On Integer Multiflows and Metric Packings in Matroids

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Abstract. Seymour [10] has characterized graphs and more generally matroids in which the simplest possible necessary condition, the "cut condition", is also sufficient for multiflow feasibility. In this work we exhibit the next level of necessary conditions, three conditions which correspond in a well-defined way to minimally non-ideal binary clutters. We characterize the subclass of matroids where the presented conditions are also sufficient for multiflow feasibility, and prove the existence of integer multiflows for Eulerian weights. The theorem we prove uses results from Seymour[10] and generalizes those results and those in Schwärzler, Sebő [7]. We then study the polar of the considered multiflow problems, and characterize the subclass where the integer metric packing theorem holds for bipartite weights; surprisingly, unlike for most of the known multiflow theorems this subclass is not the same as the class where integer multiflow theorems hold for bipartite weights, but is essentially smaller.

1 Introduction

Let M be a binary matroid defined on the finite set $E(M)$ and p a function assigning integer values to the elements of $E(M)$. We think of the negative values of p as representing *demands* and of the nonnegative values as representing *capacities*. Define $F(p) = \{e \in E(M) : p(e) < 0\}$. A *flow problem* is a pair (M,p). It has a *solution* if there exists a *multiflow*, that is a function $\Phi : \mathcal{C}_P(M) \to \mathbb{R}_+$ defined on the set $\mathcal{C}_P(M)$ of all circuits C of M with $|C \cap F(p)| = 1$ such that

$$\sum_{C \in \mathcal{C}_P, C \ni e} \Phi(C) \begin{cases} \leq \ \ p(e), & \text{if } e \in E(M) - F(p), \\ = -p(e), & \text{if } e \in F(p). \end{cases}$$

A function $m : E(M) \to \mathbb{R}_+$ is a *metric* if $m(e) \leq m(C - \{e\})$ for all circuits C of M and all elements e of C. (We use the notation $m(X) = \sum_{e \in X} m(e)$ for subsets X of $E(M)$.) Δ is a *family of metrics* if for every binary matroid M, $\Delta(M)$ is a set of metrics defined on $E(M)$. For $A \subseteq \mathbb{R}_+$, we will denote the family of all metrics $m : E(M) \to A$ by $\Delta_A(M)$, or simply by Δ_A. A metric m is *bipartite* if $m(C)$ is even for all circuits C of M. The extreme rays of the cone $(\Delta_A(M))$ are called *primitive*.

Let Δ be a family of metrics, and (M,p) be a flow problem. Consider the condition

$$m.p \geq 0 \text{ for all } m \in \Delta(M). \qquad (1)$$

It follows easily from LP duality that (1) is necessary for the existence of multiflows, even if $\Delta(M)$ is the set of all metrics on M, and, in this case, (1) is also sufficient. The question that arises is then the following one: When is (1) being true for a specific family of metrics sufficient to imply that (1) is true for all metrics?

A binary matroid M for which the condition (1) is sufficient for the existence of a solution of (M,p) for arbitrary functions p, will be called *flowing with respect to* Δ. A flow problem (M,p) is *Eulerian* if $p(D)$ is even for all cocircuits D of M. If (1) is sufficient for the existence of an integer solution for all Eulerian problems (M,p), then M is called *cycling with respect to* Δ.

A well known and easy fact to be used throughout (it is a consequence of Farkas' Lemma):

Fact 1. *[7] Let M be a matroid and $A \subseteq \mathbb{Z}_+$. M is flowing with respect to Δ_A if and only if $\Delta_{\mathbb{Z}_+} \subseteq$ cone $(\Delta_A(M))$.* □

The polar problem of the multiflow problem could be seen as the packing of a metric m into a set of primitive metrics $\Delta_A(M)$, that is we want to write m as $\lambda_1 m_1 + \ldots + \lambda_k m_k$, $\lambda_i \in \mathbb{Z}_+, m_i \in \Delta_A(M)$, $1 \leq i \leq k$. From Fact 1 it follows that if M is flowing with respect to Δ_A, then a metric m on $E(M)$ may be always written as a – fractional – sum of metrics in A. So now we are interested in a packing with integer coefficients, but with no further hypotheses this seems to be too restrictive. So we ask an integer packing whenever a given metric m is bipartite (see the analogy with "cycling"); if a binary matroid M has this property for all bipartite metrics, then we say that it is *packing with respect to* Δ_A. If the coefficients in the packing are integer multiples of 1/2, we say that M is *half packing*.

The problem of packing metrics in graphs has been raised in several papers in the past: For the case of cut-metrics, Karzanov [2] and Schrijver ([5], [6]) have proved the existence of integer "polars" of several well-known multiflow theorems, and Karzanov in [4] proves the existence of an integer packing of *bip*(2, 3)-*metrics* and cuts for graphs with a demand-set adjacent to at most five vertices. Given an undirected graph $G = (V, E)$ and a partition of V in 5 possibly classes A_1, A_2 and B_1, B_2, B_3, such that $A_1 \cup A_2$ and $B_1 \cup B_2 \cup B_3$ are non-empty, define a metric $m : E \to \mathbb{Z}_+$, a bip(2, 3)-metric, as follows:

$$m(x,y) = \begin{cases} 1, & \text{if } x \in A_i,\ y \in B_j, \\ 2, & \text{if } x \in A_i,\ y \in A_j\ (i \neq j) \text{ or } x \in B_i,\ y \in B_j\ (i \neq j), \\ 0, & \text{if } x, y \in A_i \text{ or } x, y \in B_i. \end{cases}$$

We shall denote by $\mathcal{C}(M)$ the set of cycles (that is, disjoint union of circuits) of the matroid M and by $\mathcal{C}^*(M)$ the set of cocycles. We refer to Welsh [11] for the basic concepts and facts of matroid theory.

In section 2 we give an overview of the multiflow problem in binary matroids and its relation to metrics; in section 3 we study the K_5- and F_7-metrics, showing that both are primitive and that the condition (1) restricted to K_5- and F_7-metrics is sufficient for the existence of a multiflow in a certain class of matroids.

