The Connectivity of Minimal Imperfect Graphs

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ABSTRACT

We prove that partitionable graphs are $2\omega - 2$-connected, that this bound is sharp, and prove some structural properties of cutsets of cardinality $2\omega - 2$. The proof of the connectivity result is a simple linear algebraic proof. © 1996 John Wiley & Sons, Inc.

1. INTRODUCTION

Let $G = (V, E)$ be a graph with vertex-set $V = V(G)$, and edge-set $E = E(G)$.

$V' \supseteq V$ is said to be a (vertex-) *cutset*, if $G - V'$ is not a connected graph (or $V' = V$), and similarly, $E' \subseteq E$ is an (edge-) cutset, if $G - E'$ is not a connected graph. For $k \in \mathbb{N}, G$ is said to be *k-connected* or *k-edge-connected*, if $|V(G)| \geq k + 1$ and the cardinality of every vertex-cutset or edge-cutset respectively, is at least k. Clearly, if G is k-connected, then it is also k-edge-connected, but the converse is not necessarily true. The (vertex- or edge-) *connectivity* number of a graph is the maximum k for which the graph is k (vertex or edge-) connected.

$\omega = \omega(G)$ denotes the cardinality of a maximum clique of a graph, and $r(G)$ denotes the linear rank of the set of characteristic vectors (in $\mathbb{R}^V$) of the ω-cliques of G, and is called the *clique rank* of G, it was introduced in Fonlupt and Sebő [8]. $\alpha(G)$ denotes $\omega(\bar{G})$, that is the cardinality of a maximum stable set. If $k \in \mathbb{N}$, a k-clique or k-stable set will mean a clique or stable set of size k. If $V' \subseteq V$, then $\omega(V'), \alpha(V')$, and $r(V')$ will denote the maximum clique or stable set or clique rank of the graph induced by V'.

$\chi = \chi(G)$ is the chromatic number of G, that is the minimum number of stable sets partitioning V. If there is exactly one partition into a minimum number of stable sets, G is said to be *uniquely colorable*. Subgraph means induced subgraph in this note.

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A graph G is called *perfect* if $\chi(H) = \omega(H)$ for every subgraph H of G, otherwise it is called *imperfect.* A graph is called *minimal imperfect* if it is not perfect, but all its subgraphs are perfect.

An odd hole is an odd circuit of length at least five, and an odd antihole is the complement of such a graph.

We will use the notation $n = n(G) := |V(G)|$.

Lovász [13] proved that every minimal imperfect graph is *partitionable,* that is, $n = \alpha\omega + 1, (\alpha, \omega \in \mathbb{N}, \alpha \geq 2, \omega \geq 2)$, and $G - v$ can be partitioned both into ω-cliques and into α-stable sets, for arbitrary $v \in V(G)$. If G is partitionable, then clearly, $\chi = \omega + 1, \chi(G - v) = \omega = \omega(G - v)$, and $\bar{G}$ is also partitionable.

Padberg [18] deduced from Lovász's result that the number of ω-cliques of a minimal imperfect graph G is n, moreover they are linearly independent. In fact, what Padberg proved is that the α-stable sets can be listed in the following way: fix an arbitrary α-stable set S, and consider the coloration of $G - s$ for all $s \in S$; the $\alpha\omega + 1 = n$ considered color classes together with S include every α-stable set. As a consequence $G - v$ is uniquely colorable for all $v \in V(G)$.

Bland, Huang, and Trotter [1] observed that the same properties hold for partitionable graphs.

Let G be partitionable. It follows from the unique colorability of the graphs $G - v$ $(v \in V(G))$ that the α-stable sets can be coded with the notation $S(a, b)$, meaning the color-class of $G - a$ which contains b; similarly, $K(a, b)$ denotes the ω-clique containing b in the clique partition of $G - a$. It is easy to see that the unique ω-clique disjoint from $S(a, s)$ is $K(s, a)$.

