Coloring Precolored Perfect Graphs

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Received February 21, 1995

Abstract: We consider the question of the computational complexity of coloring perfect graphs with some precolored vertices. It is well known that a perfect graph can be colored optimally in polynomial time. Our results give a sharp border between the polynomial and NP-complete instances, when precolored vertices occur. The key result on the polynomially solvable cases includes a good characterization theorem on the existence of an optimal coloring of a perfect graph where a given stable set is precolored with only one color. The key negative result states that the 3-colorability of a graph whose odd circuits go through two fixed vertices is NP-complete. The polynomial algorithms use Grötschel, Lovász and Schrijver's algorithm for finding a maximum clique in a graph, but are otherwise purely combinatorial. © 1997 John Wiley & Sons, Inc. J Graph Theory **25:** 207–215, 1997

*This author acknowledges partial support of Czech Research Grants GAUK 351 and GAČR 2167 and partial technical support of University of Oregon in Eugene, Oregon, which the first author visited in 1994/5 in terms of Fulbright grant No. 18589.
†Both authors acknowledge partial support of DIMACS and of the DO-NET project (human capital and mobility).

 CCC 0364-9024/97/030207-09

1. PRELIMINARIES

Throughout the paper, $G = (V, E)$ stands for a graph with vertex-set $V = V(G)$ and edge-set $E = E(G)$. Our terminology and notation are standard: a *clique* is a set of pairwise adjacent vertices, a *stable set* is a set of pairwise nonadjacent vertices, $\omega = \omega(G)$ denotes the maximum size of a clique in $G, \alpha = \alpha(G)$ denotes the maximum size of a stable set and $\chi = \chi(G)$ denotes the chromatic number, that is, the minimum size of a partition of the vertex set into stable sets. An ω-clique is a clique of cardinality ω and k-coloring is a partition of V into at most k stable sets.

A *family* of sets in this paper always means a multiset, that is, the members of the family are not necessarily distinct. The cardinality of such a family $\mathcal{H}$ (denoted by $|\mathcal{H}|$) is the sum of the multiplicities of the members. If $\mathcal{H}$ is a family of subsets of V and $v \in V$ is a vertex, then $d_{\mathcal{H}}(v)$ denotes the cardinality of the subfamily consisting of elements of $\mathcal{H}$ which contain v.

The graph theoretic notions extend naturally to weighted graphs. If $w: V \to \mathbb{Z}_+$ is a weight function on V ($\mathbb{Z}_+$ denotes the set of non-negative integers), the (weighted) clique number $\omega(G, w)$ is the maximum of $w(Q) = \sum_{v \in Q} w(v)$, for cliques $Q \subseteq V(G)$. The (weighted) stability number $\alpha(G, w) := \omega(\bar{G}, w)$ is then defined as the weighted clique number of the complement of G. The weighted chromatic number $\chi(G, w)$ is the minimum of $\sum_{i=1}^{k} \lambda_i$, for stable sets $S_1, \ldots, S_k$ and $\lambda_i \in \mathbb{Z}_+$, such that $\sum_{i \in \{1,\ldots,k\}, v \in S_i} \lambda_i = w(v)$ for every $v \in V$. The sets $S_1, \ldots, S_k$ above with the coefficients $\lambda_1, \ldots, \lambda_k$ are called a $\sum_{i=1}^{k} \lambda_i$-coloring of (G, w). Obviously, $\omega(G, w) \le \chi(G, w)$ for every $w, \omega = \omega(G) = \omega(G, \mathbf{1})$ and $\chi = \chi(G) = \chi(G, \mathbf{1})$, where $\mathbf{1}$ is the all-1 vector.