In section 4 we show that $M(K_5)$ and F_7 are packing and that under certain hypotheses we can get a half packing matroid out of a special 2-sum of packing matroids - and that is the best that we can get.

2 Multiflows

The incidence vector χ_D of a cocycle D of M is called a *cut-metric*, and $\Delta_{(CC)}(M)$ denotes the set of all *cut-metrics* of the binary matroid M. We say that (M,p) satisfies the so-called *cut-condition* if and only if

$$m.p \geq 0 \text{ for all } m \in \Delta_{(CC)}(M). \qquad (CC)$$

Seymour's following result (see [10]) tells us that the metrics in $\Delta_{(CC)}$ are sufficient to describe the flowingness with respect to $\Delta_{\{0,1\}}$ and characterizes the related class of matroids.

Theorem 2. *For a binary matroid M the following are equivalent:*
(i) M is cycling with respect to $\Delta_{(CC)}$;
(ii) M is flowing with respect to $\Delta_{\{0,1\}}$;
(iii) M has no F_7, R_{10} or $M(K_5)$ minor. □

F_7 is the Fano matroid on 7 elements, $M(K_5)$ is the graphic matroid of the complete graph on 5 nodes, and R_{10} is a special matroid on 10 elements used to characterize regular matroids [8], that can be represented by the node-edge incidence matrix of the complete bipartite graph $K_{3,3}$, plus a column of 1.

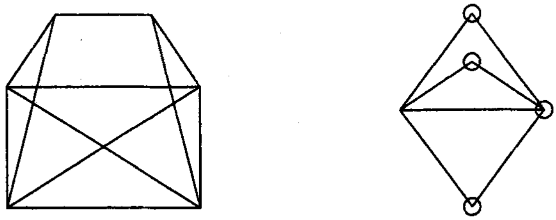
Schwärzler and Sebő [6] have shown that extending the cut condition to a larger class of metrics, called $CC3$-metrics, a statement similar to Seymour's holds for a larger class of matroids. We will deduce the following sharper form in Sect. 3, where $CC3$ is replaced by the cut-condition or either of two conditions which correspond to the only primitive metrics in $CC3$.

Theorem 3. *For a binary matroid M the following are equivalent:*
(i) M is cycling with respect to $\Delta_{(CC,F_7,K_5)}$;
(ii) M is flowing with respect to $\Delta_{\{0,1,2\}}$;
(iii) M has no $AG(2,3)$, S_8, R_{10}, $M(H_6)$, $M(K_5)\bigoplus_2 F_7$, $M(K_5)\bigoplus_2 M(K_5)$, $F_7\bigoplus_2 F_7$ minor. □

Here H_6 is the graphic matroid in Figure 1 (a), $AG(2,3)$ is the representation of a projective plane and S_8 can be represented as the node-edge incidence matrix of the graph in Figure 1 (b), with a column with the circled elements. The definition of 2-sum $M_1 \bigoplus M_2$ of binary matroids is given in [10].

3 The two conditions

Let $\Delta_{K_5}(M)$ (respectively $\Delta_{F_7}(M)$) be the class of metrics $m \in \Delta_{\{0,1,2\}}$ such that, if we contract the elements e with $m(e) = 0$, we obtain a $M(K_5)$ (respectively F_7), probably with some parallel elements, with the weights on each



(a) (b)

Fig. 1. H_6 and S_8

element of a parallel class defined below. For the K_5, if we denote by $\{1,2,3,4,5\}$ the set of vertices, and by ij the edge between the vertices i and j, then we have

$$m(ij) = \begin{cases} 2, & \text{if } ij \in \{12, 23, 13, 45\}, \\ 1, & \text{otherwise.} \end{cases}$$

If C is a three-element circuit of $\mathcal{C}(F_7)$, then we define

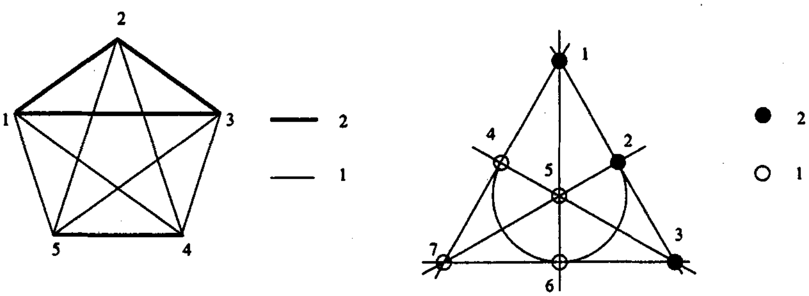


Fig. 2. K_5- and F_7-metrics

$$m(e) = \begin{cases} 2, & \text{if } e \in C, \\ 1, & \text{otherwise.} \end{cases}$$

Lemma 4. *The K_5- and F_7-metrics are primitive.*

Proof. We will show that the F_7-metric is an extreme ray of the cone $\Delta_{\mathbb{Z}_+}$ (for K_5 the proof works in the same way, and is well known, see for example Karzanov [3]). If it is not primitive, then m can be decomposed in a sum of primitive metrics, and the equalities $m(C-e) = m(e)$, $e \in C \in \mathcal{C}(F_7)$, satisfied by the F_7-metric, must be satisfied by any primitive metric in the decomposition. We check that the only solution to the system formed by these equalities is the F_7-metric, and its positive multiples.

To facilitate our task, let x_i denote the value of the function x on the element i, following Figure 2. Then we have the equalities:

$$\left.\begin{array}{l} x_1 = x_4 + x_7 = x_5 + x_6 \\ x_2 = x_5 + x_7 = x_4 + x_6 \end{array}\right\} \Rightarrow x_5 = x_4, \quad x_6 = x_7,$$

and in the same way we obtain that $x_4 = x_7$, $x_5 = x_6$ and so $x_4 = x_5 = x_6 = x_7$ and $x_1 = x_2 = x_3 = 2x_4$, and this corresponds to the F_7-metric, proving that it is the only primitive metric in the decomposition. □

Now we prepare the proof of the implication (iii) $\Rightarrow$ (i) of the Theorem 3. A twofold application of Seymour's 'Splitter Theorem' gives the following [10].

