is 1 on a $K_{2,3}$ subgraph of K_5, and 2 otherwise.

We define an $[M, F]$ *metric* to be a function m on the elements of an arbitrary matroid such that contracting the elements with m-value 0 we get the matroid M with the metric $[M, F]$ (up to isomorphism, after deleting loops and replacing parallel elements by one element with m-value equal to the minimum of the m-values of the parallel class).

A *multiflow problem*, (M, R, c), on a matroid $M = (S, \mathcal{C})$ is defined by a set of "demands" $R \subseteq S$, and a function $c : S \mapsto \mathbb{R}_+$. A *multiflow* is a function $f : \mathcal{C} \mapsto \mathbb{Q}_+$, so that if $f(C) > 0$, then $|C \cap R| = 1$, and the sum of the f-values of circuits containing a given $e \in S$ is at most $c(e)$, moreover equality holds here for $e \in R$. For the most basic facts about multiflows in matroids we refer to [25]; for their simple connection to binary clutters to [25] or [8]. If $e \in R$, $c(e)$ is called the *demand* of e, if $e \in S\setminus R$ it is called the *capacity* of e.

The connection between metrics and multiflows is provided by the following statement, well-known for graphs, which is also easy from linear programming duality (Farkas' lemma) for matroids.

Metric Criterion *For $R \subseteq S(M)$ and $c : S(M) \mapsto \mathbb{R}_+$ there exists a multiflow if and only if the following* Metric Condition *is satisfied: for every metric m,* $\sum_{e\in R} m(e)c(e) \leq \sum_{e\in S(M)\setminus R} m(e)c(e)$.

Clearly, distance functions are metrics and, for us, the only metrics which will play a role in the conditions of multiflow problems.

The Metric Condition specialized to a subclass μ of metrics will be called the μ Condition. Incidence vectors of cocycles are metrics, and the Metric Criterion specialized to them is called *Cut Condition* (see [25]). The matroid M is called *F-flowing* ([25]) for $F \subseteq S(M)$, if for arbirary $c : S(M) \mapsto \mathbb{R}_+$ for which the Cut Condition is satisfied there exists a multiflow.

A cocycle for which the Cut Condition holds with equality will be called a *tight cocycle*. It is easy to see that *switching* on a tight cocycle (that is interchanging the edges in R and those which are not in R), we get a multiflow problem which has an (integer) solution if and only if the original problem has one. If the Cut Condition holds then a tight cocycle is the disjoint union of tight cocircuits, and a sequence of switchings can be replaced by just one switching on a tight cocycle.

We state the following refinement of Seymour's conjecture:

Conjecture *Let $M = (S, \mathcal{C})$ be a binary matroid, and $R \subseteq S$. Then exactly one of the following possibilities is true:*
(i) For every $c : S \mapsto \mathbb{Z}$ for which the Cut Condition holds there exists a multiflow in (M, R).
(ii) There exists a $c : S \mapsto \mathbb{Z}$ so that possibly after switching on a tight cocycle some $[F_7, F_7 - L]$, $[K_5, K_{2,3}]$, $[R_{10}, R_{10} - C_3]$ condition is not satisfied, but the Cut Condition is satisfied.

L is a line of F_7; C_3 is a 3-element circuit of R_{10}, and R_{10} can be defined as $M_0(K_5)$. (i) and (ii) trivially exclude each other. Papernov's multiflow problem (H_6, R) shows why we have to allow switching on a tight cut: H_6 is the graph one gets from K_5 by uncontracting an edge ab so that no series edges occur; R consists of three edges ab, x_1x_2 and y_1y_2 forming a matching of H_6, and such that a is adjacent to both x_1 and x_2 and b is adjacent to both y_1 and y_2. Defining now c to be 1 everywhere, and $c(ab) := 2$, it is easy to see that the Cut Condition and the $[K_5, K_{2,3}]$ Condition are satisfied, but that there is no multiflow (the $[H_6, H_6 \backslash R]$-condition is not satisfied). On the other hand the edges adjacent to a form a tight cut, switching on which the following $[K_5, K_{2,3}]$ Condition is violated: $m(ab) := 0$, $m(e) := 1$ for $e \in H_6 \backslash R'$ and $m(e) := 2$ for $e \in R'$, where R' is the set of demand edges after the switching.