An edge $e \in E(G)$ is called *critical,* if $\alpha(G \setminus e) > \alpha + 1$. As Markossian, Gasparian, and Markossian [14] and [13] observe, if *xy* is a critical edge, then there is a unique ω-clique containing x and not y. (Indeed, if S denotes the $\alpha + 1$-stable-set of $G \setminus e$, then such a clique K_x is disjoint of the α-stable set $S \setminus \{x\}$. Conversely, the clique disjoint of $S \setminus \{x\}$ must intersect $S \setminus \{y\}$, whence it contains x and not y.)

If G is partitionable, $T \subseteq V(G)$, is called a *small transversal,* if $|T| \leq \alpha + \omega - 1$, and the intersection of T with every ω-clique and every α-stable set is non-empty. Chvátal [6] pointed out that no minimal imperfect graph has a small transversal, because if T was one, then $\alpha(G - T) \leq \alpha - 1, \omega(G - T) \leq \omega - 1$, and $n - |T| \geq (\alpha - 1)(\omega - 1) + 1$, implying that $\chi(G - T) > \omega(G - T)$. There exist partitionable graphs which do have small transversals, and those which don't are not necessarily minimal imperfect.

All these facts about partitionable and minimal imperfect graphs will be used without further reference in the sequel.

An (α, ω)-web is a graph $G = (V, E)$ on $\alpha\omega + 1$ vertices so that $\omega(G) = \omega, \alpha(G) = \alpha$, and there is a cyclical order of V so that every set of ω consecutive vertices in this cyclical order is an ω-clique. (It follows then that there is no other ω-clique in G, and that the definition is symmetric in α and ω : if G is a web, then it also has another cyclic order, where exactly the sequences of α consecutive vertices form α-stable sets.) Chvátal [6] proved that (α, ω)-webs have a small transversal, provided both $\alpha \geq 3$ and $\omega \geq 3$, implying that among webs, only odd holes and antiholes can be minimal imperfect.

We prove in this note a lower bound on the (vertex-) connectivity number of partitionable graphs. The bound will turn out to be sharp for both the vertex- and edge-connectivity number of webs, in particular for odd holes and antiholes. (This provides us with examples for arbitrary α and ω.)

The connectivity results that have been known so far are the following:

— The minimum degree of a minimal imperfect graph is at least $2\omega - 2$, a result proved by Olaru [17], and independently reproved by Markossian and Karapetian [16], and Reed [19].
— The connectivity number of a minimal imperfect graph is at least ω , a result proved in Hougardy [12].

Let us finish this introduction with a relation we will need between the rank $r(G)$ of the set of characteristic vectors of the ω-cliques of a graph, observed by Fonlupt and Sebő [8], as the trivial part of a characterization of perfectness and unique colorability. We include the proof for the sake of completeness.

$(*)$ If $\chi(G) = \omega(G)$, then $r(G) \leq n - \omega + 1$,

and if in addition equality holds, then G is uniquely colorable.

Indeed, let $\chi_1, \ldots, \chi_\omega$ be the characteristic vectors of the color classes (as sets of vertices) in an ω-coloration. The vectors $\chi_1 - \chi_2, \ldots, \chi_1 - \chi_\omega$ are all orthogonal vectors to every ω-clique, and they are linearly independent: the bound follows.

If there exists another coloration, then let $\chi_{\omega+1}$ be the characteristic vector of a color class in this coloration, which is different from $\chi_i (i = 1, \ldots, \omega)$. Clearly, $\chi_1 - \chi_{\omega+1}$ is linearly independent from $\{\chi_1 - \chi_2, \ldots, \chi_1 - \chi_\omega\}$, and it is also orthogonal to the ω-cliques. But then the bound is not tight. ∎

2. CONNECTIVITY AND TIGHTNESS

The intuition behind the high connectivity number of minimal imperfect graphs is that their clique rank is high $(= n)$, but the clique rank of perfect graphs is small $(\leq n - \omega + 1)$: hence, if the ω-cliques are split up into those of two proper subgraphs, then the vertex sets of these must have a big intersection. This is the key to the proof of the following theorem:

Theorem 1. If $G = (V, E)$ is *partitionable,* then it is $2\omega - 2$-connected. Furthermore, if $C \subseteq V$ is a cutset of cardinality $2\omega - 2$, then $\omega(C) \leq \omega - 1, G - C$ has exactly two components C_1 and C_2, and both $C_1 \cup C$ and $C_2 \cup C$ induce uniquely colorable graphs.