Proposition 5. *Every binary matroid with no $AG(2,3), S_8, R_{10}$ or $M(H_6)$ minor may be obtained by 1- and 2-sums from matroids cycling with respect to $\Delta_{(CC)}$ and copies of F_7 and $M(K_5)$.* □

And we can use it to prove that

Proposition 6. *Any 2-sum $M_1 \bigoplus_2 M_2$ of a matroid M_1 cycling with respect to $\Delta_{(CC,K_5,F_7)}$ and a matroid M_2 cycling with respect to $\Delta_{(CC)}$ is cycling with respect to $\Delta_{(CC,K_5,F_7)}$.*

Proof. Let $E(M_1) \cap E(M_2) = \{f\}$ and $M = M_1 \bigoplus_2 M_2$. Choose $p : E(M) \to \mathbb{Z}$ such that (M,p) is Eulerian and (CC, K_5, F_7) is satisfied. We define functions $p_i : E(M_i) \to \mathbb{Z}$ $(i \in \{1,2\})$ in the following way:

$$p_i(e) = \begin{cases} p(e), & \text{if } e \in E(M_i) - f, \\ (-1)^{i-1}q, & \text{if } e = f, \end{cases}$$

where $q = \min\{p(D - f) : f \in D \in C^*(M_2)\}$. Let D_0 be a cocycle of M_2 with $p(D_0 - f) = q$.

Claim 1. *p_i $(i \in \{1,2\})$ is an Eulerian function.*

Proof. Let D_i be a cocycle of M_i. If $f \notin D_i$, then $p_i(D_i) = p(D_i) \equiv \bmod 2$, because D_i is also a cocycle of M. If $f \in D_i$, then

$$\begin{aligned} p_i(D_i) &= p_i(D_i - f) + p_i(f) \\ &\equiv p(D_i - f) + p(D_0 - f) \equiv p(D_i \triangle D_0) \bmod 2, \end{aligned}$$

because $D_i \triangle D_0$ is a cocycle of M. □

Claim 2. *(M_2, p_2) satisfies (CC).*

Proof. Let $D \in C^*(M_2)$. If $f \notin D$, then again D is a cocycle of M and $p_2(D) = p(D) \geq 0$, because we assumed that (CC, K_5, F_7) and so in particular (CC) is satisfied for (M,p). If $f \in D$, then the definition of q implies the following inequality: $p_2(D) = p_2(D - f) + p_2(f) = p(D - f) - p(D_0 - f) \geq 0$. □

Claim 3. *(M_1, p_1) satisfies (CC, K_5, F_7).*

Proof. We have to show that $pm_1 \geq 0$ for every choice of $m_1 \in \Delta_{(CC,K_5,F_7)}(M_1)$.

If m_1 is a CC-metric, then everything works as in Claim 2. Otherwise we associate to m_1 a metric $m \in \Delta_{(CC,K_5,F_7)}(M)$ defined as

$$m(e) = \begin{cases} m_1(e), & \text{if } e \in E(M_1) - f, \\ m_1(f), & \text{if } e \in D_0, \\ 0, & \text{otherwise.} \end{cases}$$

It is not difficult to see that if m_1 is a K_5- or F_7-metric on M_1, then m is a K_5- or F_7-metric on M. And so we have that

$$\begin{aligned} p_1 m_1 &= \sum_{e \in E(M_1)} p_1(e) m_1(e) \\ &= \sum_{e \in E(M)} p_1(e) m_1(e) + p_1(f) m(f) \\ &= \sum_{e \in E(M) - D_0} p(e) m(e) + p(D_0 - f) m(f) \\ &= \sum_{e \in E(M)} p(e) m(e) \\ &\geq 0, \end{aligned}$$

because (M, p) satisfies (CC, K_5, F_7). Thus Claim 3 is proved. □

As M_1 (respectively M_2) was assumed to be cycling with respect to $\Delta_{(CC,K_5,F_7)}$ (respectively $\Delta_{(CC)}$), the above claims guarantee the existence of integer flows ϕ_i in (M_i, p_i) ($i \in \{1,2\}$). ϕ_i consists of a list of cycles of $\mathcal{C}_{p_i}(M_i)$. Suppose without loss of generality that precisely the first k_i cycles of each list contain the element f. It follows from the definition of a flow that $k_i \leq q = k_2$. After deleting the first $k_2 - k_1$ cycles from the second list ϕ_2, the union of the two lists contains exactly k_1 cycles of $\mathcal{C}(M_1)$ and k_1 cycles of $\mathcal{C}(M_2)$ passing through the element f. Build k_1 pairs (C_1, C_2) ($C_i \in \mathcal{C}(M_i)$) of the cycles passing through f and replace each pair by $C_1 \Delta C_2$. It is easy to see that the list of cycles obtained in this way represents an integer flow of (M, p). □

Proof. (of Theorem 3:)
(i) $\Rightarrow$ (ii) trivial.
(ii) $\Rightarrow$ (iii) There are several ways of proving this implication. [7] checks it by showing multiflow problems that have no solution, but whose multiflow functions satisfy (1) for $\Delta_{\{0,1,2\}}$. We show that there are primitive metrics for these matroids that are not in $\Delta_{\{0,1,2\}}$, which is a shorter and easier way of proving the implication.

