JOURNAL OF COMBINATORIAL THEORY, Series B **67**, 62–85 (1996)
ARTICLE NO. 0034

On Critical Edges in Minimal Imperfect Graphs

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Received November 25, 1993; revised March 16, 1995

An edge of a graph is called *critical*, if deleting it the stability number of the graph increases, and a nonedge is called *co-critical*, if adding it to the graph the size of the maximum clique increases. We prove in this paper, that the minimal imperfect graphs containing certain configurations of two critical edges and one co-critical nonedge are exactly the odd holes or antiholes. Then we deduce some reformulations of the *strong perfect graph conjecture* and prove its validity for some particular cases. Among the consequences we prove that the existence in every minimal imperfect graph G of a maximum clique Q, for which $G-Q$ has one unique optimal coloration, is equivalent to the strong perfect graph conjecture, as well as the existence of a vertex v in $V(G)$ such that the (uniquely colorable) perfect graph $G-v$ has a "combinatorially forced" color class. These statements contain earlier results involving more critical edges, of Markossian, Gasparian and Markossian, and those of Bacsó and they also imply that a class of partitionable graphs constructed by Chvátal, Graham, Perold, and Whitesides does not contain counterexamples to the strong perfect graph conjecture. © 1996 Academic Press, Inc.

INTRODUCTION

If G is a graph $\omega=\omega(G)$ denotes the cardinality of a maximum clique and $\alpha=\alpha(G)$ denotes the cardinality of a maximum stable set. n will always stay for $|V(G)|$. A k-stable set or k-clique ($k\in\mathbb{N}$) will mean a clique or stable set of size k, and a k-coloration is a partition into k stable sets. $\chi=\chi(G)$ is the chromatic number of G, that is the minimum of k for which a k-coloration exists. Subgraph means induced subgraph in this paper.

Let us replace $\{x\}$ by x throughout the paper. Paths and circuits go through every vertex at most once. They will be considered to be subgraphs or edge-sets. The vertex-set of the graph G will be denoted by $V(G)$, the edge-set by $E(G)$. $N(v)=N_G(v)$ ($v\in V(G)$) will denote the set of neighbors of v (not including v). $\bar{G}$ denotes the complementary graph, and for $V'\subseteq V(G)$ the graph induced by V' will be denoted by $G(V')$.

* This research was partially supported by the European project DONET.

0095-8956/96 $18.00

A graph G is called *perfect* if $\chi(H) = \omega(H)$ for every subgraph H of G; otherwise it is called *imperfect*.

A graph is called *minimal imperfect* if it is not perfect, but all its subgraphs are perfect.

A co-NP characterization of perfectness immediately follows from Lovász [13 or 14], but perfect graphs are not known to belong to NP.

However, the direction of co-NP characterizations is not closed either. The strongest co-NP characterization would be given by the following conjecture of Berge [3, 4]:

A graph isomorphic to an odd circuit of length at least five is called an *odd hole*, and the complement of such a graph is called an *odd antihole*.

Berge's Strong Perfect Graph Conjecture (SPGC). If a graph is minimal imperfect, then it is an odd hole, or an odd antihole.

Lovász [12, 13] proved that every minimal imperfect graph is *partitionable*; that is, $n = \alpha\omega + 1$ ($\alpha, \omega \in \mathbb{N}$, $\alpha \geqslant 2$, $\omega \geqslant 2$), and $G - v$ can be partitioned both into ω-cliques and into α-stable sets, for arbitrary $v \in V(G)$. If G is partitionable, then clearly, $\chi = \omega + 1$, $\chi(G - v) = \omega = \omega(G - v)$, and $\overline{G}$ is also partitionable.

Padberg [22] deduced from Lovász's result that the number of ω-cliques of a minimal imperfect graph G is n; moreover, they are linearly independent. In fact, what Padberg proved is that the set of all α-stable sets is the following: fix an arbitrary α-stable set S and consider the coloration of $G - s$ for all $s \in S$; the $\alpha\omega + 1 = n$ considered color classes, together with S, include every α-stable set. As a consequence $G - v$ is uniquely colorable for all $v \in V(G)$. A graph is called *uniquely colorable* if it has exactly one partition into a minimum number of stable sets.

Bland, Huang, and Trotter [5] observed that the same properties hold for partitionable graphs.

Tucker [25] has noted that the graph whose vertices are the ω-cliques of G and two vertices are joined, if and only if the intersection of the corresponding ω-cliques is nonempty, is also partitionable, with the same parameters. (Easy to check.) Let us call this graph the *intersection graph* of G and denote it by $I(G)$.

Let us call an edge of G *determined*, if there exists an ω-clique containing both of its endpoints. Clearly, $I(I(G))$ is the graph we get from G after deleting the nondetermined edges. Let G be partitionable.

It follows from the unique colorability of the graphs $G - v$ ($v \in V(G)$) that the α-stable sets of G can be coded with the notation $S(a, b)$, meaning the color-class of $G - a$ which contains b; similarly, $K(a, b)$ denotes the ω-clique which is the color class containing b in the optimal coloration of $\overline{G} - a$. It is easy to see that the unique ω-clique disjoint from $S(a, s)$ is $K(s, a)$. If K is an ω-clique, the unique α-stable set disjoint from K will be denoted by $S(K)$; the unique ω-clique disjoint from the α-stable set S will be denoted by $K(S)$.

An edge $e \in E(G)$ is called *critical* if $\alpha(G \backslash e) = \alpha + 1$.

If S is an $(\alpha - 1)$-stable set, then the relation xSy $(x, y \in V(G))$ will mean that $x, y \notin S$ and $S \cup x$, $S \cup y$ are stable-sets. Then xy is a critical edge.

Similarly a nonedge xy of G, that is, $xy \in E(\bar{G})$, will be called *co-critical* if it is a critical edge of $\bar{G}$. If K is an $(\omega - 1)$-clique of G, then the relation xKy $(x, y \in V(G))$ means that x, $y \notin K$ and $K \cup x$, $K \cup y$ are ω-cliques.

If xKy, then x and y must have the same color in every ω-coloration of $G - v$, provided $v \notin K$. Therefore the relation xKy is called a *forcing* (by cliques). We say that x and y are *forced* in the graph G, if they are in the same component of the graph consisting of the co-critical nonedges (that is, if they can be joined by a chain of forcings). Clearly, *if* $xy \in E(G)$ *and* x, y *are forced, then* $\chi(G) > \omega(G)$, *and in particular*, G *is not perfect*. That is, if $\chi(G) = \omega(G)$ then any two forced vertices of G are nonadjacent.

If S is an $(\alpha - 1)$-stable set of G, then the relation xSy will be called a *coforcing*, if it is a forcing in $\bar{G}$, that is x, $y \notin S$ and $S \cup x$, $S \cup y$ are α-stable sets. Two vertices are *coforced*, if they are in the same component of the graph which consists of the critical edges (that is, if and only if they are forced in $\bar{G}$). If $\chi(\bar{G}) = \alpha(G)$ then coforced vertices are adjacent.