Proof. Suppose that $C \subseteq V$ is a cutset, that is $\{V_1, C, V_2\}$ is a partition of $V(G), V_1 \neq \emptyset \neq V_2$, and there is no edge between V_1 and V_2. Let $n_1 := |V_1|, n_2 := |V_2|, k = |C|$.

Clearly, every ω-clique is either in $V_1 \cup C$ or in $V_2 \cup C$. Hence:

$$r(G) \leq r(V_1 \cup C) + r(V_2 \cup C).$$

According to $(*), r(V_i \cup C) \leq n_i + k - (\omega(V_i \cup C) - 1)$, where $\omega(V_i \cup C) = \omega$, because $V_i \cup C$ contains all ω-cliques intersecting V_i $(i = 1, 2)$. (Since $V_i \neq \emptyset$, there exists at least one such ω-clique.) Substituting this, we get

$$n \leq n_1 + k - (\omega - 1) + n_2 + k - (\omega - 1).$$

Finally, using $n_1 + n_2 + k = n :$ we get $k \geq 2(\omega - 1)$ as claimed. In case of equality, we must have equality in all our bounds, in particular $r(G) = r(V_1 \cup C) + r(V_2 \cup C)$, which means exactly that on the right-hand-side no ω-clique is counted more then once, that is, C contains no ω-clique.

In case of equality, we also have $r(V_1 \cup C) = n_i + k - (\omega(V_i \cup C) - 1)$, and then, according to $(*)$, $V_i \cup C$ induces a uniquely colorable graph $(i = 1, 2)$.

At last, we have to prove that both V_1 and V_2 induce connected graphs. Suppose indirectly that $\{V_1', V_2', V_3', C\}$ is a partition of $V(G)$, where none of the classes is empty, and there is no edge between V_1', V_2', and V_3'. Since any of the V_i' can play the role of V_1, $V_i' \cup C$ is uniquely colorable, and $\omega(V_i' \cup C) = \omega (i = 1, 2, 3)$. Moreover since $V_i' \cup V_j' \cup C$ is a proper subgraph,

$$\omega \geq \chi(V_i' \cup V_j' \cup C) \geq \omega(V_i' \cup V_j' \cup C) = \omega,$$

and then clearly, we have equality throughout. On the other hand, the ω-coloration of $V_i' \cup V_j' \cup C$ induces a coloration of both $V_i' \cup C$ and $V_j' \cup C$, which is the unique coloration of these graphs, and therefore the unique coloration of $V_i' \cup C$ and that of $V_j' \cup C$ induce the same coloration of C.

We have now that the unique coloration of $V_i' \cup C$ induces the same coloration of C for all $i = 1, 2, 3$, proving that G is ω-colorable, a contradiction. ∎

We cannot say more about tight cutsets in general than what is claimed in Theorem 1. However, we will say more about a particular kind of tight cutset: those, where on one of the two separated sides there is exactly one vertex, corresponding to the tightness of the Olaru, Markossain, Gasparian, Reed bound. This seems to be a reasonable first treatable case; the neighbors of a vertex are a most natural candidate for a $2\omega - 2$ element cutset. Let us denote the neighbors of $v \in V(G)$ (without v) by $N(v)$.

According to a conjecture of Ravindra (see Chvátal [7]), if a vertex $v \in V(G)$ has exactly $2\omega - 2$ neighbors, then $N(v)$ can also be covered by two $\omega - 1$-cliques. We prove that this latter implies that G is "locally" a web:

Theorem 2. If $G = (V, E)$ is partitionable, $v_0 \in V(G)$, and $N(v_0)$ is covered by two $\omega-1$-cliques, then there exists a (unique) order $v_{-(\omega-1)}, \ldots, v_{-1}, v_0, v_1, \ldots, v_{\omega-1}$ of $\{v_0\} \cup N(v_0)$ so that $\{v_i, v_{i+1}, \ldots, v_{i+\omega-1}\}$ is an ω-clique in G for all $i = -(\omega - 1), \ldots, -1, 0$.