Let S_8, R_{10} and $AG(2,3)$ be represented by the matrices in Figure 3. We will prove the result for the S_8 case, the other ones follow similarly.

$$
\begin{bmatrix}
1&0&0&0&1&1&1&1\\
0&1&0&0&1&0&0&1\\
0&0&1&0&0&1&0&1\\
0&0&0&1&0&0&1&1
\end{bmatrix}
\quad
\begin{bmatrix}
1&1&1&1&1&1&0&0&0&0\\
1&1&1&0&0&0&1&1&1&0\\
1&0&0&1&1&0&1&1&0&1\\
0&1&0&1&0&1&1&0&1&1\\
0&0&1&0&1&1&0&1&1&1
\end{bmatrix}
\quad
\begin{bmatrix}
0&0&0&1&0&1&1&1\\
0&0&1&1&1&0&1&0\\
0&1&1&1&0&0&0&1\\
1&1&1&1&1&1&1&1
\end{bmatrix}
$$

Fig. 3. Matrix representation of S_8, R_{10} and $AG(2,3)$.

Now let $m := (2,1,1,1,1,1,1,3)$. For this metric m we have the following equalities arising from the inequality $m(e) \leq (C - e)$, where C is a circuit in S_8:

$$
\begin{array}{lll}
m_1 = m_2 + m_5 & m_1 = m_3 + m_6 & m_1 = m_4 + m_7 \\
m_8 = m_2 + m_3 + m_7 & m_8 = m_2 + m_4 + m_6 & m_8 = m_3 + m_4 + m_5 \\
m_8 = m_5 + m_6 + m_7. & &
\end{array}
$$

These equations are affinely independent and all solutions for this system are vectors of the form $(2a, a, a, a, a, a, a, 3a)$, $a \geq 0$, which is exactly the extreme ray of the cone of metrics $\Delta_{\mathbb{Z}_+}(S_8)$ defined by m. Therefore m is primitive, but is is not a $(0,1,2)$-vector, so, by Fact 1, S_8 is not flowing with respect to $\Delta_{\{0,1,2\}}$.

In the same way we can show that $m = (3,3,1,1,3,1,1,1,1,1)$ and $m = (1,1,1,4,\ 1,1,1,1)$ define extreme rays of the cone of metrics $\Delta_{\mathbb{Z}_+}(R_{10})$ and $\Delta_{\mathbb{Z}_+}(AG(2,3))$, respectively, proving that they are not flowing with respect to $\Delta_{\{0,1,2\}}$.

A primitive metric for H_6 is represented in Figure 4, and one can check that it is primitive in the same way as in the cases above.

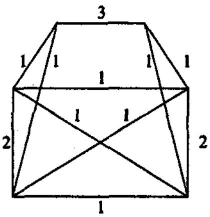


Fig. 4. H_6

Now let $M(K_5) \bigoplus_2 M(K_5)$ be as in Figure 5, the marker is indicated in dashed line, and let $m : E(M(K_5) \bigoplus_2 M(K_5)) \to \mathbb{Z}$ be as follows

$$
m(x) := \begin{cases} 4, & \text{if } x \in \{j, l, o, r\}, \\ 2, & \text{if } x \in \{a, c, i, k, m, n, p, q\}, \\ 1, & \text{otherwise.} \end{cases}
$$

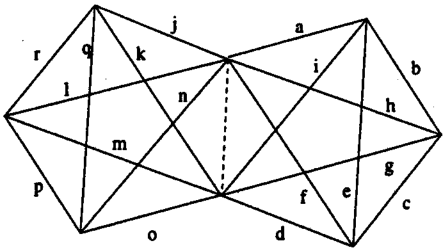


Fig. 5. $M(K_5) \bigoplus_2 M(K_5)$

The following are tight inequalities for m

$$\begin{array}{llll}
r = p + q & r = m + k & j = n + q & j = k + f + d \\
j = k + g + h & l = p + n & l = m + f + d & l = m + g + h \\
o = k + q & o = p + m & o = n + f + d & o = g + h + n \\
a = f + e & a = h + b & i = b + g & i = d + e \\
c = e + b & c = f + h & c = d + g &
\end{array}$$

and it is not difficult to find that the solutions for the system are

$$s(x) := \begin{cases} 4\delta, & \text{if } x \in \{j, l, o, r\}, \\ 2\delta, & \text{if } x \in \{a, c, i, k, m, n, p, q\}, \\ 1\delta, & \text{otherwise,} \end{cases}$$

where $\delta \geq 0$.

Let $F_7 \bigoplus_2 M(K_5)$ and $F_7 \bigoplus_2 F_7$ be as in Figure 6, the markers are indicated by dashed lines, and $m_1 : E(F_7 \bigoplus_2 M(K_5)) \to \mathbb{Z}$ and $m_2 : E(F_7 \bigoplus_2 F_7) \to \mathbb{Z}$ be as follows

$$m_1(x) := \begin{cases} 4, & \text{if } x \in \{g, h, i, m\}, \\ 2, & \text{if } x \in \{e, f, j, k, l, n, o\}, \\ 1, & \text{otherwise.} \end{cases}$$

$$m_2(x) := \begin{cases} 4, & \text{if } x \in \{a, b, e\}, \\ 2, & \text{if } x \in \{c, d, f, l, m\}, \\ 1, & \text{otherwise.} \end{cases}$$

We can check in the same way as above that both metrics m_1 and m_2 are primitive, and as they are not $(0,1,2)$-vectors, and together with the Fact 1 they show that the matroids are not flowing with respect to $\Delta_{\{0,1,2\}}$.

(iii) $\Rightarrow$ (i) Now we know that K_5 and F_7 are flowing with respect to $\Delta_{\{CC,K_5,F_7\}}$ (see [4] and [7]). Using these results with Proposition 6 we get easily the desired implication. $\square$

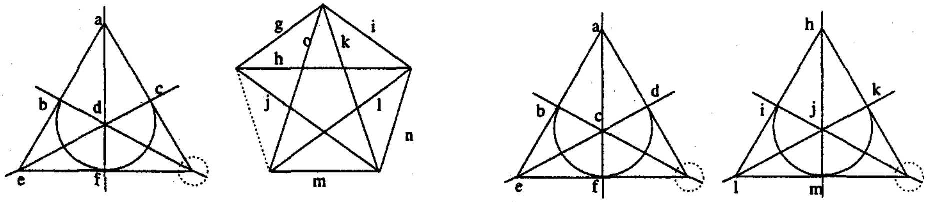


Fig. 6. $F_7 \bigoplus_2 M(K_5)$ and $F_7 \bigoplus_2 F_7$

4 Packing matroids

We prove now two statements showing that K_5 and F_7 are packing with respect to $\Delta_{\{CC,K_5,F_7\}}$. The first is in fact a consequence of a theorem of Karzanov ([4]), but for the sake of completeness and because of the analogy of our proof for K_5 and F_7 we include a simple proof. We say that we *can subtract* a metric m_2 from m_1 if $m_1 - m_2$ is a metric. If both m_1 and m_2 are bipartite, then obviously $m_1 - m_2$ is also bipartite.