If G is partitionable, $T \subseteq V(G)$, is called a *small transversal*, if $|T| \leqslant \alpha + \omega - 1$, and T meets every ω-clique and every α-stable set of G. Chvátal [5, 6] pointed out that no minimal imperfect graph has a small transversal, because if T was one, then $\alpha(G - T) \leqslant \alpha - 1$, $\omega(G - T) \leqslant \omega - 1$, and $n - |T| \geqslant (\alpha - 1)(\omega - 1) + 1$, implying $\chi(G - T) > \omega(G - T)$; he also provided partitionable graphs called webs which have small transversals. A graph $G = (V, E)$ on $\alpha\omega + 1$ vertices is called a *web*, if $\omega(G) = \omega$, $\alpha(G) = \alpha$, and there is a cyclical order of V so that every set of ω consecutive vertices in this cyclical order is an ω-clique. It is easy to see that the critical edges of webs are exactly those between the neighboring vertices in their (uniquely determined) cyclical order, whereas the cocritical nonedges are exactly between vertices at distance $\omega + 1$ in this order. Webs are also rich in small transversals. They satisfy the condition of every result or conjecture in this paper, and the reader may find them to be useful examples throughout.

All these facts about partitionable and minimal imperfect graphs will be used without further reference in the sequel. The results we prove in the present paper are the following.

THEOREM 1. *If G is partitionable, v_1, $v_2 \in V(G)$, $v_1 K v_2$, where K is an $(\omega - 1)$-clique, and there exist u_1, $u_2 \in K$ (not necessarily distinct), such that $u_1 v_1$, $u_2 v_2$ are critical edges, then G is an odd hole or antihole or has a small transversal.*

This theorem and the following two corollaries have been presented in Sebő [23].

COROLLARY 1.1. *If G is a partitionable graph which contains a path consisting of ω critical edges, then G is an odd hole or antihole or has a small transversal.*

Indeed, if a and b are the first and last edges of the path in the condition and K is the rest of its vertices, then it is easy to see aKb, and Theorem 1 can be applied.

Corollary 1.1 slightly sharpens a theorem of Markossian, Gasparian, and Markossian [17, 18], where the condition is asked from both G and $\bar{G}$. This "Armenian theorem" played a pioneering role in the subject of critical edges. However, the original proof is quite complicated.

A relatively short proof of Theorem 1 has appeared in Sebő [23], another short proof will be given in Section 1 below. A third proof of the Armenian theorem (Corollary 1.1) follows from Theorem 2 below (see Corollary 4.2), with another kind of sharpening: ω can be replaced by $\omega - 1$ in the condition. This slight surplus will tell us that the *strong perfect graph conjecture is implied by a statement on uniquely colorable perfect graphs* (see Corollary 4.2 below and the subsequent remark).

After reformulation and restriction to minimal imperfect graphs Corollary 1.1 becomes:

COROLLARY 1.2. *If G is minimal imperfect and there exists an* $xy \in E(G)$ *so that x and y are forced, then G is an odd hole or antihole.*

Forcings and their relation to uniquely colorable graphs deserve more attention, as was pointed out by Fonlupt and Sebő [8]. However, in the present paper we wish to concentrate solely on proving theorems about critical edges themselves. This "forcing" aspect of the results on critical edges is explained in Sebő [23].

A graph is called *strongly perfect*, if in all of its induced subgraphs there exists a stable set which meets every (inclusionwise) maximal clique.

Olaru [21] proved that a minimal imperfect graph whose proper subgraphs are strongly perfect has a long path of critical edges (see Lemma 1.2). At the *Workshop on Perfect Graphs* he pointed out that this result and Theorem 1 imply:

COROLLARY 1.3. *If G is not perfect but* $G - v$ *is strongly perfect for every* $v \in V(G)$ *then G is an odd hole or antihole.*

The following theorem has been conjectured by Bacsó and Sebő and is a common generalization of their results involving four critical edges. (See

Bacsó [1] for an account.) The proof method of Theorem 1 can be adopted to prove this conjecture.

THEOREM 2. *Let G be partitionable, and suppose that there exists a vertex of G which is adjacent to two critical edges and one cocritical nonedge. Then G is an odd hole, odd antihole, or has a small transversal.*

The proofs of earlier results show a set of cardinality $\alpha + \omega - 1$, which, if G is not an odd hole or antihole, is a small transversal. The proofs of Theorems 1 and 2 separate the problem into two cases, in both of which there is a small transversal for different reasons (see Sections 1 and 2).

COROLLARY 2.1. *Suppose that G is a partitionable graph and there exists* $v_0 \in V(G)$, $N(v_0) \subseteq A \cup B$, *where A and B are* ω*-cliques, and* $A \cap B = \emptyset$. *Then G is an odd hole or antihole or has a small transversal.*

Corollary 2.2 is an application of Corollary 2.1 to a particular situation. However, according to Theorem 3 below, the condition of Corollary 2.2 always holds whenever the seemingly weaker condition of Corollary 2.1 holds. In other words (knowing Theorem 3), conversely, Corollary 2.1 is also an evident consequence of Corollary 2.2.

COROLLARY 2.2. *If* $G = (V, E)$ *is partitionable and* $v_{-(\omega-1)}, ..., v_{-1}, v_0, v_1, ..., v_\omega$ *are vertices so that* $v_{i+1}, ..., v_{i+\omega}$ *is an* ω*-clique for every* $i = -\omega, ..., 0$, *then G is an odd hole or antihole or has a small transversal.*

The condition here means exactly that in $I(G)$ there exists a path consisting of ω critical edges (see Lemma 3.2) and consequently Corollary 2.2 can be reformulated as follows:

COROLLARY 2.3. *If G is a partitionable graph and* $I(G)$ *contains a path consisting of* ω *critical edges then G is an odd hole or antihole or has a small transversal.*

Of course again, like in Corollary 1.2 one can replace the condition by "adjacent forced vertices" (see (6) below). Note that the only difference between Corollaries 1.1 and 2.3 is that here it is $I(G)$ which is supposed to have critical edges.

Remark. Chvátal, Graham, Perold, and Whitesides [7] give two methods of constructing partitionable graphs, and these seem to be the only general methods up to now. We would like to note that the "first method" produces graphs which satisfy the condition of Corollary 2.2 and, thus, cannot be minimal imperfect. (The second, more sophisticated method, can also not produce minimal imperfect graphs, as was shown in a restricted sense by Grinstead [10], and recently, in general, by Bacsó *et al.* [2].)

As mentioned above, the conditions of Corollary 2.1 and 2.2 are related by the following theorem [24] which will be reproved in Section 3:

THEOREM 3. *If $G=(V, E)$ is partitionable, $v_0 \in V(G)$ and $N(v_0) \subseteq A \cup B$, where A and B are cliques, then there exists a (unique) order $v_{-(\omega-1)}, \ldots, v_{-1}, v_0, v_1, \ldots, v_{\omega-1}$ of $\{v_0\} \cup N(v_0)$ so that $\{v_i, v_{i+1}, \ldots, v_{i+\omega-1}\}$ is an ω-clique in G for all $i=-(\omega-1), \ldots, -1, 0$.*

We prove Theorem i in Section i ($i=1, 2, 3$). Some of the lemmas and simple statements of Section 1 about critical edges will be used throughout.
The main result of Section 3 is Theorem 4 below, which uses all of the preceding results of the paper, and some new steps. It is motivated by the study of the following bound of Olaru [20] (also found by Markossian and Karapetian [16] and Reed [22a]:

(0) *If G is partitionable, then the degree of every vertex is at least $2\omega-2$.*

The proof of (0) is not difficult; see, for instance, Sebő [24] (or apply the first part of Lemma 3.1 below to $I(G)$).