Proof. Since the cardinality of $\{v_0\} \cup N(v_0)$ is by Theorem 1 at least $2\omega - 1$ and on the other hand this set contains all the ω different ω-cliques containing v_0, we can apply $(*)$ (or Theorem 1): $\{v_0\} \cup N(v_0)$ induces a uniquely colorable graph, and the set $\mathcal{K}$ of ω-cliques of this graph is exactly the set of ω-cliques of G which contain v_0. Clearly, $N(v_0)$ is also uniquely colorable.

Claim 1. In the (unique) $\omega - 1$-coloration of $N(v_0)$ the cardinality of every color class is 2.

Indeed, since $|N(v_0)| = 2\omega - 2$ and $\omega(N(v_0)) = \omega - 1$, if not all the $\omega - 1$ color classes are of cardinality 2, then there is one of cardinality 1, let $\{v\}$ $(v \in N(v_0))$ be this color class. $\{v\}$ intersects all the maximum cliques of $N(v_0)$, that is $v \in K$ for every $K \in \mathcal{K}$. Thus v and v_0 are contained in the same ω-cliques. But this is impossible, because v is contained in an ω-clique of the partition of $G - v_0$ into ω-cliques.

Claim 2. If $\{a, b\} \subseteq N(v_0)$ is a color class in the (unique) $\omega - 1$-coloration of $N(v_0)$, then there exists an $\omega - 1$-clique K so that $K \cup \{a\}, K \cup \{b\} \in \mathcal{K}$.

Indeed, let S be an α-stable set, $v_0 \in S$. The α-stable sets of G are S, and the color classes of $G - s$ $(s \in S)$. For $s \in S \setminus \{v_0\}$ one of the color classes of $G - s$ contains v_0, and the others provide an $\omega - 1$-coloration of $N(v_0)$: if a and b are as described in the condition of the claim, then by Claim 1, $S(s, a) = S(s, b)$. Thus $K(a, s) = K(b, s)$ for every $s \in S \setminus \{v_0\}$. We have obtained: in the ω-coloration of $G - a$ and $G - b$ all cliques are the same, except $K(a, v_0)$ and $K(b, v_0)$. But then the symmetric difference of these two is $\{a, b\}$ and the statement follows. (In Markossian, Gasparian, and Markossian's [14], [15] terminology Claim 2 states that ab is a critical edge in $\bar{G}$, and the last two sentences of the proof are equivalent to their remark: ab is a critical edge of $\bar{G}$ if and only if the set $\{a, b\}$ is contained in $\alpha - 1$ α-stable sets.)

Let the color classes provided by Claim 1 be $\{a_i, b_i\}$ and let the corresponding $\omega - 1$-cliques provided by Claim 2 be K_i $(i = 1, \dots, \omega - 1)$.

Let us denote now the two $(\omega - 1)$-cliques of the condition by A and B, respectively, that is $N(v_0) = A \cup B$. Since $\{a_i, b_i\}$ is a stable set, it intersects both A and B in at most one element, so we can choose the notation so that $A = \{a_1, \dots, a_{\omega-1}\}$, and $B = \{b_1, \dots, b_{\omega-1}\}$.

Claim 3. The list $K_i \cup \{a_i\}, K_i \cup \{b_i\}$ $(i = 1, \dots, \omega - 1)$ of ω-cliques in $\mathcal{K}$ lists all elements of $\mathcal{K}$ exactly twice, except for $A \cup v_0$, and $B \cup v_0$, which are listed once.

We only have to prove that every $K \in \mathcal{K}$ is listed at most twice, and $A \cup v_0, B \cup v_0$ are listed at most once, because then, since the list is of length $2\omega - 2$, we must have equality everywhere.

Let $K \in \mathcal{K}$ be arbitrary, and let S be the α-stable set disjoint from K.