Lemma 7. *In the matroid $M(K_5)$ every bipartite metric can be expressed as a positive integer sum of metrics in $\Delta_{(CC,K_5)}(M(K_5))$.*

Proof. Let m be a bipartite metric on $M(K_5)$. We want to write it as an integer sum of cuts and K_5-metrics. By Theorem 3 and Fact 1 we know that m can be expressed as $m = \nu_1\chi_{C_1} + \ldots + \nu_n\chi_{C_n} + \lambda_1 m_1 + \ldots + \lambda_k m_k$, where C_i is a cut, m_i is a K_5-metric, and $\nu_i, \lambda_i > 0$.

Claim 1. *Let D be a cut on K_5. If $(m-\chi_D)(C-e)-(m-\chi_D)(e) < 0$, for a circuit C and $e \in C$, then there exists a triangle T (a circuit of cardinality 3), and an element $f \in T$, such that $(m-\chi_D)(T-f)-(m-\chi_D)(f) < 0$.*

Proof. Let $l = m - \chi_D$. If $|C| = 4$, then there exists a triangle T such that $|T \cap C| = 2$, $T \setminus C = \{f\}$, $e \in T \cap C$. Let $T' = C \Delta T$. Then $0 > l(C-e) - l(e) = l(T-e) - l(e) + l(T'-f) - l(f)$ and so we have that either $l(T-e)-l(e) < 0$ or $l(T'-f)-l(f) < 0$.

If $|C| = 5$, then there exists a triangle T, $|T \cap C| = 2$, $T \setminus C = \{f\}$, $e \in T \cap C$ and a circuit $C' = C \Delta T$, $|C'| = 4$, such that $0 > l(C-e) - l(e) = l(T-e) - l(e) + l(C'-f) - l(f)$. Now either we get directly the conclusion, or we use the previous case. □

Claim 2. *If C_i is a cocircuit, then χ_{C_i} can be subtracted from m.*

Proof. If χ_{C_i} cannot be subtracted from m, then by Claim 1, there is a triangle T and $e \in T$, such that

$$(m - \chi_{C_i}(T-e) - (m-\chi_{C_i})(e) < 0. \qquad (*)$$

As $m(T-e)-m(e)=0$ implies that $\chi_{C_i}(T-e)-\chi_{C_i}(e)=0$, we may suppose that $m(T-e)-m(e)\geq 2$. As $\chi_{C_i}(T-e)-\chi_{C_i}(e)\leq 2$, we cannot have $(*)$. □

Claim 3. *If m_i and m_j are different K_5-metrics, then m_i+m_j can be written as a sum of cut-metrics.*

Proof. The Figure 7 shows the sum of two K_5-metrics, which has the following cut decomposition (we use the numeration of Figure 2): $\{15,25,35,14,24,34\}$ $+\{15,25,45,13,23,34\}+\{12,13,15,24,34,45\}+\{12,14,15,23,34,35\}$. The other cases are similar. □

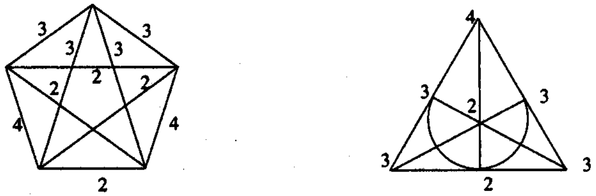


Fig. 7. Sums of two K_5-metrics and two F_7-metrics

The three Claims above imply that we can obtain from a fractional decomposition for m an integer decomposition for m consisting on cut-metrics and K_5-metrics. □

Lemma 8. *In the F_7 matroid every bipartite metric may be expressed as a positive integer sum of metrics in $\Delta_{(CC,F_7)}$.*

Proof. Let m be a bipartite metric on F_7. We proof as above:

Claim 1. *If C_i is a cocircuit, then χ_{C_i} can be subtracted from m.*

Proof. χ_{C_i} may be subtracted from m if and only if

$$m(C-e)-m(e)\geq \chi_{C_i}(C-e)-\chi_{C_i}(e), \quad \text{for all } e\in C\in \mathcal{C}(F_7).$$

$m(C-e)-m(e)=0$ implies that $\chi_{C_i}(C-e)-\chi_{C_i}(e)=0$. If $m(C-e)-m(e)\geq 2$, then $|C_i\cap C|\equiv 0 \bmod 2$, because F_7 is a binary matroid. So we have only to consider when $|C_i\cap C|=4$. In this case $C=C_i$ and $e\in C_i\cap C$, so $\chi_{C_i}(C-e)-\chi_{C_i}(e)\leq 2$, and we have the desired inequality. □

Claim 2. *If m_i and m_j are different F_7-metrics, then m_i+m_j can be written as a sum of cut metrics.*

Proof. In Figure 7 we show the sum of two F_7-metrics, and its decomposition in metric-cuts is: $\{1,2,4,5\}+\{1,3,5,7\}+\{1,2,6,7\}+\{1,3,4,6\}+\{2,3,4,7\}$ (with the same numeration of Figure 2). All the others are symmetric. □

Now putting together Claims 1 and 2 we obtain that m is an integer sum of cut-metrics and perhaps one F_7-metric, with an integer coefficient too – because as we can always subtract the cut-metrics from m, repeating this procedure we will end up with a F_7-metric and necessarily with an integer coefficient. □

We will introduce now a concept that will help us prove some new characterizations of flowing and packing matroids. We say that M has *the sums of circuits property* (see [9]) if the following are equivalent for all $p : E(M) \rightarrow \mathbb{Z}_+$:
(i) There is a function $\alpha : \mathcal{C}(M) \rightarrow \mathbb{R}_+$ such that $\sum \alpha(C)\chi_C = p$.
(ii) For every cocircuit D and $f \in D$, $p(f) \leq p(D - \{f\})$.