THEOREM 4. *If G is partitionable, $I(G)$ has a vertex of degree $2\omega-2$ and $I(\bar{G})$ has a vertex of degree $2\alpha-2$ then G is an odd hole or antihole, or has a small transversal.*

Note that, conversely, all of the results of the paper are reversible: odd holes, antiholes, and, more generally, webs (after deleting the nondetermined edges) satisfy the conditions of Theorems 1, 2, 3, 4, and of their corollaries.
Theorem 4 tells us, that the tightness of the bound of (0) for *some* vertex of $I(G)$ for every minimal imperfect graph G implies the strong perfect graph conjecture.
Corollary 4.1 is implied by the fact that $I(G)$ has a vertex of degree $2\omega-2$ if and only if G has an ω-clique K so that $G-K$ is uniquely colorable (see Lemma 3.1):

COROLLARY 4.1. *If G is partitionable and G has an ω-clique Q and an α-stable set S, such that $G-Q$ and $\bar{G}-S$ are uniquely colorable, then G is an odd hole, antihole, or has a small transversal.*

Corollary 4.1 sharpens Sebő [23, Theorem 1.3]. The main lemma (Lemma 3.1) of Section 3 will generate the following reformulation:

COROLLARY 4.2. *If G is partitionable and G has an ω-clique Q and an α-stable set S, such that the critical edges induced by Q and the cocritical* *nonedges induced by S respectively form a connected graph, then G is an odd hole, antihole, or has a small transversal.*

The corresponding reformulations of the strong perfect graph conjecture are stated at the end of Section 3. Here we only mention a related remark which we find more important:

Remark. According to Corollary 4.2, *the strong perfect graph conjecture is true for any class of graphs $\mathcal{G}$ closed under taking induced subgraphs, provided the uniquely colorable perfect graphs of $\mathcal{G}$ and of $\{\bar{G}: G \in \mathcal{G}\}$ have a forced color class* (*that is, a stable set which meets every ω-clique and whose induced cocritical nonedges form a connected graph*). Is there *any* "interesting" class of graphs for which the condition of this statement can be proved? Does it not hold for the class of *all* graphs? Equivalently, we are asking for a uniquely colorable perfect graph which does not have a forced color class.

We did not succeed replacing $I(G)$ by G in the condition of Theorem 4, so we state the related statement as a conjecture. We believe that the following Conjecture 1 might be reachable, whereas Conjecture 2 is probably more difficult.

Conjecture 1. *If a minimal imperfect graph G and its complement have a vertex incident to $2\omega - 2$ and $2\alpha - 2$ determined edges, respectively, then G is an odd hole or antihole.*

Conjecture 2. *If G is minimal imperfect, then it has a vertex of degree* $2\omega - 2$.

1. Proving Theorem 1 and Its Corollaries

The following simple but basic observations about critical edges are due to Markossian, Gasparian and Markossian [18]. They will be useful throughout the paper. We suppose that G is partitionable, and nothing else in statements (1)–(6):

(1) *If $xy \in E(G)$ is a critical edge, then the $(\alpha - 1)$-stable-set S for which xSy is a coforcing is uniquely determined: $S \cup y = S(x, y)$. Moreover, the ω-clique K_x containing x and not containing y is also uniquely determined and $K_x = K(S \cup y)$.*

Proof. Suppose that xSy is a coforcing, and let $K_x := K(y, x)$. Clearly, every ω-clique containing x is disjoint from S. Since in addition $y \notin K_x$, we have $K_x \cap (S \cup y) = \emptyset$. The unicity of both K_x and S follows. Q.E.D.

(2) *Let* $x, y \in V(G)$. xy *is a critical edge in* G *if and only if there are* $\omega - 1$ *different* ω*-cliques containing* $\{x, y\}$.

Proof. First let xy be a critical edge. Among the ω different ω-cliques containing x, by (1) there is one which does not contain y, so there are $\omega - 1$ different ω-cliques which contain both x and y.

Conversely, if there are $\omega - 1$ different ω-cliques containing $\{x, y\}$, then the colorations of $G - x$ and that of $G - y$ have $\omega - 1$ common color classes. The set U of vertices which are in none of these color classes contains both x and y, and both $U \backslash x$ and $U \backslash y$ are stable sets. Since $\alpha(G) = \alpha$ and $|U| = \alpha + 1$, $xy \in E(G)$, and then it is a critical edge. Q.E.D.

(3) *If* $x_0x_1, x_1x_2, ..., x_{k-1}x_k$ *are critical edges,* $1 \leqslant k < \omega$, *and the corresponding coforcings are* $x_0S_1x_1, ..., x_{k-1}S_kx_k$, *then* $S_i \cup x_i$ *is the color class of* x_i *in the coloration of* $G - x_0$ $(i = 1, ..., k)$.

Proof. Applying (1) to the critical edge $x_{i-1}x_i$ we get that $S_i \cup x_i$ is disjoint from a ω-clique containing x_{i-1}, whence it is a color class of $G - x_{i-1}$. If $i = 1$ the claim is proved, so suppose $i \geqslant 2$.

Now clearly, the coloration of $G - x_{j-1}$ can be derived from that of $G - x_j$ by replacing $x_{j-1} \cup S_j$ by $S_j \cup x_j$ $(j = 1, ..., k)$. Applying this consecutively to $j = i - 1, ..., 0$ we get that $S_i \cup x_i$ is a color class of $G - x_0$ as claimed. Q.E.D.

(4) *If* $x_0x_1, ..., x_{k-1}x_k$ *are critical edges,* $1 \leqslant k < \omega$, *then* $\{x_0, ..., x_k\}$ *is a clique, and for the corresponding coforcings* $x_{i-1}S_ix_i$ $(i = 1, ..., k)$, $S_i \cap S_j = \emptyset$ *if* $i \neq j \in \{1, ..., k\}$.

Proof. Let $x_{i-1}S_ix_j$ be the coforcings corresponding to these critical edges $(i = 1, ..., k)$. Then $\{x_0, ..., x_k\} \cup S_1 \cup \cdots \cup S_k$ is a proper subset of the vertices of G, because $k + 1 + k(\alpha - 1) = k\alpha + 1 < \omega\alpha + 1$. But a proper subset of an (α, ω)-graph has a partition into at most α cliques, and since coforced vertices must be in the same clique class, they are adjacent, as claimed.

Now since $x_ix_j \in E(G)$, $x_j \notin S_i \cup x_i$, $i \neq j \in \{1, ..., k\}$. But then according to (3), $S_i \cup x_i$ and $S_j \cup x_j$ are different color classes of $G - x_0$, and the claim follows. Q.E.D.

(5) *Suppose* H *is a connected subgraph of the graph which consists of the critical edges of* G, $|V(H)| = \omega + 1$. *Then* H *is a path, and if* $u, v \in V(H)$ *denote its endpoints,* $u(V(H) \backslash \{u, v\})v$ *is a forcing.*

Proof. Since $\omega(H) < |V(H)|$, H has two nonadjacent vertices u and v. By (4), the distance of u and v in H is at least ω, and because

$|V(H)| = \omega + 1$, it is exactly ω. But then a shortest path between u and v is a Hamiltonian path of H, and H has no other edges but the edges of this path.