If $K = A \cup v_0$, then $S \cap N(v_0) \subseteq B$, whence S intersects $N(v_0)$ in at most one element. But then $K = K_1 \cup a_1 = K_2 \cup a_2$ for some $a_1, a_2 \in A$ is impossible, because it would imply $S \supseteq \{b_1, b_2\}$. (By definition, S is disjoint from K_1 and K_2, and then it is not disjoint from $K_1 \cup b_1$ or $K_2 \cup b_2$.) Thus K is in the list at most once, and the case $K = B \cup v_0$ is the same.

Suppose now indirectly that $K = K_1 \cup \{x_1\} = K_2 \cup \{x_2\} = K_3 \cup \{x_3\}$ say, where $x_1, x_2, x_3 \in N(v_0)$ are three different vertices. Let y_1, y_2, y_3 be the other vertex of the same color as x_1, x_2, x_3 in the coloration provided by Claim 1; that is, according to Claim 2, $K_i \cup \{y_i\}$ $(i = 1, 2, 3)$ are also ω-cliques. S must contain $\{y_1, y_2, y_3\}$, because it is disjoint from $K_i \cup x_i$, but intersects $K_i \cup \{y_i\}$ $(i = 1, 2, 3)$. On the other hand, since $N(v_0)$ is covered by two cliques, a stable set cannot contain 3 elements of $N(v_0)$, and this contradiction proves Claim 3.

Let us define now a bipartite graph, whose vertices are K_i $(i = 1, \dots, \omega - 1)$ constituting one of the two classes of the bipartition, and $\mathcal{K}$ constituting the other; we define an edge between K_i and $K \in \mathcal{K}$ if and only if $K = K_i \cup \{x_i\}$ $(x_i \in \{a_i, b_i\})$. According to Claim 3 the degree of every vertex of this graph is two, except that of $A \cup v_0$ and $B \cup v_0$: the graph can be decomposed into a path between these two, and circuits, all pairwise vertex-disjoint. But since $|(A \setminus B) \cup (B \setminus A)| = 2\omega - 2$, the path here must have at least $2\omega - 2$ edges, which is the number of edges of the graph. Thus the defined graph has a Hamiltonian path between $A \cup v_0$ and $B \cup v_0$, and there are no circuits in it: this means that one can reach $B \cup v_0$ from $A \cup v_0$ in $\omega - 1$ steps by exchanging an $a \in A$ to a $b \in B$ in each step. If we define a_i to be the element of A which is deleted in the ith step, and b_i the element of B which is added, we get the theorem. ∎

3. MUSINGS

In this section we study some connections of the results we proved to the Strong Perfect Graph Conjecture.

First let us note that according to Berge's Strong Perfect Graph Conjecture [2, 3] the bound on the connectivity number provided by Theorem 1 is always tight:

Berge's Strong Perfect Graph Conjecture. A graph is minimal imperfect if and only if it is an odd hole or antihole.

In an odd hole and antihole *every degree is exactly* $2\omega - 2$, whence *Theorem 1 is tight for arbitrary* ω (even if we replace vertex-connectivity with edge-connectivity). Thus the following conjecture follows from Berge's conjecture.

Conjecture 1. If a graph $G = (V, E)$ is minimal imperfect, then it has cutsets of cardinality $2\omega - 2$, and the neighborhood of every vertex is such a cutset. ∎

Conversely, supposing Conjecture 2, Conjecture 1 implies the perfect graph conjecture.

Conjecture 2. Let G be partitionable, and suppose G and $\bar{G}$ have a cutset of cardinality $2\omega - 2$, then G is either an odd hole or antihole or it contains a small transversal. ∎

Conjecture 2 might be proved before the Strong Perfect Graph Conjecture, but we do not think the same about Conjecture 1.

Webs provide examples to the tightness of the bound in Theorem 1 for an arbitrary *pair* ω, α : consider a web where the only edges are those contained in some ω-clique, and let $C = C_1 \cup C_2$, where C_1 and C_2 are both $\omega - 1$ consecutive vertices in the cyclical order defining the web, but they are not next to each other in this cyclic order. Then C *is a cutset,* $|C| = 2\omega - 2$. The neighbors of each vertex are of this form.