In [10] Seymour characterized matroids that have the sums of circuits property - they are the duals of those flowing with respect to $\Delta_{(CC)}$, and conjectured the following result, proved by Alspach, Goddyn and Zhang ([1]).

Theorem 9. *If M is a binary matroid and has no F_7^*, R_{10}, $M^*(K_5)$ or $M(P_{10})$ minor, and p satisfies (ii) and is Eulerian, then there is an integral α satisfying (i).* □

$M(P_{10})$ is the graphic matroid of the Petersen graph. Dualizing this result, we get a class of packing matroids:

Corollary 10. *If M is a binary matroid and has no F_7, R_{10}, $M(K_5)$ or $M^*(P_{10})$ minor, then M is packing with respect to $\Delta_{(CC)}$.* □

Notice that this is the class of matroids cycling with respect to $\Delta_{(CC)}$, except for those containing $M(P_{10})$ as minor. We know, by Lemmas 7 and 8, that the $M(K_5)$ and F_7 matroids are packing with respect to $\Delta_{(CC,K_5,F_7)}$. We would like to join these classes of matroids and obtain something "bigger". The 2-sum of matroids could help us in this direction, but the packing property is not preserved by the 2-sum. We show in Figure 8 an example of a bipartite metric on a 2-sum of matroids that are each packing, but its decomposition in primitive metrics is half-integer.

Notice that in the Proposition 6 we do not "2-sum" two matroids $M(K_5)$ or F_7, or a $M(K_5)$ with an F_7, so we might suppose that the matroid $M = M_1 \bigoplus_2 M_2$ is in fact a "big" 2-sum of several matroids packing with respect to $\Delta_{(CC)}$ and one $M(K_5)$ or (exclusive) one F_7. This fact will be used in the following proposition, which is not a recursive result, but shows a way of decomposing a metric in such a "big" 2-sum.

Lemma 11. *A matroid M resulting from 2-sums of one copy of $M(K_5)$ or F_7, and matroids packing with respect to $\Delta_{(CC)}$ is half packing with respect to $\Delta_{(CC,K_5,F_7)}$.*

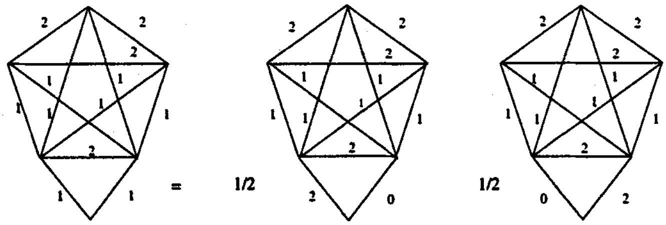


Fig. 8. A half integer metric-packing

Proof. We will find a half integral metric packing in M from integral metric packings of each piece of the 2-sums.

Let $M = R_1 \bigoplus_2 R_2 \bigoplus_2 \dots \bigoplus_2 R_n$ where R_1 is either a $M(K_5)$ or a F_7, and each R_i, $2 \leq i \leq n$, is a matroid packing with respect to $\Delta_{(CC)}$ that is being "2-summed" with R_1. Let $f_i = E(R_1) \cap E(R_i)$, $2 \leq i \leq n$.

Choose $m : E(M) \to \mathbb{Z}_+$ such that m is a bipartite metric. Let $m_1' : E(R_1) \to \mathbb{Z}_+$ be such that:

$$m_1'(e) := \begin{cases} m(e), & \text{if } e \in E(R_1) - \{f_j,\ 2 \leq j \leq n\}, \\ q_i, & \text{if } e = f_j, \end{cases}$$

where $q_i := \min\{m(C - f_i) : C \in \mathcal{C}(R_i)\}$, $2 \leq i \leq n$. Now we define functions $m_i : E(R_i) \to \mathbb{Z}_+$, $1 \leq i \leq n$, in the following way: $m_1(e) := \min\{m_1'(X) : X = \{e\}$ or $X = C - e$ for some $C \in \mathcal{C}(R_1)$ with $e \in C\}$ and

$$m_i(e) := \begin{cases} m(e), & \text{if } e \in E(R_i) - f_i, \\ m_1(f_i), & \text{otherwise.} \end{cases}$$

Let C_i^0 be the circuit of $\mathcal{C}(R_i) \cup \mathcal{C}(R_1 \bigoplus_2 \dots \bigoplus_2 R_{i-1} \bigoplus_2 R_{i+1} \bigoplus_2 \dots \bigoplus_2 R_n)$ with $f_i \in C_i^0$ and $m(C_i^0 - f_i) = m_i(f_i)$, $2 \leq i \leq n$.

Claim 1. *m_i, $2 \leq i \leq n$, is a bipartite metric.*

Proof. Let C_i be a cycle of R_i. If $f_i \notin C_i$, then $m_i(C_i - e) \leq m_i(e)$, $\forall e \in C_i$, and $m_i(C_i) = m(C_i) \equiv 0 \bmod 2$, because C_i is also a cycle of M.