Now by (4) $V(H)\backslash u$ and $V(H)\backslash v$ are ω-cliques, proving the last assertion. Q.E.D.

(6) *The following statements are equivalent*:

(i) *There exists in G a path of cocritical nonedges of length* α.

(ii) $\bar{G}$ *has a subgraph on* $\alpha + 1$ *vertices whose critical edges form a connected graph.*

(iii) *In G there exist two adjacent vertices which are forced.*

Proof. (ii) follows obviously from (i). If (ii) holds, then any two vertices of the claimed connected subgraph are forced, and since there is no $(\alpha + 1)$-stable set in G, two of them are adjacent. Thus (iii) holds.

Last, to prove (i) from (iii), suppose that $xy \in E(G)$, x and y are forced. That is, $xy \notin E(\bar{G})$, and x and y are joined by a path P of critical edges of $\bar{G}$. Now applying (4) to $\bar{G}$, $|P| \geqslant \alpha$. Q.E.D.

We prove now the following simple lemma of technical character, replacing the linear algebra in the proof in Sebő [23].

LEMMA 1.1. *If* S_1 *and* S_2 *are* α*-stable sets,* $S_1 \cap S_2 = \emptyset$, *and* $S \subseteq S_1 \cup S_2$ *is an* α*-stable set, then* $(S_1 \cup S_2)\backslash S$ *meets every* ω*-clique except, maybe,* $K(S_1)$ *and* $K(S_2)$.

Proof. If K is an ω-clique, $K \neq K(S_1)$, $K \neq K(S_2)$, then let $s_1 = K \cap S_1$ and $s_2 = K \cap S_2$. $\{s_1, s_2\} \subseteq S$ is impossible, because K is a clique, and S is a stable set. Q.E.D.

THEOREM 1. *If G is partitionable,* $v_1, v_2 \in V(G)$, $v_1 K v_2$, *where K is an* $(\omega - 1)$*-clique, and there exist* $u_1, u_2 \in K$ *(not necessarily distinct) such that* $u_1 v_1$, $u_2 v_2$ *are critical edges, then G is an odd hole or antihole or has a small transversal.*

Proof. If $u_1 = u_2$, then $\omega = 2$, and G is an odd hole or antihole (see, for instance, [19]). Therefore, we suppose $u_1 \neq u_2$ for the rest of the proof. Let S_1 and S_2 be the uniquely defined $(\alpha - 1)$-stable sets (see (1)) for which $(u_1 S_1 v_1)$ and $(u_2 S_2 v_2)$. By (1) $S_1 \cup v_1 = S(K \cup v_2)$ and $S_2 \cup v_2 = S(K \cup v_1)$. $S_1 \cap S_2 = \emptyset$ follows. Let $U_1 := S(u_2, u_1)$ and $U_2 := S(u_1, u_2)$; see Fig. 1. We distinguish two cases.

Case 1. There exists an α-stable set S not contained in $S_1 \cup S_2 \cup \{u_1\}$, $u_1 \in S \neq U_1$.

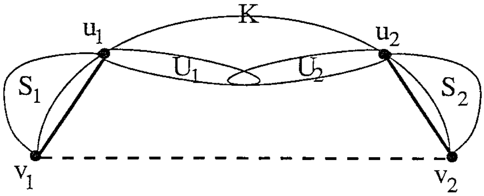

FIGURE 1

Choose an α-stable set S with $u_1 \in S$ (then of course $u_2 \notin S$), and let $s \in S \backslash (S_1 \cup S_2)$, $s \neq u_1$. Let $Q = K(s, v_2)$.

Claim. $s \in S(Q)$; $\{u_2, v_2\} \subseteq Q$; $Q \neq K \cup v_2$.

Indeed, we see $s \in S(Q)$ and $v_2 \in Q$ directly from the definition of Q. If $u_2 \notin Q$ then by (1), $u_2 \cup S_2 = S(Q)$. But then $s \in S(Q)$ and $s \in S \backslash S_2$ contradict each other. Finally, if $Q = K \cup v_2$ then similarly, by (1), $v_1 \cup S_1 = S(Q)$, and $s \in S(Q)$ and $s \in S \backslash S_1$ contradict each other. The claim is proved.

To finish now the proof of the theorem when Case 1 holds, suppose that $S \neq U_1$. Because of $s \in S(Q)$ and $u_2 \in Q$ (see the claim), $Q = K(s, u_2)$. Then $S \cap Q \neq \emptyset$ because $u_1 \in S = S(Q) = S(u_2, s)$ would imply $S = S(u_2, u_1) = U_1$.

Define now $T = (S \backslash u_1) \cup (Q \backslash v_2) \cup v_1$. $(S \backslash u_1) \cap (Q \backslash v_2) \neq \emptyset$, because we have just proved $S \cap Q \neq \emptyset$; $v_2 \notin S$ is clear; moreover, using (1) $u_1 \in Q$ would imply that $Q = K \cup v_2$, contradicting the claim. Thus $|T| = \alpha + \omega - 2$.

Clearly, T meets all ω-cliques, except maybe $K(S)$ and those containing u_1. But according to (1) the ω-cliques containing u_1 also contain $v_1 \in T$, or are equal to $K \cap v_2$ and thus contain u_2: according to the claim, $u_2 \in Q \backslash v_2 \subseteq T$. Thus the only ω-clique maybe disjoint from T is $K(S)$.

Similarly, T meets all α-stable sets, except maybe those containing v_2 or $S(Q)$. But $s \in S(Q) \cap T$ according to the claim. Since $v_1 \in T$, using (1), we see that the only α-stable set disjoint from T is $S_2 \cup v_2$.

Since $S \neq S_2 \cup v_2$, $K(S)$ meets $S_2 \cup v_2$. Adding their intersection point to T we get a small transversal.

The theorem is proved in Case 1, and also in the case when u_1 and U_1 of Case 1 are replaced by u_2 and U_2, respectively. So we can suppose:

Case 2. Every α-stable set containing either u_1 or u_2 but different from both U_1 and U_2 is contained in $S_1 \cup S_2 \cup \{u_1, u_2\}$.

If $u_1 = u_2 =: u$ then $\omega = 2$, because $\omega \geqslant 3$ implies that at least three ω-cliques contain u, but by (1) only one of these does not contain v_1, and another does not contain v_2. Thus at least one ω-clique contains $\{v_1, u, v_2\}$, in contradiction with $v_1 v_2 \notin E(G)$.

Suppose now that $u_1 \neq u_2$. Then clearly $\omega \geqslant 3$, and suppose indirectly that $\alpha \geqslant 3$ too. Then there exists an α-stable set S, $u_1 \in S$ but $S \neq U_1$, $S \neq S_1 \cup u_1$. By the constraint of Case 2, $S \subseteq (S_1 \cup u_1) \cup S_2$, whence, by Lemma 1.1, $S' := (S_1 \cup u_1) \cup (S_2 \cup v_2) \backslash S$ meets every ω-clique except, perhaps $K(S_1 \cup u_1)$ and $K(S_2 \cup v_2)$.

Let $T := S' \cup K \backslash u_2$. Clearly, $|T| = \alpha + \omega - 2$, u_1, $v_2 \in T$ ($u_1 \in K$, $v_2 \in S'$).