A graph G is called *perfect*, if $\chi(H) = \omega(H)$ for every induced subgraph H of G. Lovász [10] proved that perfect graphs also satisfy the more general equality $\chi(G, w) = \omega(G, w)$ for arbitrary $w: V \to \mathbb{Z}_+$. As a consequence, clique polytopes of perfect graphs are described by the stable set constraints. It follows by simple polarity (antiblocking relation see [3]) that the stable set polytopes are described by the clique constraints, and hence that $\bar{G}$ is perfect. In particular, the equality $\chi(\bar{G}, w) = \alpha(G, w)$ follows, and will be important in the sequel. (The reader is referred for details to [8] (9.2.4) pages 277–278.) Polyhedral arguments will only be used here for providing the intuition behind some of the proofs.

2. RESULTS

Roughly speaking, according to Lovász's result, perfect graphs are those in which the weighted chromatic number is well-characterized by the means of the clique number. Still, to find the chromatic number of perfect graphs is not easy. A polynomial time algorithm was given by Grötschel, Lovász and Schrijver [7], using the ellipsoid method. It can be expected that other problems related to colorings can be well-characterized in perfect graphs. Answering a question asked by Paul Seymour [11] at the DIMACS meeting on polyhedral combinatorics in Morristown, 1989, our Theorem 2.1 gives such a good characterization for the case when a perfect graph is to be ω-colored so that a given stable set is monochromatic.

Theorem 2.1. *Let G be a perfect graph, and let $X \subseteq V(G)$ be a stable set. Then G has an ω-coloring with X monochromatic if and only if for an arbitrary family of cliques $\mathcal{Q}$ and a family of at most $|V|$ distinct ω-cliques $\mathcal{K}$ satisfying*

$$d_{\mathcal{Q}}(v) = d_{\mathcal{K}}(v) \qquad \textit{for all } v \in V - X,$$

and

$$d_{\mathcal{Q}}(v) = d_{\mathcal{K}}(v) + 1 \qquad \text{for all } v \in X,$$

we have $|\mathcal{Q}| \geq |\mathcal{K}| + |X|$.

Theorem 2.1 does provide a good-characterization: a coloring can be checked in polynomial time, and there is an obstacle of polynomial size if there is no feasible coloring. ($|\mathcal{K}| \leq |V|$ implies $|\mathcal{Q}| \leq |V|^2 + 1$.)

In fact, we will consider a slightly more general question: if another set of vertices $Y \subseteq V(G)$ is given, one can ask, in addition, that the color of X is different from all colors occurring in Y. The good-characterization theorem is then the following:

Theorem 2.2. *Let G be a perfect graph, and $X, Y \subseteq V$, $X \cap Y = \emptyset$. Then G has an ω-coloring such that all elements of X are colored with the same color different from all the colors used on Y, if and only if for an arbitrary family of cliques $\mathcal{Q}$ and a family of at most $|V|$ distinct ω-cliques $\mathcal{K}$ satisfying*

$$d_{\mathcal{Q}}(v) = d_{\mathcal{K}}(v) \qquad \text{for all } v \in V - (X \cup Y),$$

and

$$d_{\mathcal{Q}}(v) = d_{\mathcal{K}}(v) + 1 \qquad \text{for all } v \in X,$$

we have $|\mathcal{Q}| \geq |\mathcal{K}| + |X|$.

A coloring satisfying the condition (X is colored with one color different from all colors in Y, where, if Y is not given, $Y = \emptyset$) will be called *feasible*. The proof of Theorem 2.2 (which also implies Theorem 2.1, as this is a special case for $Y = \emptyset$) is presented in Section 3.1. The proof also provides a polynomial algorithm for finding a feasible coloring (if one exists), as it is shown in Section 3.2. Our algorithm uses Grötschel, Lovász and Schrijver's geometric algorithm [7] which finds a maximum clique in a graph. Of course such an algorithm can be used in particular to color perfect graphs, choosing the precolored set of vertices to be empty. Note the difference between this specialization of our algorithm and the coloring algorithm of Grötschel, Lovász and Schrijver [7]: our variant only uses a ''black box'' which finds a maximum clique in a graph; it does not use the ellipsoid method, or any kind of polyhedral argument besides that. Furthermore, our algorithm is purely combinatorial—no fractional coloring is needed to start with.