If $f_i \in C_i$, then $m_i(C_i - f_i) - m_i(f_i) = m(C_i) - m(C_i^0) \geq 0$, and for $e \neq f_i$, $m_i(C_i - e) - m_i(e) = m(C_i - e) + m(C_i^0) - m(e) \geq 0$, because of the definition of C_i^0; and $m_i(C_i) = m_i(C_i - f_i) + m_i(f_i) = m(C_i - f_i) + m(C_i^0 - f_i) \equiv m(C_i \Delta C_i^0) \equiv 0 \bmod 2$, because $C_i \Delta C_i^0$ is a cycle of M. □

Claim 2. *m_1 is a bipartite metric.*

Proof. By the definition of m_1 as an induced metric, we have only to verify that m_1 is bipartite. Let C be a cycle of R_1 and $L := \{j : \exists f_j \in C\}$, then

$$\begin{aligned} m_1(C) &= m_1(C\backslash\{f_j : j \in L\}) + \sum_{j\in L} m_1(f_j) \\ &= m(C \setminus \{f_j : j \in L\}) + \sum_{j\in L} m(C_j - f_j) \\ &\equiv m(C\Delta_{j\in L}C_j^0) \\ &\equiv 0 \bmod 2, \end{aligned}$$

because $C\Delta_{j\in L}(C_j^0)$ is a cycle of M. □

As R_1 (respectively R_i, $2 \leq i \leq n$) was assumed to be packing with respect to $\Delta_{(CC,K_5,F_7)}$ (respectively to $\Delta_{(CC)}$), the above Claims guarantee the existence of cocircuits $C_1^i, \ldots, C_{r_i}^i \in \mathcal{C}^*(R_i)$ such that $\sum_{j=1}^{r_i} \chi_{C_j^i} = m_i$ and cocircuits $C_1^1, \ldots, C_{r_1}^1 \in \mathcal{C}^*(R_1)$, and K_5- or F_7-metrics $l_1, \ldots, l_t$ such that $\sum_{j=1}^{r_1} \chi_{C_j^1} + \sum_{j=1}^{t} l_j = m_1$.

Now we make the packing of cocircuits, K_5- and F_7-metrics for M and f. We can already put in the list the cocircuits C_j^1 $(j = 1, \ldots, r_1)$.

We will consider two cases:

Case I : $l_i(f_j) = 1$, $i \in \{1, \ldots, t\}$, $j \in \{2, \ldots, n\}$

To each K_5- or F_7-metric l_i such that $l_i(f_j) = 1$, we associate a cocircuit C_k^j, $k \in \{1, \ldots, r_j\}$, where $f_j \in C_k^j$ and we join l_i to C_k^j to create a K_5- or F_7-metric l_i' in the matroid M. It is not difficult to see how to "glue" l_i and C_k^j: the elements in C_k^j will replace f_j, creating perhaps some parallel elements in the new K_5- or F_7-metric. We will replace l_i by l_i', if there is some j' such that $l_i(f_{j'}) = 2$; otherwise l_i' goes to the list of the packing.

Case II : $l_i(f_j) = 2$, $i \in \{1, \ldots, t\}$, $j \in \{2, \ldots, n\}$

In this case, if we have, for every circuit $K \in \mathcal{C}(R_j)$ with $f_j \in K$,

$$m_j(K - f_j - e) - m_j(e) \geq 0, \quad \text{for all } e \in K,$$

then we can simply contract f_j in both matroids R_1 and R_j and find a packing of cocircuits for l_i in $R_j/\{f_j\}$ in R_1 and a new packing of cocircuits in $R_j/\{f_j\}$ for m_j, and add this last packing to the list. If the corresponding K_5- or F_7-metric has already been transformed into cocircuits, we have only to find a packing in $R_j/\{f_j\}$.

For these metrics l_i that have been transformed, and for all f_j such that $l_i(f_j) = 2$, but f_j has not been contracted, there is a natural association between cocircuits in the decomposition of l_i and cocircuits in the decomposition of m_j which contain f_j; the symmetric difference of such pairs of cocircuits is a cocycle in M, and we put them all in the list.

Now we have only metrics l_i not yet decomposed for which there exists a f_j with $l_i(f_j) = 2$ and one can not contract f_j as above.

To each l_i and each R_j for which we have $l_i(f_j) = 2$, as $m_1(f_j) = m_j(f_j)$, we know that we can associate two cocircuits $C_1^j, C_2^j \in \mathcal{C}(R_j)$, such that $f_j \in C_1^j \cap C_2^j$. We now build two lists L_k, $k = 1, 2$ of all these C_j^k, for all j with $l_i(f_j) = 2$. We will transform the metric l_i and the lists L_1, L_2 into two single K_5- or F_7-metrics in the following way ($k = 1, 2$):

$$l_i^k(e) := \begin{cases} l_i(e), & \text{if } e \neq f_j,\ \forall j \in \{2, \dots n\}, \\ 2, & \text{if } e \in C \in L_k,\ e \neq f_j, \\ 0, & \text{otherwise.} \end{cases}$$

It is not difficult to see that l_i^1, l_i^2 are K_5- or F_7-metrics in M, and that

$$\frac{1}{2}l_i^1 + \frac{1}{2}l_i^2 = l_i + \sum_{C \in L_1} \chi_{C - f_j} + \sum_{C \in L_2} \chi_{C - f_j}.$$

And this completes the packing.

At the end we add to the list the metrics corresponding to the remaining cocircuits that do not contain the marker. □

Remark: In the second case of the proof, if $l_i^1 = l_i^2$, then the sum $\frac{1}{2}l_i^1 + \frac{1}{2}l_i^2$ is integer, and so, the result of the decomposition may be integer.

As a conclusion we get that

Theorem 12. *If a binary matroid M has no $M^*(P_{10})$ minor, then M is cycling with respect to $\Delta_{(CC,F_7,K_5)}$ if and only if M is half-packing with respect to this set of metrics.*

Proof. If M is cycling with respect to $\Delta_{(CC,F_7,K_5)}$, then, by Theorem 3 and Proposition 5, M may be obtained by 1- and 2-sums from matroids cycling with respect to $\Delta_{(CC)}$ and one copy of F_7 or (exclusive) one copy of $M(K_5)$. From Corollary 10 and the hypothesis that M has no $M^*(P_{10})$ minor, we conclude that M may be obtained by 1- and 2-sums from matroids packing with respect to $\Delta_{(CC)}$ and one copy of F_7 or (exclusive) one copy of $M(K_5)$.