By our previous remark about S', since $K(S_2 \cup v_2) = K \cup v_1$ meets T, *the only ω-clique disjoint from T is $K(S_1 \cup u_1)$.*

Since $(K \cup v_2) \backslash u_2 \subseteq T$, we immediately see that the α-stable sets not containing u_2 and different from $S(K \cup v_2)$ meet T. Since $S(K \cup v_2) = S_1 \cup v_1$, and S_1 is not a subset of S (because $u_1 \in S \neq S_1 \cup u_1$), $S(K \cup v_2)$ meets $S' \subseteq T$. So we only have to examine the α-stable sets containing u_2. Let U be such an α-stable set. Because of the constraint of Case 2, $U \backslash u_2 \subseteq S_1 \cup S_2$ unless $U = U_2$. Since $S_1 \cup S_2 \backslash S \subseteq T$ we get that T meets U unless $U \backslash u_2 = S \backslash u_1$ or $U = U_2$. But the latter possibility contains the former one, because in the former case $u_1(U \backslash u_2) u_2$ is a co-forcing: then, according to (1), $U = S(u_1, u_2) = U_2$. *Thus U_2 is the only α-stable set disjoint from T.*

Since $S_1 \cup u_1 \neq U_2$, $K(S_1 \cup u_1)$ meets U_2. Adding their common element to T we get a small transversal. Q.E.D.

Corollary 1.1. *If G is a partitionable graph which contains a path consisting of ω critical edges, then G is an odd hole or antihole or has a small transversal.*

Proof. Let $x_0 x_1$, $x_1 x_2$, ..., $x_{\omega-1} x_\omega$ be the critical edges of the condition. By (the last sentence of) (5): $v_1 := x_0$, $v_2 := x_\omega$, and $K := \{x_1, ..., x_{\omega-1}\}$, $u_1 := x_1$ and $u_2 := x_{\omega-1}$ satisfy the condition of Theorem 1. Q.E.D.

Corollary 1.2. *If G is minimal imperfect, and there exists an $xy \in E(G)$ so that x and y are forced, then G is an odd hole or antihole.*

Proof. According to (6) the condition of Corollary 1.1 is satisfied by $\bar{G}$. Q.E.D.

Corollary 1.3. *If G is not perfect but $G - v$ is strongly perfect for every $v \in V(G)$, then G is an odd hole or antihole.*

Proof. As Professor Olaru observed, the following lemma from Olaru [21] reduces Corollary 1.3 to Theorem 1.

Lemma 1.2. *If G is not perfect and $G - v$ is strongly perfect for all $v \in V(G)$, then for every stable-set $S \neq \varnothing$ of G there exists a cocritical non-edge which has exactly one endpoint in S.*

Proof. Let $S \neq \emptyset$ be an arbitrary stable-set. Since $G - S$ is strongly perfect, there exists a stable-set $T \subseteq V(G) \setminus S$ meeting every maximal clique of $G - S$.

Since G is not perfect, $G - T$ contains an ω-clique Q. $K := Q \setminus S$ is not a maximal clique in $G - S$, because it is disjoint from T, whence for some $t \in T$, $K \cup t$ is an ω-clique. Denoting the unique element of $Q \setminus K = Q \cap S$ by s, we get that st is a cocritical nonedge, $s \in S$, $t \notin S$ as claimed. Q.E.D.

Corollary 1.3 follows now by applying Lemma 1.2 successively. First to $S_1 = \{s_1\}$, where $s_1 \in V(G)$ is arbitrary. Lemma 1.2 provides us with the cocritical nonedge $s_1 s_2$. After $k-1$ applications of Lemma 1.2 we get the $S_k = \{s_1, ..., s_k\}$, whose induced cocritical nonedges form a connected graph. Applying (4) to $\bar{G}$ we get that S_k is a stable set for $k \leqslant \alpha$. So we can apply Lemma 1.2 a last time to S_α, to get a graph on $\alpha + 1$ vertices whose induced cocritical nonedges form a connected graph. But then by (6), the result follows from Corollary 1.2. Q.E.D.

2. Proving Theorem 2 and Its Corollaries

We first add four other claims to our collection of simple statements about critical edges in partitionable graphs, the first of them is from Markossian, Gasparian and Markossian [18]. (We suppose again that G is partitionable throughout (7)–(10).)

(7) *If $x_0 x_1, x_1 x_2, ..., x_{k-1} x_k$ are critical edges, where $1 \leqslant k < \omega$, and Q is an ω-clique, then $Q \cap \{x_0, ..., x_k\}$ is either a starting sequence or an end sequence of $x_0, ..., x_k$. ($\emptyset$ is also considered to be a starting sequence; $x_0 = x_k$ is allowed.)*

Proof. If indirectly, $Q \cap \{x_0, ..., x_k\}$ is neither a starting sequence nor and end sequence of $x_0, ..., x_k$ then it contains a sequence $x_i, x_{i+1}, ..., x_j$, $1 \leqslant i \leqslant j \leqslant k-1$, such that $x_{i-1} \notin Q$, $x_{j+1} \notin Q$.

Since $x_i, x_j \in Q$ and $S_i \cup x_i$, $x_j \cup S_{j+1}$ are α-stable sets, $S_i \cap Q = \emptyset$, $S_{j+1} \cap Q = \emptyset$. But then Q is disjoint of both $x_{i-1} \cup S_i$ and $S_{j+1} \cup x_{j+1}$, and since by (4) $S_i \cap S_{j+1} = \emptyset$, these two α-stable sets are distinct, a contradiction. Q.E.D.

The fact that a minimal imperfect graph with a circuit of critical edges is an odd hole or antihole is well known and can be easily proved in various ways; see Markossian and Karapetian [16] or Giles, Trotter, and Tucker [9]. The latter paper contains in fact an elegant proof of the following sharper statement:

(8) *If C is a circuit of critical edges in G, then it is a Hamiltonian circuit of G, and G is a web.*

I heard the following proof from Grigor Gasparian; it is probably the shortest of all using the statements we already have at hand. In fact, the inequality in the claim of the proof—an easy consequence of (7)—is sufficient for the use we will make of it later.

Proof.

Claim. $|C| \geqslant \omega + 1$, and if K is an ω-clique, $K \cap V(C) \neq \emptyset$, then K is a subpath of C.

We first prove the second part of the claim. *If $V(C) \backslash K \neq \emptyset$ then the statement is obvious*, because then C has a subpath $x_0, ..., x_{k+1}$, where x_0, $x_{k+1} \notin K$, but $x_i \in K$ $(1 \leqslant i \leqslant k)$: according to (7) for any such subpath we have $k = \omega$ and $K = \{x_1, ..., x_\omega\}$.

Let now x_0 and x_1 be two arbitrary neighboring vertices of C, and let K be the ω-clique containing x_1 and not containing x_0; see (1). Then $V(C) \backslash K \neq \emptyset$, and for such ω-cliques K we have just proved $K \subset V(C)$. Thus $|V(C)| \geqslant |\{x_0\} \cup K| = \omega + 1$.

But then for an arbitrary ω-clique K, $V(C) \backslash K \neq \emptyset$, and the claim is proved.