Let us also note that Marcus and Sebő [14] characterize matroids for which the Cut Condition and the $[F_7, F_7 - L]$ and $[K_5, K_{2,3}]$ conditions are necessary and sufficient for the existence of a multiflow. This implies the validity of the above conjecture for these matroids, and the corresponding class is contained in the classes provided by theorems 4.5 and 4.6 below.

We define now a partial order on the set of Metric Conditions according to which the Cut Condition is the smallest metric, and the Conjecture is equivalent to asserting that the only metrics on the following, *"second lowest level"* are $[F_7, F_7 - L]$, $[K_5, K_{2,3}]$, $[R_{10}, R_{10} - C_3]$:
Given the metrics m_1 and m_2 on M, we say that $m_1 < m_2$, if for every (M, R) ($R \subseteq E(M)$) for which an m_2-metric occurs as a non-satisfied condition for some multiflow problem (M, R, c), where $c : S(M) \mapsto \mathbb{R}+$, m_1 also occurs with some (maybe different) capacity function and possibly after switching on a tight cut.
This induces a partial order on metric types and can be reformulated in terms of only m_1 and m_2 (without using "every (M, R)"). Some properties of this second level can be deduced from Lehman's theorem [26] on minimal non-ideal clutters; these have to be omitted here.

If in the matroid $M = (E, \mathcal{C})$ for $R \subseteq E$ either (i) or (ii) of the conjecture holds we will say that M is *settled* with respect to R. M is settled if it is settled with respect to every $R \subseteq E$. The class of settled matroids is minor-closed. We say that for a binary clutter $\mathcal{H}$ *Seymour's conjecture holds*, if it is ideal, or if it contains a minor in $\mathcal{S}$.

Note also that our conjecture actually characterizes the following refinement of the notion of idealness. Let $\mathcal{A} \subseteq 2^S$ be a binary clutter, and let $\mathcal{B}$ be its blocker. We will say that $A \in \mathcal{A}$ is ideal (with respect to $\mathcal{A}$), or that $\mathcal{A}$ is (locally) ideal in $A \in \mathcal{A}$, if $w : S \mapsto \mathbb{R}_+$, $w(A) \leq w(A')$ for all $A' \in \mathcal{A}$ implies that there exist $B_1, \dots, B_k$ and $\lambda_1, \dots, \lambda_k \in \mathbb{R}_+$ such that $\sum_{i=1}^k \lambda_i B_i \leq w$, and $\sum_{i=1}^k \lambda_i = w(A)$.

Clearly, $\mathcal{A}$ is ideal, if and only if every $A \in \mathcal{A}$ is ideal (with respect to $\mathcal{A}$). In both $\mathcal{K}_5$ and its blocker the minimum cardinality sets in the clutter (the core) are not, the other elements are ideal, and similarly for $\mathcal{F}_7$. An explanation of this lies in the notion of "cores" (see [3]) related to Lehman's theorem. In the parlence of polyhedral combinatorics, every $A \in \mathcal{A}$ is a vertex of the polyhedron $P = \{x \in \mathbb{R}^n : x(B) \geq 1, x \geq 0\}$, but $Q = conv(A) + \mathbb{R}^n$, $Q \subseteq P$, may be a proper subset; the idealness of $A \in \mathcal{A}$ means that "locally" the two polyhedra are the same, that is, the facets of Q containing the vertex A are the same as those of P.

(4.1) *Let $\mathcal{A}$ be a binary clutter, $\mathcal{B} = b(\mathcal{A})$. The following are equivalent:*
(i) $\mathcal{A}$ is ideal.
(ii) $\mathcal{A}$ is ideal in every $A \in \mathcal{A}$.
(iii) $\mathcal{B}$ is ideal in every $B \in \mathcal{B}$. □

The following is a formulation of the connection between the cut condition and the max-flow-min-cut property pointed out by Seymour (for instance [25]).