Let us consider now the following problem: we are given a perfect graph G, a stable set $X \subseteq V(G)$ which has already been colored black, and $y \in V(G) - X$ colored already with some color (black or other). Is there an ω-coloring, which respects the colors that have already been assigned? If the color of y is also black, the answer is in Theorem 2.1. If it is another color, then Theorem 2.2 with the particular choice $Y = \{y\}$ solves the problem.

The generalization of this problem, where a set of vertices has already been arbitrarily ''precolored'' (so that each color class is a stable set) and the question is to continue this coloring so that the number of colors used altogether is minimum, is called the *precoloring extension* problem. It has been studied in [1, 4, 5] and [9] for general graphs. The case when two colors of arbitrary size are used in the precoloring has been mentioned to be NP-complete by Seymour [11]. In this paper we are going to separate polynomial and NP-complete subproblems of precoloring extension problems concerning perfect graphs. The subproblems restrict the number of colors used in the precoloring, and the number of precolored vertices. Formally, we consider the following **PRECOLORING EXTENSION** problem parametrized by positive integers $m_1 \geq \cdots \geq m_k$:

$\mathbf{PE}(m_1, \ldots, m_k)$
Instance: A graph G and pairwise disjoint stable sets $X_1, \ldots, X_k$ such that $|X_i| \leq m_i$ $(i = 1, \ldots, k)$.
Question: Is there an ω-coloring of G such that each X_i is contained in a different color class?

We have the following sharp separation theorem for the complexity of PRECOLORING EXTENSION restricted to perfect graphs:

Theorem 2.3. *The problem $PE(m_1, \ldots, m_k)$ restricted to perfect graphs is polynomially solvable if and only if $k \leq 2$ and $m_2 \leq 1$, and it is NP-complete otherwise.*

The polynomial part of the theorem follows directly from Theorems 2.1 and 2.2 and Algorithm 3.1. The NP-completeness part is proved in Section 3.3.

3. PROOFS

3.1. Good Characterizations

In this subsection we prove the good characterization theorems for the chromatic number of precolored perfect graphs, in the most general not NP-complete case. Note that setting $Y = \emptyset$, Theorem 2.1 becomes a special case of Theorem 2.2. In the formulation of the proof, we already have in mind the algorithm which we then describe in the next section. Subsets of vertices and their characteristic vectors as vertex sets will not be distinguished.

The essence of Theorem 2.2 can be quickly understood using polyhedral combinatorics: a feasible coloring exists if and only if the face of the stable set polyhedron of G defined with the equalities $x(K) = 1$ for every ω-clique $K \subseteq V(G), x(v) = 1$ for $v \in X$ and $x(v) = 0$ for $v \in Y$, and the inequalities $x(K) \leq 1$ for every other clique K and $x \geq 0$, is non-empty. A necessary and sufficient condition for this face to be non-empty can be read out from Farkas' lemma, and this is exactly how the condition of Theorem 2.2 arises.

Proof of Theorem 2.2. The 'only if' part is easy, and we prove the following statement: *If a feasible coloring exists, then for any family of cliques $\mathcal{Q}$ and any family of ω-cliques $\mathcal{K}$ satisfying $d_{\mathcal{Q}}(v) \geq d_{\mathcal{K}}(v)$ for all $v \in V - (X \cup Y)$, and $d_{\mathcal{Q}}(v) = d_{\mathcal{K}}(v) + 1$ for all $v \in X$, we have $|\mathcal{Q}| \geq |\mathcal{K}| + |X|$.*