Odd holes and antiholes are in the class of webs we considered. In webs we saw minimum cutsets which are not neighbors of a vertex, but all have the following property. This is a slight modification of Ravindra's conjecture for arbitrary cutsets:

Conjecture 3. If C is a cutset of cardinality $2\omega - 2$ in a partitionable graph, then there exist two disjoint ω-cliques K and L and $a \in K, b \in L$ so that $C = (K \setminus \{a\}) \cup (L \setminus \{b\})$. ∎

Let G be partitionable. Tucker [22] has noted that the graph whose vertices are the ω-cliques of G, and two vertices are joined if and only if the intersection of the corresponding ω-cliques is nonempty, is also partitionable. (Easy to check.) Let us call this graph the intersection graph of G, and denote it by $I(G)$.

Conjecture 2 is true if we ask the condition for the intersection graph:

Theorem 3. Let G be a partitionable graph, and suppose that both $I(G)$ has a vertex of degree $2\omega - 2$, and $I(\bar{G})$ has a vertex of degree $2\alpha - 2$. Then G is an odd hole or antihole, or has a small transversal.

Note that the constraint of this theorem means exactly the existence of an ω-clique K and an α-stable set S so that $G - K$ and $\bar{G} - S$ are uniquely colorable. If in addition $K \cap S = \emptyset$, then the proof becomes considerably easier, and has already been proved in Sebő [20]. For a proof of Theorem 3 we refer to Sebő [21].

From Theorem 2 we can immediately read out the following Corollary.

Corollary 1. The Strong Perfect Graph Conjecture is equivalent to the existence of a vertex of degree $2\omega - 2$ in $I(G)$, for every minimal imperfect graph G.

Edges which are contained in ω-cliques are called *determined*, the number of determined edges adjacent to a vertex is the *determined degree*. The results of this note hold for arbitrary partitionable graphs, so they hold in particular if we delete non-determined edges.

Corollary 2. Let G be a partitionable graph, and suppose that both G has a vertex of determined degree $2\omega - 2$, and $\bar{G}$ has a vertex of determined degree $2\alpha - 2$. Then G is an odd hole or antihole, or $I(G)$ has a small transversal.

Applying Theorem 1 to the partitionable graph formed by the determined edges, we get that every edge adjacent to a vertex of degree $2\omega - 2$ of a partitionable graph is determined. So we can also delete "determined" in Corollary 2.

Proof. Note that $I(I(G))$ is the graph we get from G by deleting the non-determined edges. Using this, we get the statement by applying Theorem 2 to $I(G)$ in the place of G, and noting also that $I(G)$ is an odd hole or antihole, if and only if G is so. ∎

Corollary 2, which is the "polar" of Theorem 2 is not a useful statement, because there is not yet a result saying that $I(G)$ has no small transversal if G is minimal imperfect. So we cannot deduce from Corollary 2 the "polar" of Corollary 1, that is we cannot yet replace $I(G)$ by G in Corollary 1. However, we believe that this polar can be shown by proving the following conjecture, the "skew polar" of Theorem 2:

Conjecture 4. Let G be a partitionable graph, and suppose that both G has a vertex of determined degree $2\omega - 2$, and $\bar{G}$ has a vertex of determined degree $2\alpha - 2$. Then G is an odd hole or antihole, or has a small transversal. ∎

Remark. It is not a good idea to study when the degree bound is tight at the same time for a partitionable graph and its complement, because this involves the too strong assumption that both G and $\bar{G}$ have a vertex with only determined edges adjacent to them. A more interesting problem is to characterize *when the bound is tight for the determined degrees.* For example, the assumption of Theorem 2 concerns the determined edges of $I(G)$ and of its complement. (The edge-set of $I(\bar{G})$ is exactly the set of determined edges of the complement of $I(G)$.)

However, we do not have to deal with determined edges: we can always delete them before applying the theorems, and we should do so in order to get the sharpest possible result.

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