(4.2) *Let $\mathcal{A}$ be a binary clutter, $\mathcal{B} = b(\mathcal{A})$. For the matroid $M := M_1(\mathcal{A})$, $c : S \mapsto \mathbb{R}_+$ and $R \in \mathcal{B}$ the Cut Condition is satisfied if and only if $c(R) \leq c(B)$ for all $B \in \mathcal{B}$. Moreover, $\mathcal{B}$ is ideal in $B \in \mathcal{B}$ if and only if $M_1(\mathcal{A})$ is B-flowing.*

Proof. Let $c : S \mapsto \mathbb{R}_+$. Since by 2.C.2 $\mathcal{B} = \mathcal{B}(M, R)$, we know that $\mathcal{B}$ consists of the sets of minimal support of the form $R + C$, where C is a cut of M. Hence the cut condition holds for (M, R, c), that is $c(Q \cap R) \leq c(Q \backslash R)$ for every cut Q of M if and only if (adding $c(R \backslash Q)$ to both sides) $c(R) \leq c(R + Q)$, that is $c(R) \leq c(B)$ for all $B \in \mathcal{B}$.

Suppose now that M is B-flowing, and let us show that $\mathcal{B}$ is ideal in B. Let $c : S \mapsto \mathbb{R}_+$ be such that $c(B) \leq c(B')$ for all $B' \in \mathcal{B}$. Since the circuits in a multiflow of M contain one edge of B each, they are all in $\mathcal{A}(M, R) = \mathcal{A}$, and provide the packing proving that $\mathcal{B}$ is ideal in B. The proof of the converse is similar. (Each of the sets of $\mathcal{A}$ which are in the packing proving that $B \in \mathcal{B}$ is ideal contains exactly one element of B, by complementary slackness.) □

The following statement is a restatement of (4.1) and (4.2):

(4.3) *Let $\mathcal{A}$ be a binary clutter, $\mathcal{B} = b(\mathcal{A})$. The following statements are equivalent:*
(i) $\mathcal{A}$ is ideal.
(ii) $M_0^*(\mathcal{A})$ is A-flowing for all $A \in \mathcal{A}$.
(iii) $M_1(\mathcal{A})$ is B-flowing for all $B \in \mathcal{B}$.

The following theorem shows that the conjecture above implies Seymour's conjecture. (We do not see the converse.)

Theorem 4.1 *For a binary clutter $\mathcal{H} \subseteq 2^S$ Seymour's conjecture holds if and only if $M_0^*(\mathcal{H})$ is settled with respect to every $E \in \mathcal{H}$.*

Proof. We know from (4.1) that $\mathcal{H}$ is ideal if and only if it is locally ideal for every $E \in \mathcal{H}$, and we know from (4.2) that this happens if and only if $M_1(b(\mathcal{H})) = M_0^*(\mathcal{H})$ is E-flowing for every $E \in \mathcal{H}$. What remains to be proved is that $\mathcal{H}$ *contains a clutter minor $\mathcal{F} \in \mathcal{S}$ if and only if for $M := M_0^*(\mathcal{H}) = M_1(b(\mathcal{H}))$ and some $R \in \mathcal{H}$ (ii)* *of the conjecture holds.* Note that *the existence* of R for which (ii) holds is simply equivalent to the existence of R and c for which some $[F_7, F_7 - L]$, $[K_5, K_{2,3}]$ or $[R_{10}, R_{10} - C_3]$ condition is not satisfied, but the Cut Condition is satisfied. (If this holds only after switching on a tight cocircuit Q of M, we replace R by $R + Q$. It follows from 2.C.2 that $R + Q \in \mathcal{H}$ (choose $\mathcal{B} := \mathcal{H}$).)