Indeed, suppose that $S, V \supseteq S \supseteq X$ is a color class of a feasible ω-coloring. Since S is a stable-set, $|S \cap Q| \leq 1$ for all $Q \in \mathcal{Q}$, whence $|\mathcal{Q}| \geq \sum_{s \in S} d_{\mathcal{Q}}(s)$. By the conditions on $\mathcal{Q}$ and $\mathcal{K}$, and since $S \supseteq X$ and $S \cap Y = \emptyset$, $\sum_{s \in S} d_{\mathcal{Q}}(s) = \sum_{s \in X} d_{\mathcal{Q}}(s) + \sum_{s \in S-X} d_{\mathcal{Q}}(s) \geq |X| + \sum_{s \in S} d_{\mathcal{K}}(s)$. Finally, since S is a color class of an ω-coloring, $|S \cap K| = 1$ for all $K \in \mathcal{K}$, and consequently $\sum_{s \in S} d_{\mathcal{K}}(s) = |\mathcal{K}|$. Hence $|\mathcal{Q}| \geq \sum_{s \in S} d_{\mathcal{Q}}(s) \geq |X| + \sum_{s \in S} d_{\mathcal{K}}(s) \geq |X| + |\mathcal{K}|$ as claimed.

To prove the 'if' part of Theorem 2.2, let $\mathcal{K}$ be an arbitrary family of ω-cliques. Define a weight function w by

$$w(v) = \begin{cases} d_{\mathcal{K}}(v) + 1 & \text{if } v \in X \\ 0 & \text{if } v \in Y \\ d_{\mathcal{K}}(v) & \text{if } v \in V(G) - (X \cup Y) \end{cases}$$

Let S be a stable-set of maximum weight, that is, $w(S) = \alpha(G, w)$, and subject to this choose S to be of minimum size (namely, $w(v) \neq 0$ for every $v \in S$). Any stable set contains at most one vertex of each clique, therefore $w(S) \leq |\mathcal{K}| + |X|$.

(1) If $\alpha(G,w) < |\mathcal{K}|+|X|$, then since G is perfect, we have $\chi(\bar{G},w) = \alpha(G,w) < |\mathcal{K}|+|X|$ and the cliques participating in the $\alpha(G,w)$-coloring of $(\bar{G},w)$ contradict the condition of the theorem.

(2) Suppose $\alpha(G,w) = w(S) = |\mathcal{K}| + |X|$. Then S obviously contains X, it is of course disjoint from Y, and $|S \cap K| = 1$ for all $K \in \mathcal{K}$. We distinguish two possibilities:

(2.1) If $\omega(G-S) = \omega - 1$ we are done: since $G-S$ is perfect it has an $(\omega-1)$-coloring, and adding S to this we get an ω-coloring of G in which S is a color class.

(2.2) Suppose $\omega(G-S) = \omega$ and let K be an ω-clique of $G-S$. Then K must be linearly independent (over the rationals) of the set $\mathcal{K}$. (If it were not, we could express K as a linear combination of $\mathcal{K}, K = \sum_{L\in\mathcal{K}} \lambda_L L$. Since K and all members of $\mathcal{K}$ have the same cardinality ω, the sum of the coefficients $\Lambda = \sum_{L\in\mathcal{K}} \lambda_L$ equals 1. Using also $|S \cap L| = 1$ for all $L \in \mathcal{K}$, it follows that $|K \cap S| = \Lambda = 1$, a contradiction with $K \subseteq G - S$.) It follows that if $\mathcal{K}$ is an arbitrary generating system of ω-cliques (for instance the set of all ω-cliques, but a basis is already enough), then this last possibility cannot hold, and the statement is proved. ∎

Let us remark that the 'if' part of Theorem 2.2 was proved in a sharper form involving linear independence instead of cardinality: *If a feasible ω-coloring does not exist, then one can find a family of cliques $\mathcal{Q}$ and a set of linearly independent ω-cliques $\mathcal{K}$ such that $d_{\mathcal{Q}}(v) = d_{\mathcal{K}}(v)$ for all $v \in V-(X\cup Y)$, $d_{\mathcal{Q}}(v) = d_{\mathcal{K}}(v)+1$ for all $v \in X$, and $|\mathcal{Q}| < |\mathcal{K}|+|X|$.* Note that linearly independent cliques are always distinct and one cannot find more than $|V|$ of them. In fact, it is easy to prove (this is the trivial part of the ''clique-rank'' characterization of perfect graphs [2]) that one cannot find more than $|V| - \omega + 1$ linearly independent ω-cliques in G.