If now C is not Hamiltonian, then let $u \in V(G) \backslash V(C)$, $v \in V(C)$. According to the claim, some of the ω-cliques which are color classes in the optimal coloration of $\bar{G} - u$ partition $V(C)$, and some of the ω-cliques which are color classes in the optimal coloration of $\bar{G} - v$ partition $V(C) \backslash v$. But it is not possible that both $|C|$ and $|C| - 1$ are divisible by ω. Thus C is Hamiltonian.

Applying now (4) to C we get that G is a web. Q.E.D.

The following statement occurs both in Markossian and Karapetian [16] and Giles, Trotter, and Tucker [9].

(9) *If G is partitionable and $v \in V(G)$, and the degree of v is $2\omega - 2$, then there are two different critical edges adjacent to v.*

Proof. The coloration of $G - v$ has ω color classes and the constraint implies that at least two of them, S_1 and S_2 contain at most one vertex adjacent to v. Since S_1 and S_2 are maximum stable sets, they both contain exactly one vertex adjacent to v; let these be $s_1 \in S_1$ and $s_2 \in S_2$. Clearly, vs_1 and vs_2 are critical edges. Q.E.D.

We will also need the following statement:

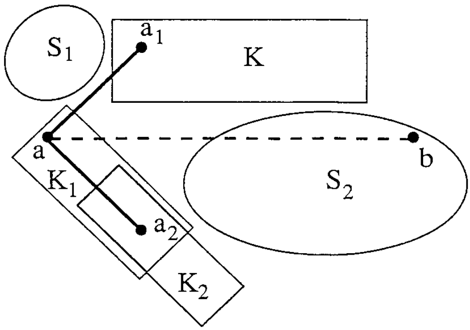

FIGURE 2

(10) *If aQb is a forcing and aSc is a co-forcing, then exactly one of* $b \in S$ *and* $c \in Q$ *holds.*

Proof. Since $Q \cup a$ is a clique and $S \cup a$ is a stable set, their only common point is a, whence $Q \cap S = \emptyset$.

If $b \notin S$, $c \notin Q$, then both $a \cup Q$ and $Q \cup b$ are disjoint from $S \cup c$, a contradiction.

If $b \in S$, $c \in Q$, then $Q \cup b$ and $S \cup c$ both contain $\{b, c\}$, which is not possible, because one of them is a stable-set and the other is a clique. Q.E.D.

THEOREM 2. *Suppose G is a partitionable graph, and* $a, a_1 \neq a_2, b \in V(G)$ *are such that* aa_1, aa_2 *are critical edges in G and ab is a critical edge in* $\overline{G}$. *Then G is an odd hole or antihole or has a small transversal.*

Proof. Let K be the $(\omega - 1)$-clique for which aKb, and similarly, S_1 and S_2 are the $(\alpha - 1)$-stable sets for which aS_1a_1 and aS_2a_2. If G is an odd hole or antihole we are done, so we can suppose $\alpha, \omega \geqslant 3$. Follow the proof on Fig. 2.

Claim 1. $S_1 \cup a_1$ and $S_2 \cup a_2$ are distinct color classes of the coloration of $G - a$.

Indeed, by (1) they are color classes, and they are distinct, because by (4), $a_1 a_2 \in E(G)$.

Claim 2. Exactly one of $a_1 \in K$ and $a_2 \in K$ holds.

Indeed, if $a_1, a_2 \in K$ then $K \cup b$ is an ω-clique not containing a, and containing both a_1 and a_2, in contradiction with (7). Similarly, if $a_1, a_2 \notin K$ then $K \cup a$ contradicts (7).

According to Claim 2 we can suppose without loss of generality $a_1 \in K$ and $a_2 \notin K$. Let us summarize the containment relations which follow:

Claim 3. $a_1 \in K$, $a_2 \notin K$

— $b \notin S_1$, $b \in S_2$

— S_1, S_2, and K are pairwise disjoint

— The list of ω-cliques containing at least one of a_1, a, a_2 is the following: $K(S_1 \cup a) = K \cup b$, containing a_1; $K(S_2 \cup a_2) = K \cup a$ containing a_1 and a; $\omega - 2$ different ω-cliques containing all of a_1, a, a_2; $K_1 := K(S_1 \cup a_1)$ containing a and a_2; $K_2 := K(S_2 \cup a)$ containing a_2 (Fig. 2).

We have the first item by assumption (relying on Claim 2). Then by (10), $b \notin S_1$ because of $a_1 \in K$ and $b \in S_2$ because of $a_2 \notin K$.

We have $S_1 \cap S_2 = \emptyset$ by Claim 1, and $S_1 \cap K = \emptyset$, $S_2 \cap K = \emptyset$ follow from the fact that $S_1 \cup a$, $S_2 \cup a$ are α-stable sets, whereas $K \cup a$ is an ω-clique.

The last item immediately follows from (7) (applied to $k = 2$), and (1) or (2).

In order to prove the theorem we distinguish two cases:

Case 1. Every α-stable set containing a is a subset of $S_1 \cup S_2 \cup a$. Since $\alpha \geqslant 3$ there exists an α-stable set S, such that $a \in S$ and $S_1 \cup a \neq S \neq S_2 \cup a$. Since $\omega \geqslant 3$ (the last item of) Claim 3 implies that there exists an ω-clique $A \supseteq \{a, a_1, a_2\}$.

Let $T := (A \cup S) \setminus a$. Clearly, $|T| = \alpha + \omega - 2$. We show now that adding a vertex to T we can get a small transversal. Note that $a_1, a_2, b \in T$. ($b \in S$ by (2), because $a \in S \neq S_1 \cup a$.)

T meets every ω-clique besides $K(S)$: since $T \supseteq S \setminus a$ we only have to check the ω-cliques containing a, and according to Claim 3, these either contain a_1 or a_2 (or both).

T meets every α-stable set besides $S(A)$: since S contains $Q \setminus a$ we only have to check the α-stable sets U containing a. If $b \in U$ we are done (because $b \in T$), so suppose $b \notin U$. But then U is the unique α-stable set (see (1)) containing a and not containing b, and because $S_1 \cup a$ is such an α-stable set by Claim 3, we get that $U = S_1 \cup a$. On the other hand, $S \subseteq S_1 \cup S_2 \cup a$ and $S \neq S_2 \cup a$ imply together that $(S \setminus a) \cap S_1 \neq \emptyset$, thus $U \cap T \neq \emptyset$ holds in this case as well. Thus adding $K(S) \cap S(A)$ to T we get a small transversal.

Remark. This type of small transversal and the condition of its usability was exhibited by Bacsó [1] and Gurvich and Temkin [11]. The last paragraph above proves that S satisfies this “Bacsó–Gurvich–Temkin” condition. One can arrive at the same result by applying (9) to $\bar{G}$, a comment of Grigor Gasparian.

Case 2. There exists an α-stable set S: $a \in S$, $S \not\subseteq S_1 \cup S_2 \cup a$. Let $s \in S \backslash (S_1 \cup S_2 \cup a)$, and $A := K(s, a)$. Since $a_1 S_1 a$ and $aS_2 a_2$ are forcings in $\bar{G} - s$, $\{a_1, a, a_2\} \subseteq A$. Define $T := (A \cup S) \backslash a$. Clearly, $|T| = \alpha + \omega - 2$. We show now that adding a vertex to T we can get a small transversal. Note that $a_1, a_2, s, b \in T$.