To prove the only if part of this statement suppose there exists $X, Y \subseteq S$, $X \cap Y = \emptyset$ such that $\mathcal{F} := \mathcal{H}/X\backslash Y \in \mathcal{S}$. Then by (1.3) $M_0^*(\mathcal{F}) = M\backslash X/Y$.

$$c(e) := \begin{cases} 0 & \text{if } e \in X \\ |S| & \text{if } e \in Y \\ 1 & \text{if } e \in S - (X \cup Y) \end{cases}$$

Let $R \subseteq S$ be in the core of $\mathcal{F}$ (that is, R is a minimal cardinality set in $\mathcal{F}$) and let

$$m(e) = \begin{cases} 0 & \text{if } e \in Y \\ 2 & \text{if } e \in X \cup R \\ 1 & \text{if } e \in S\backslash(X \cup Y \cup R). \end{cases}$$

It is not difficult to check now that m is a $[F_7, F_7 - L]$, $[K_5, K_{2,3}]$ or $[R_{10}, R_{10} - C_3]$ metric depending on whether $\mathcal{F} = \mathcal{F}_7$, $\mathcal{F} = \mathcal{K}_5$, $\mathcal{F} = \mathcal{K}_5^*$ respectively, and the Cut Condition is satisfied for the defined multiflow problem, but the Metric Condition with the metric m is not satisfied.

To prove now the if part, suppose that $M = M_0^*(\mathcal{H})$ is settled with respect to every $E \in \mathcal{H}$. Let and $R \in \mathcal{H}$ $w : S \mapsto \mathbb{R}_+$ be so that $w(R) \leq w(E)$ for all $E \in \mathcal{H}$, but for a metric in (ii), denote it by $m : S \mapsto \mathbb{R}_+$, the Metric Condition is not satisfied. Suppose also that $\mathcal{H}$ is (minorwise) minimal: if not we apply the result to a minor by induction.

Claim $m(e) > 0$ for all $e \in S$.

Indeed, suppose $m(e) = 0$ for some $e \in S$. Then $R\backslash e \in \mathcal{H}/e$ and according to (1.3) $M_0^*(\mathcal{H}/e) = M_0^*(\mathcal{H})\backslash e = M\backslash e$. The restriction of m to $S\backslash e$ is a metric of $M_0^*(\mathcal{H})\backslash e$: the Metric Condition does not hold for this metric, whereas the cut condition for $M\backslash e, R\backslash e)$ with the restriction of c to $S\backslash e$ follows from the cut condition for (M, R, c).

Since m is one of the metrics $[F_7, F_7 - L]$, $[K_5, K_{2,3}]$, $[R_{10}, R_{10} - C_3]$ it follows that $M_0^*(\mathcal{H})$ is one of F_7, K_5, R_{10}. With case checking one gets that the only clutter $\mathcal{H}$ with this property and also having an $E \in \mathcal{H}$ for which (ii) of the conjecture holds is $\mathcal{F}_7$, $\mathcal{K}_5$ and $b(\mathcal{K}_5)$ respectively. □

Since Seymour's conjecture holds for $\mathcal{A}$ if and only if it holds for $\mathcal{B}$, Theorem 4.1 means that this happens concurrently when $M_0^*(\mathcal{A}) = M_1(\mathcal{B})$ is settled with respect to every $A \in \mathcal{A}$ or $M_1(\mathcal{A}) = M_0^*(\mathcal{B})$ is settled with respect to every $B \in \mathcal{B}$.

Theorem 4.1 has two useful properties: first, it deals with settled instead of ideal clutters, the former being easier to work with; second, it reduces the global property to local ones which are easier to "sum":

Lemma 4.2 *If M is the 1-, 2-, or 3-sum of the two matroids M' and M'', then M is settled if and only if both M' and M'' are settled.*

Proof. (Sketch) The only if part follows from the fact that M' and M'' are minors of M, and the class of settled matroids is minor-closed.

In order to prove the if part suppose that M' and M'' are settled let $F \subseteq S(M)$, and $c : S(M) \to \mathbb{R}_+$ so that (M, F, c) satisfies the Cut Condition. We define now F' and $c' : S(M') \to \mathbb{R}_+$, F'' and $c'' : S(M'') \mapsto \mathbb{R}_+$, F'' by copying c and F for everything but the marker(s) and on the markers we define them as dictated by the Cut Condition. In lack of space we omit the details of this definition, and refer to [25] (7.2) and (7.3). In (M', F', c') and (M'', F'', c'') the Cut Condition is satisfied. We are going to prove that either there exists a multiflow for (M, F, c), or there exists a $\bar{c} : S(M) \mapsto \mathbb{R}_+$ for which some $[F_7, F_7 - L]$, $[K_5, K_{2,3}]$, $[R_{10}, R_{10} - C_3]$ condition is violated.