3.2. Algorithm

A polynomial algorithm which finds a feasible coloring (or gives a polynomial size obstacle that such a coloring does not exist) can be read out from the proof of Theorem 2.2:

Algorithm 3.1
Input: A perfect graph G, a stable set $X \subset V(G)$ and a set $Y \subset V(G) - X$.
Output: An ω-coloring of G such that X is monochromatic and its color is different from all colors used on Y, or a certificate that such a coloring does not exist.
Subroutine GLS(G,w): For arbitrary perfect graph G and $w: V(G) \to \mathbb{Z}_+$ as input, determines $\omega(G,w)$, and a clique of this size, in polynomial time. (Since $\bar{G}$ can also be given as input, it can also determine $\alpha(G,w)$ in polynomial time.)
Subroutine COL(G,w): For arbitrary perfect graph G and $w: V(G) \to \mathbb{Z}_+$ as input, determines $\omega(G,w)$, and an $\omega(G,w)$-coloration of (G,w), in polynomial time.
Variables: $\mathcal{K}$ a set of ω-cliques of G;
w a weight function on $V(G)$;
$S \subset V(G)$ a stable set;
$Q \subset V(G)$ a clique.

begin
$\mathcal{K} := \emptyset$;
loop
  for $v \in V(G)$ **do**
    case of
    $v \in V(G) - (X \cup Y)$: $w(v) := d_{\mathcal{K}}(v)$;
    $v \in X$: $w(v) := d_{\mathcal{K}}(v) + 1$;
    $v \in Y$: $w(v) := 0$