Now, using Lemma 11 one sees that M is half-packing with respect to $\Delta_{(CC,F_7,K_5)}$.

In the other direction, from the proof of Theorem 3 one can conclude that the matroids $AG(2,3)$, S_8, R_{10}, $M(H_6)$, $M(K_5) \bigoplus_2 F_7$, $M(K_5) \bigoplus_2 M(K_5)$ and $F_7 \bigoplus_2 F_7$ are not half-packing with respect to $\Delta_{(CC,F_7,K_5)}$, since there is a primitive metric for each of them that is not in $\Delta_{\{0,1,2\}}$. □

As $M^*(P_{10})$ is not graphic, for graphic matroids we get the following sharper result:

Theorem 13. *For a graph G the following are equivalent:*
(i) G is cycling with respect to $\Delta_{(CC,K_5)}$;
(ii) G is flowing with respect to $\Delta_{\{0,1,2\}}$;
(iii) G has no H_6, $K_5 \bigoplus_2 K_5$ as minor;
(iv) G is half-packing with respect to $\Delta_{(CC,K_5)}$. □

We will see that the example given in Fig. 8 is essentially the only one, where we 2-sum a $M(K_5)$ or a F_7 with a matroid consisting on two circuits C_1, C_2 such that $C_1 \cap C_2 = \{f\}$, where f is the marker of the 2-sum, and $|C_1| \geq 3$. We call this configuration a $\bar{K}_5$ and a $\bar{F}_7$, respectively.

For the characterization we will need first the following lemma.

Lemma 14. *The matroid M, resulting from the 2-sum of a matroid M_1 that is packing with respect to $\Delta_{(CC,K_5,F_7)}$ with a matroid M_2, that is a circuit, is metric packing with respect to $\Delta_{(CC,K_5,F_7)}$.*

Proof. If M_1 does not contain $M(K_5)$ or F_7 as a minor, the result is trivial. So let M_1 contain one of $M(K_5)$ and F_7 as minor. Given a bipartite metric m on the matroid M, we will find an integral decomposition for m as a sum of cocircuits, F_7- or K_5-metrics. Let $M_1 \cap M_2 = \{f\}$.

As in the proof of Lemma 11, we define two metrics $m_1 : E(M_1) \rightarrow \mathbb{Z}_+$ and $m_2 : E(M_2) \rightarrow \mathbb{Z}_+$ such that

$$m_i(e) = \begin{cases} m(e), & \text{if } e \in E(M_i) - f, \\ q, & \text{if } e = f, \end{cases}$$

where $q = \min\{m(C - f) : C \in \mathcal{C}(M_1) \cup \mathcal{C}(M_2)\}$.

With a similar reasoning to the proof of Lemma 11 it can be seen that m_i, $i = 1,2$, is a bipartite metric, and so there are cocircuits $C_1, \ldots, C_r \in \mathcal{C}^*(M_2)$ such that $\sum_{j=1}^{r} \chi_{C_j} = m_2$, and we assume that the first k_2 cocircuits contain f; and there are cocircuits $D_1, \ldots, D_s \in \mathcal{C}^*(M_1)$, and K_5- or F_7-metrics $l_1, \ldots, l_t$ such that $\sum_{j=1}^{r} \chi_{D_j} + \sum_{j=1}^{t} l_j = m_1$. We suppose that the first k_1 cocircuits contain f, and that the first k_3 F_7- or K_5-metrics l_i are such that $l_i(f) = 1$. Notice that $k_2 = k_1 + k_3 + 2(t - k_3)$.

To each l_j, $1 \leq j \leq k_3$, and to each D_i, $1 \leq i \leq k_1$, we associate a cocircuit $C_k \in \mathcal{C}^*(M_2)$, in each of them we replace f by C_k, and the result is clearly a cocircuit, a K_5- or an F_7-metric.

Now we associate to each l_i, $k_3+1 \leq i \leq t$, two cocircuits C_j, C_k, and replace all of them by B_1 and B_2 defined as follows. Let $B_1 = C_j \Delta C_k$, B_1 is a cocircuit in M_2, and so in M. Let $B_2 = E(M_2) - (C_j \Delta C_k)$; if $|B_2| \equiv 0 \bmod 2$, then since M_2 is a circuit, B_2 is a cocycle in M_2, and we replace in l_i the element f with $B_2 - f$. If $|B_2| \equiv 1 \bmod 2$, then we decompose l_i with $l_i(f) = 0$ into cocircuits, and put them in the list, with a coefficient 2. We add $B_2 - f$ to the list, with a coefficient 2, if $B_2 - f \neq \emptyset$, since in this case $B_2 - f$ is a cocycle in M_2 and so in M. Proceeding this way we get an integer packing of cocircuits and F_7- or K_5-metrics for m. □

Combining Corollary 10 and Lemma 14 one gets the following:

A binary matroid M flowing with respect to $\Delta_{\{0,1,2\}}$ is packing with with respect to $\Delta_{(CC,F_7,K_5)}$ if and only if M has no $M^*(P_{10})$, $\bar{K}_5$, $\bar{F}_7$, R_{10} minor. □

Just before submitting this article, the authors have realized that in fact a characterization of packing matroids follows. (A matroid is *packing* if it is integer packing with respect to its primitive metrics.)

For a binary matroid M the following statements are equivalent (provided R_{10} is packing):
(i)M is packing,
(ii) M is packing with respect to $\Delta_{(CC,F_7,K_5,R_{10})}$,
(iii) M has no $M^*(P_{10})$, $\bar{F}_7$, $\bar{K}_5$, $\bar{R}_{10}$ minor.

The matroid $\bar{R}_{10}$ is defined in the same way as $\bar{F}_7$, $\bar{K}_5$. The fact that R_{10} is packing is being checked (if not, that makes only trivial changes to the theorem).

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