If in (M', F', c') there is no multiflow, then, since M' is settled, after switching on a tight cut some $[F_7, F_7 - L]$, $[K_5, K_{2,3}]$, $[R_{10}, R_{10} - C_3]$ condition is violated. It is then easy to find the tight cut in (M, R, c) and extend the violated condition to M so that the same type of condition shows the non-existence of a multiflow. The same holds if in M'' with the demands F'' for some $c'' : S(M') \mapsto \mathbb{R}_+$ the cut condition is satisfied but there is no multiflow.

We can thus suppose that in both (M', R', c') and (M'', R'', c'') there exists a multiflow. But then it is easy to reconstruct a multiflow of M: in order to spare space we refer again to [25] (7 2) and (7.3). □

Using Theorem 4.1 this Lemma can be restated in terms of Seymour's conjecture:

Corollary 4.3 *If Seymour's conjecture holds for an arbitrary odd circuit (signing) clutter of M' and M'', then it also holds for an arbitrary odd circuit (signing) clutter of M, where M is their* 1-, 2- *or* 3-*sum*. We cannot prove the same replacing odd circuit and signing clutters by clutters of t-cuts or t-joins. An odd circuit clutter of M' is clutter of t-cuts of M'^*, and a signing clutter of M' is clutter of t-joins of M'^*:

Corollary 4.4 *If Seymour's conjecture holds for an arbitrary t-join (t-cut) clutter of M' and M'', then it also holds for an arbitrary t-join (t-cut) clutter of M, where M is their* 1-, 2- *or dual* 3-*sum*.

For the definition of the matroids $AG(2,3)$ S_8 see [25].

In the proof of the following theorems we use Seymour's splitter theorem [24], the above Lemma and some extra work. Note that neither of the conditions of Theorem 4.5 or 4.6 imply that $\mathcal{H}$ is ideal. We omit the proof of Theorem 4.5, and write out full details of that of Theorem 4.6 which we find more interesting:

Theorem 4.5 *Let $\mathcal{H}$ be a binary clutter. If at least one of $M_0(\mathcal{H})$ and $M_1^*(\mathcal{H})$ have no $AG(2,3)$, S_8 or H_6^* minors, then Seymour's conjecture holds for $\mathcal{H}$.*

Theorem 4.5 contains Edmonds and Johnson's minimax theorem [5] on the idealness of join-clutters, whereas Theorem 4.6 contains Lins' theorem on one sided paths on projective plane [13] and particular (but not simply trivial) instances of of Edmonds and Johnson's theorem; on the other hand the proofs *use* these theorems. We stress, though, that Theorems 4.5 and 4.6, concern matroids M whose t-joins are ideal for some t but *contain a minor in $\mathcal{S}$ for some other t'*.

Note the symmetric role of the up and down matroids in the statement of Theorem 4.6. A trick in the proof is to decompose according to the down matroid using the Lemma, and when we cannot any more, then we switch to the down matroid of the blocking clutter. Since then the up matroid becomes the down matroid we can decompose again, and for the "bricks" we can prove the statement directly.

Theorem 4.6 *Let $\mathcal{H}$ be a binary clutter. If neither $M_0(\mathcal{H})$ nor $M_1(\mathcal{H})$ have*

$AG(2,3)$ or S_8 minors, then Seymour's conjecture holds for $\mathcal{H}$.

Proof. Suppose $\mathcal{H}$ is a counterexample with $|S(\mathcal{H})|$ minimum.

Claim 1 Both $M_0(\mathcal{H})$ and $M_1(\mathcal{H})$ are either graphic or cographic, or equal to R_{10}.

Indeed, it follows from Seymour's splitter theorem (see [25]) that $M_0(\mathcal{H})$ is either graphic or cographic, or R_{10}. Since AG$(2,3)$ and S_8 *are self-dual*, the fact that $M_1(\mathcal{H})$ does not contain them as minors, implies that $M_1^*(\mathcal{H}) = M_0(b(\mathcal{H}))$ also does not contain them as minors. Since R_{10} is also self-dual, It follows now in the same way as for $\mathcal{H}$ that $M_0(b(\mathcal{H}))$ $(= M_1^*(\mathcal{H}))$ is also either graphic or cographic or R_{10} and since this set of minors is also self-dual, we finally get that $M_1(\mathcal{H})$ is also either graphic or cographic or equal to R_{10}.