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      endcase
   endfor;
   use GLS(Ḡ, w) to find α(G, w) and a stable set S with this weight;
   if α(G, w) < |𝒦| + |X| then goto end1
   else
      use GLS(G − S, 1) to find ω(G − S), and a clique Q ⊂ V(G) − S of this size;
      if |Q| = ω(G) then 𝒦 := 𝒦 ∪ {Q}
      else goto end2
      endif
   endif
endloop;
end1: output(''There is no feasible coloring'', and the coloration provided by COL(Ḡ, w)
        shows an obstacle).
end2: Use COL(G − S, 1) to find an (ω − 1)-coloring of G − S,
        add S as a color class to this coloring,
        output(the resulting ω-coloring of G)
end.
```

It is clear from the proof of Theorem 2.2 that Algorithm 3.1 correctly finds a feasible ω-coloring, or derives that no such coloring exists. Theorem 2.2 also guarantees that the loop of the algorithm is run at most $|V|$ times and the polynomiality of the algorithm follows if GLS and COL can be executed in polynomial time. Consequently, substituting for GLS and COL Grötschel, Lovász and Schrijver's polynomial algorithm [7], Algorithm 3.1 has polynomial running time. Another choice of substitution for COL is to call Algorithm 3.1 recursively with $X = \emptyset$. This special case deserves more explanation:

If there is no precolored vertex, Algorithm 3.1 becomes a coloring algorithm slightly different from Grötschel, Lovász and Schrijver's [7]. We describe it in a form adapted to the weighted case: if a function $c: V \to \mathbb{Z}_+$ is given, we replace $\omega(G - S)$ in the proof by $\omega(G - S, c)$ respectively. (In the algorithm COL$(G - S, \mathbf{1})$ is replaced by COL$(G - S, c)$).

Using Grötschel, Lovász and Schrijver's algorithm to determine $\omega(G - S, c)$ as a subroutine, we determine, as before, more and more linearly independent cliques K with $c(K) = \omega(G, c)$ until determining a color class S. (Since now $|\mathcal{K}| + |X| = |\mathcal{K}|$, and since it follows from the definition of w and the perfectness of G that $\alpha(G, w) = |\mathcal{K}|$, **end1** is never reached in this application.)

The coefficient of S in the coloring can be defined as Grötschel, Lovász and Schrijver do in [8], to be $\omega(G, c) - \omega(G - S, c)$. This makes sure that the set of stable sets used, similarly to the cliques used, will be linearly independent, and their number will be at most $|V(G)|$. Of course, the weighted generalization of Theorem 2.2 (about precolored weighted graphs) works in a similar way, and a polynomial algorithm follows then as well.

This algorithm uses the ellipsoid method, but only in the subroutine finding $\omega(G, c)$, otherwise it is purely combinatorial—no fractional coloring is needed to start with. It follows the spirit of the algorithm of Fonlupt and Sebő [2] based on the ''clique rank'' of perfect graphs. (This algorithm does not use the ellipsoid method, however, its worst case bound is only n^{ω}.)

3.3. NP-Complete Cases

In this section we prove the NP-completeness part of Theorem 2.3, which says that PE$(m_1, \ldots, m_k)$ is NP-complete on perfect graphs if $k \geq 3$ or $k = 2$ and $m_1 \geq m_2 \geq 2$. Since PE(1, 1, 1) is

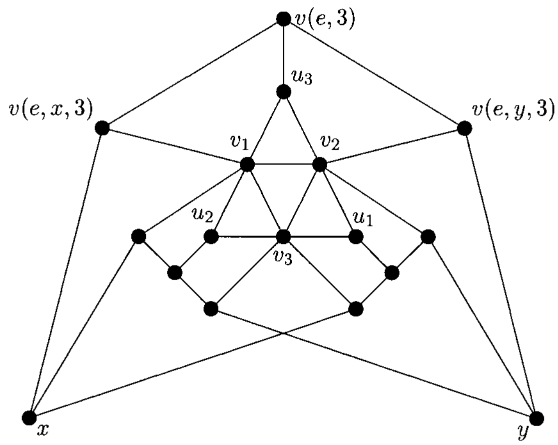

FIGURE 1. Illustration to the proof of Lemma 3.2.

a subproblem of PE($m_1, \ldots, m_k$) with $k \geq 3$, and PE(2, 2) is a subproblem of PE($m_1, \ldots, m_k$) with $m_1 \geq m_2 \geq 2$, it suffices to prove that PE(1, 1, 1) and PE(2, 2) are NP-complete for perfect graphs. We first prove two auxiliary complexity results which might be interesting on their own:

Lemma 3.2. *Given a graph G with three edges forming a triangle so that deleting these edges we get a bipartite graph, it is NP-complete to decide whether $\chi(G) = 3$ is true.*

Proof. We reduce GRAPH 3-COLORABILITY (see Garey, Johnson [6], page 84) to the restricted instances of the theorem by ''local replacement'' of edges:

Suppose $G = (V, E)$ is an instance of GRAPH 3-COLORABILITY. The basis of the reduction is the gadget of Figure 1:

Add the vertices v_1, v_2, v_3 forming a triangle, to G, and let us denote the colors of these vertices by 1, 2, 3 respectively. Add now u_1, u_2, u_3 and join u_i to $\{v_1, v_2, v_3\} - \{v_i\}$ $(i = 1, 2, 3)$. The color of u_i is forced to be i $(i = 1, 2, 3)$.