T meets every ω-clique besides $K(S)$: since T contains $S \backslash a$ we only have to check the ω-cliques containing a, and according to Claim 3, these either contain a_1 or contain a_2 (or both).

$S(A) \cap T \neq \emptyset$, because $s \in S(A)$. If U is an α-stable set, $U \neq s(A)$, then $U \cap A \neq \emptyset$, whence $U \cap T \neq \emptyset$, unless $a \in U$, $b \notin U$. According to (1) there is exactly one α-stable set with this property, and it is $S_1 \cup a$.

Thus T meets every ω-clique besides $K(S)$ and every α-stable set besides $S_1 \cup a$. These two are not disjoint, because $S_1 \cup a \neq S$ by the assumption of Case 2. Adding their intersection to T we get a small transversal.

Since either Case 1 or Case 2 holds the theorem is proved. Q.E.D.

COROLLARY 2.1. *Suppose that G is a partitionable graph and there exists* $v_0 \in V(G)$, $N(v_0) \subseteq A \cup B$, *where* A *and* B *are* ω*-cliques and* $A \cap B = \emptyset$. *Then G is an odd hole or antihole or has a small transversal.*

Proof. $|N(v_0)| \leqslant 2\omega - 2$, because $\omega(N(v_0)) \leqslant \omega - 1$, and by the constraint, $N(v_0)$ is covered by two ω-cliques. By (0) we have equality here, and by (9) there are two different critical edges adjacent to v_0.

Since $A \cap B = \emptyset$ we can suppose $v_0 \notin A$ (see Fig. 3). Clearly, $|N(v_0) \cap A| = |N(v_0) \cap B| = \omega - 1$. Let a be the (unique) vertex of $A \backslash N(v_0)$. Clearly, $v_0(A \backslash a)$ a is a forcing; that is, $v_0 a$ is a cocritical nonedge, and Theorem 2 can be applied; Q.E.D.

With some more work one can prove the same assertion under the somewhat weaker assumption that $G(v_0 \cup N(v_0) \cup a)$ *is uniquely colorable for some* $a \in V(G) \backslash N(v_0)$ (see the last paragraph of the paper before the acknowledgment).

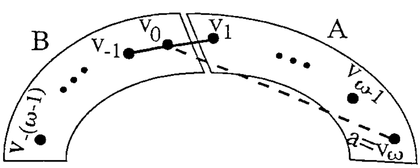

FIGURE 3

COROLLARY 2.2. *If $G=(V, E)$ is partitionable and $v_{-(\omega-1)}, ..., v_{-1}, v_0, v_1, ..., v_\omega$ are vertices so that $v_{i+1}, ..., v_{i+\omega}$ is an ω-clique for every $i=-\omega, ..., 0$, then G is an odd hole or antihole or has a small transversal.*

Proof. This statement is immediate from the previous corollary ($A=\{v_1, ..., v_\omega\}$, $B=\{v_{-(\omega-1)}, ..., v_0\}$; see Fig. 3). Let us see another proof directly from Theorem 2:

$v_{-1}v_0$ and v_0v_1 are both critical edges, because they are contained in $\omega-1$ of the cliques in the condition, and then (2) can be applied. Moreover, note that v_0Av_ω is a forcing, where $A:=\{v_1, ..., v_{\omega-1}\}$ (see Fig. 3). Thus according to Theorem 2, G is an odd hole, antihole, or has a small transversal. Q.E.D.

COROLLARY 2.3. *If G is a partitionable graph and $I(G)$ contains adjacent forced vertices, then G is an odd hole or antihole or has a small transversal.*

Corollary 2.3 is an immediate reformulation of Corollary 2.2 (see (6) and Lemma 3.2 in Section 3 below).

As mentioned in the Introduction, Corollary 2.2 shows that a class of partitionable graphs constructed by Chvátal *et al.* [7] does not contain counterexamples to the strong perfect graph conjecture. (This is the only known class of partitionable graphs for which the strong perfect graph conjecture has been open since Bacsó *et al.* [2] proved it for the other class of Chvátal *et al.*)

3. PROVING THEOREM 3, THEOREM 4, AND THEIR COROLLARIES

The key lemma of this section is the following:

LEMMA 3.1. *Let G be partitionable, and let Q be an ω-clique in G. Then the number of ω-cliques different from Q and meeting Q is at least $2\omega-2$, and the following statements are equivalent:*

(i) *The number of ω-cliques different from Q which meet Q is exactly $2\omega-2$.*

(ii) *$G-Q$ is uniquely colorable.*

(iii) *The critical edges of G induced by Q form a connected spanning subgraph of $G(Q)$.*

(iv) *The critical edges of G induced by Q form a spanning tree of Q.*

Proof. The inequality, and the implication (i) ⇒ (ii) follows from the trivial part of a result in Fonlupt and Sebő [8], but we include the proof for the sake of completeness.

The linear rank of the (characteristic vectors as vertex sets of the) ω-cliques of $G-Q$—which is the same as the number of these ω-cliques—is at most $n-1-2(\omega-1)=(n-\omega)-\omega+1$, and the stated inequality follows.

Suppose now that (i) holds. The characteristic vector χ_S of any color class S of an arbitrary ω-coloration of $G-Q$ satisfies the equation $x(K)=1$ for every ω-clique K of $G-Q$. But $G-Q$ has $n-\omega$ vertices, and $(n-\omega)-\omega+1$ linearly independent equations in the $(n-\omega)$-dimensional space have no more than ω linearly independent solutions. Since the classes of an ω-coloration provide already ω linearly independent solutions and a different color class of any other coloration is linearly independent of these, there is no other coloration.

The implication (ii) $\Rightarrow$ (iii) is from Sebő [23, (3.2)], we repeat the proof for the sake of completeness: We suppose that (ii) holds and prove that there is a critical edge between the classes of any bipartition of Q:

Let $\{X_1, X_2\}$ be an arbitrary bipartition of Q, and $x_1 \in X_1$, $x_2 \in X_2$. The coloration of $G-x_1$ consists of $S(Q)$, of $|X_1|-1$ color classes meeting X_1, and $|X_2|$ color classes meeting X_2. Similarly, the coloration of $G-x_2$ consists of $S(Q)$, of $|X_1|$ color classes meeting X_1, and $|X_2|-1$ color classes meeting X_2. Thus, since by assumption the restriction of these two colorations to $G-Q$ is the same, there exists an $(\alpha-1)$-stable A, and $a_1 \in X_1$, $a_2 \in X_2$ such that $A \cup a_2$ is a color class of $G-x_1$ and $A \cup a_1$ is a color class of $G-x_2$. Hence $a_1 a_2$ is a critical edge between X_1 and X_2, as claimed.

(iii) $\Rightarrow$ (iv) is trivial from (8).

To prove (iv) $\Rightarrow$ (i) now, suppose that H is a spanning tree of Q consisting only of critical edges. We have to prove that there are at most $2\omega-2$ different ω-cliques meeting Q.

Indeed, let $K \neq Q$, $K \cap Q \neq \emptyset$ be an ω-clique. It follows that there exists a critical edge $ab \in E(H)$ so that $a \in K$, $b \notin K$. Let S be the $\alpha-1$-stable set for which aSb. Then by (1), $S(K)=S \cup b$. Thus $K=K(S \cup a)$ or $K=K(S \cup b)$ for some $ab \in E(H)$, aSb; that is, K is one of $2(\omega-1)$ ω-cliques, as claimed. Q.E.D.