Claim 2 $M_0(\mathcal{H})$ is cographic and $M_1(\mathcal{H})$ is graphic.

Indeed, by Claim 1 there are only three possibilities for each of $M_0(\mathcal{H})$ and $M_1(\mathcal{H})$. If $M_0(\mathcal{H})$ is graphic, then regardless of $M_1(\mathcal{H})$, $\mathcal{H}$ is ideal, since, then, $\mathcal{H}$ is a join clutter. If $M_1(\mathcal{H})$ is cographic, then regardless of $M_0(\mathcal{H})$, $\mathcal{H}$ is ideal, since, then, $\mathcal{H}$ is a cut clutter. If $M_0(\mathcal{H}) = R_{10}$, then it is easy to check that $\mathcal{H}$ is ideal unless $M_1(\mathcal{H}) = K_5$. If $M_1(\mathcal{H}) = R_{10}$, then applying the previous sentence to $b(\mathcal{H})$ we get that $\mathcal{H}$ is ideal unless $M_0(\mathcal{H}) = K_5^*$. But we know from Section 2 that $M_0(\mathcal{H}) = R_{10}$ $M_1(\mathcal{H}) = K_5$ if and only if $\mathcal{H} = \mathcal{K}_5$, and $M_1(\mathcal{H}) = R_{10}$, $M_0(\mathcal{H}) = K_5^*$ if and only if $\mathcal{H} = b(K_5)$. Since ideal clutters, $\mathcal{K}_5$ and $b(\mathcal{K}_5)$ are all settled, and $\mathcal{H}$ is a counterexample, the claim is proved.

Claim 3 [7] If $M_0(\mathcal{H})$ is cographic and $M_1(\mathcal{H})$ is graphic, then $\mathcal{H}$ is either a 1-clutter, or the one-path clutter of a planar graph.

Indeed, let G be a graph whose circuit matroid is $M_1(\mathcal{H})$. Assume $M_1(\mathcal{H})$ is connected (G is a 2-connected graph), otherwise we proceed by components.

Since $M_0(\mathcal{H})$ is cographic it has a basis of circuits $\mathcal{C}_0$ so that every $e \in E(G) := S(\mathcal{H})$ is contained in exactly two circuits of $\mathcal{C}_0$. It follows that for every vertex v of G there exists a partition $\mathcal{P}_v$ of the edges incident to v in such a way that every $P \in \mathcal{P}_v$ has a cyclical order so that for any two neighboring edges in this order there exist circuits in $\mathcal{C}_0$ containing both.

$|\mathcal{P}_v| \leq 2$ for every vertex v.

Indeed, if for vertex v there are three different classes $P_1, P_2, P_3 \in \mathcal{P}_v$ then let $e_i \in P_i$ $(i = 1,2,3)$. Since $M_1(\mathcal{H})$ is connected there exists a circuit C_{ij} containing e_i and e_j, $(i \neq j \in \{1,2,3\})$. Since every cycle in the span of $\mathcal{C}_0$ has an even number of edges in every equivalence class of each vertex, the circuits in the span of $\mathcal{C}_0 \cup \{C_{12}\}$ do not span C_{13}, but then $M_1(\mathcal{H})$ cannot be the cycle matroid of G.

To finish the proof now note that if $|\mathcal{P}_u| = 2$ then we get in a similar way that every circuit using edges from two different classes of $|\mathcal{P}_u|$ uses edges from two different classes of every vertex v for which $|\mathcal{P}_v| = 2$. Split up u into two vertices and let the edges of the two partition-classes be the stars of the two new vertices. The circuits of $\mathcal{C}_0$ generate all the circuits of the obtained graph G_u, whence G_u is both graphic and cographic, and the claim follows. □

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[7] This claim has been known by the authors of [10], see a related comment in Section 3.

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