Replace each edge $e = xy \in E(G)$ by three vertex-disjoint paths $P(e, i)$ of length 4, adding 3 new vertices for each: the vertices of $P(e, i)$ are, in order, $x, v(e, x, i), v(e, i), v(e, y, i), y$ $(i = 1, 2, 3)$. Join $v(e, i)$ to u_i and $v(e, x, i), v(e, y, i)$ to different elements of $\{v_1, v_2, v_3\} - \{v_i\}$, choosing between the two possibilities in an arbitrary way $(i = 1, 2, 3)$. In other words, the graph $G' = (V', E')$ we have constructed with local replacements is defined by

$$V' = \{u_1, u_2, u_3, v_1, v_2, v_3\} \cup V(G) \cup \bigcup_{e=xy \in E} \bigcup_{i=1}^{3} \{v(e, x, i), v(e, i), v(e, y, i)\},$$

$$E' = \bigcup_{e \in E} \bigcup_{i=1}^{3} E(G'(e, i))$$

where $G'(e, i)$ is the subgraph consisting of $P(e, i)$ and the edges connecting vertices of $P(e, i)$ to $\{u_1, u_2, u_3, v_1, v_2, v_3\}$. The illustrative Figure 1 shows the subgraph induced by $\{u_1, u_2, u_3, v_1, v_2, v_3\} \cup \bigcup_{i=1}^{3} P(e, i)$ for some $e = xy \in E(G)$.

The graph $G' - \{v_1v_2, v_2v_3, v_3v_1\}$ is bipartite (the bipartition is $W = V(G) \cup \{v_1, v_2, v_3\} \cup \{v(e,i) | e \in E, i = 1,2,3\}, B = \{u_1, u_2, u_3\} \cup \{v(e,x,i) | x \in e \in E, i = 1,2,3\}$). We show that G' is 3-colorable if and only if G is 3-colorable:

Note first that $G'(e,i)$ has no 3-coloring with both x and y having color i. Indeed, say on the example of $G'(e,3)$ of Figure 1, if x and y are both colored 3, then $v(e,x,3)$ is forced to have color 2, and $v(e,y,3)$ to have color 1. But then $v(e,3)$ has neighbors of three different colors and cannot be colored. (As mentioned in the construction of G', without loss of generality we consider only colorings that color the vertices v_1, v_2, v_3 by colors 1, 2, 3, respectively.) Thus restricting a 3-coloring of G' to $V(G)$ we get a 3-coloring of G.

On the other hand, it is easy to check that $G'(e,i)$ has a 3-coloring with x having color i' and y having color j', for any choice of $(i', j') \in \{1,2,3\}^2 - \{(i,i)\}$ (again, we only consider colorings in which v_j, u_j are colored by color j for $j = 1,2,3$). It follows that any 3-coloring of G can be extended to a 3-coloring of G'. ∎

Corollary 3.3. *Given a graph G with two vertices the deletion of which leads to a bipartite graph, it is NP-complete to decide whether $\chi(G) = 3$ is true.*

Proof. If G is a graph as in Lemma 3.2, then deleting any two vertices of the triangle v_1, v_2, v_3 one gets a bipartite graph. Thus the statement follows from Lemma 3.2. ∎

Note that Corollary 3.3 is the best possible in the sense that every graph that contains a vertex the deletion of which results in a bipartite graph, is 3-colorable.

Corollary 3.4. *PE$(1,1,1)$ is NP-complete when restricted to perfect graphs.*

Proof. Given a graph G with a triangle $v_1v_2v_3$ such that deletion of the edges v_1v_2, v_2v_3, v_3v_1 results in a bipartite graph, delete these edges and precolor the vertices v_1, v_2, v_3 by colors 1, 2, 3, respectively. The precolored graph G' obtained in this way allows a 3-coloring extending its precoloring if and only if G was itself 3-colorable. However, $\omega(G') = 2$, though we only try to 3-color G'. Therefore we add a dummy triangle to G'. Lemma 3.2 then implies that PE(1, 1, 1) is NP-complete even for graphs which are the vertex-disjoint union of a triangle and a bipartite graph. ∎

Corollary 3.5. *PE$(2,2)$ is NP-complete when restricted to perfect graphs.*

Proof. Let G' be the input precolored perfect graph for PE(1, 1, 1) and let v_1, v_2, v_3 be the three precolored vertices. Add two new vertices of degree one, both adjacent to v_3, precolor one of them with the color of v_1, the other one with the color of v_2 and release the precoloring of v_3. The resulting graph allows a 3-coloring extending its precoloring if and only if the precoloring of G' was extendable. Thus Corollary 3.4 implies that PE(2, 2) is NP-complete even when restricted to graphs which are the vertex-disjoint union of a triangle and of a bipartite graph. ∎

We thank Zsolt Tuza for simplifying our proof of Corollary 3.5.

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