The following lemma will ensure the translation between $I(G)$ and G. Let us denote by $\Omega(v)$ the ω-clique of $I(G)$, corresponding to $v \in V(G)$, and by $\phi(Q)$ the vertex of $I(G)$, corresponding to the ω-clique $Q \subseteq V(G)$.

LEMMA 3.2. *Let K and L be ω-cliques of G, and suppose that there exists a path between $\phi(K)$ and $\phi(L)$ consisting of k critical edges of $I(G)$. Then $k=|K \backslash L|=|L \backslash K|$, and there is a (unique) order $v_{-k}, ..., v_{-1}$ of the vertices of $K \backslash L$, and $v_1, ..., v_k$ of those of $L \backslash K$ so that*

$$\{v_{-k+i}, ..., v_{-1}\} \cup (K \cap L) \cup \{v_1, ..., v_i\}$$

is an ω-clique for every $i=1, ..., k-1$ (Fig. 4).

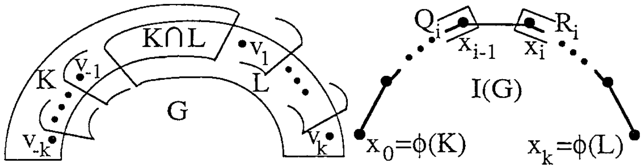

FIGURE 4

Proof. Let $x_0, ..., x_k$ be the vertices (in order) of the path of critical edges joining $x_0 := \phi(K)$ and $x_k := \phi(L)$ in $I(G)$.

The vertices of $K\backslash L$ correspond to those ω-cliques of $I(G)$ which contain $\phi(K)$, but do not contain $\phi(L)$. We show that *there is a natural one-to-one correspondance between such ω-cliques and the edges $x_{i-1}x_i$ $(i=1, ..., k)$.*

Indeed, if an ω-clique contains $\phi(K)$, but does not contain $\phi(L)$, then it contains x_{i-1} and does not contain x_i for some $i \in \{1, ..., k\}$. Conversely, if an ω-clique contains x_{i-1} and does not contain x_i for some $i=1, ..., k$, then, according to (7), it contains $\phi(K)$ and does not contain $\phi(L)$. (In fact it contains $x_0, ..., x_{i-1}$ and does not contain $x_i, ..., x_k$.) But according to (1) there is exactly one ω-clique Q_i containing x_{i-1} and not containing x_i for each $i=1, ..., k$, and $k=|K\backslash L|=|L\backslash K|$ follows.

Moreover, the order Q_i $(i=1, ..., k)$ defines an order on $K\backslash L$. Let $v_{-(k+1)+i} \in V(G)$ be the vertex for which $\Omega(v_{-(k+1)+i})=Q_i$. According to what has been proved above, $\{v_{-k}, ..., v_{-1}\}=K\backslash L$.

Similarly, let R_i be the (by (1) unique) ω-clique which contains x_i and does not contain x_{i-1}. R_i $(i=1, ..., k)$ is the list of ω-cliques which contain $\phi(L)=x_k$ and do not contain $\phi(K)=x_0$. Let $v_i \in V(G)$ be the vertex for which $\Omega(v_i)=R_i$. Similarly to the symmetric, already considered case, $\{v_1, ..., v_k\}=L\backslash K$.

After $K\backslash L$ and $L\backslash K$ let us consider the ω-cliques of $I(G)$ which correspond to the vertices of $K\cap L$. They all contain $\phi(K)=x_0$ and $\phi(L)=x_k$, and then, according to (7), they contain $\{x_0, x_1, ..., x_k\}$. Conversely, if an ω-clique contains $\{x_0, x_1, ..., x_k\}$, then the corresponding vertex of G is in $K\cap L$.

Finally, we just have to note that $\{v_{-k+i}, ..., v_{-1}\} \cup (K\cap L) \cup \{v_1, ..., v_i\}$ is an ω-clique for every $i=1, ..., k-1$, because the corresponding ω-cliques all contain x_i. (The order is unique, because for any other order $v'_{-k}, ..., v'_{-1}$ of the vertices of $K\backslash L$, and $v'_1, ..., v'_k$ of those of $L\backslash K$, there exist $i \in \{1, ..., k\}$ such that Q_i corresponds to a vertex in $\{v'_{-k+i}, ..., v'_{-1}\}$ and R_i corresponds to a vertex in $\{v'_1, ..., v'_i\}$. But $Q_i \cap R_i=\emptyset$, because by (1) $|S(Q_i)\cap S(R_i)|=\alpha-1>0$. Hence there is no edge between the vertices corresponding to Q_i and R_i, and this prevents $\{v'_{-k+i}, ..., v'_{-1}\} \cup \{v'_1, ..., v'_i\}$ from being a clique, contradicting the requirement that $\{v'_{-k+i}, ..., v'_{-1}\} \cup (K\cap L) \cup \{v'_1, ..., v'_i\}$ be an ω-clique.) Q.E.D.

Remark. These lemmas make clear that the following weakening of Conjecture 1 is also a consequence of Theorem 2:

> If G is partitionable and there exists $uv \in E(G)$ such that $|N(u)| = |N(v)| = 2\omega - 2$ then G is an odd hole or antihole or has a small transversal.

Indeed, then the ω-cliques $\Omega(u)$ and $\Omega(v)$ of $I(G)$ are nondisjoint. (uv is a determined edge in G because (0) also holds for the partitionable graph defined by the determined edges.) By Lemma 3.1 the critical edges form a connected spanning subgraph in both $\Omega(u)$ and $\Omega(v)$. But the union of these two ω-cliques contains at least $\omega + 1$ vertices, and then, by (6) (applied to $\bar{I}(G)$) and by Corollary 2.3, G is an odd hole, antihole, or has a small transversal.

As a consequence one easily deduces that *an odd hole and antihole free graph in the following class of graphs, is also perfect.*

$\mathcal{G} := \{G$: for every $V' \subseteq V(G)$ and for either $H = G(V')$ or $H = \bar{G}(V')$it is true that either $E(H) = \emptyset$, or there exists $uv \in E(H)$, so that $\alpha(N_H(u)) \leqslant 2$, and $\alpha(N_H(v)) \leqslant 2\}$.

Maffray and Preissmann [15] proved the strong perfect graph conjecture for "split neighborhood graphs" and raised the question of proving it for "claw-free neighborhood graphs," that is for graphs each subgraph H of which, or its complement, contains a vertex $v \in V(H)$ such that $\alpha(N_H(v)) \leqslant 2$.

In order to extend the above claim from $\mathcal{G}$ to claw-free neighborhood graphs, one should delete the condition $A \cap B = \emptyset$ from the condition of Corollary 2.1, or equivalently, one should delete the two "extreme cliques" in the condition of Corollary 2.2. However, these stronger statements seem to be difficult to prove.

A variant of this argument might be helpful for proving Conjecture 1 (see the remark after Theorem 4).

THEOREM 3. *If $G = (V, E)$ is partitionable, $v_0 \in V(G)$, and $N(v_0) \subseteq A \cup B$, where A and B are cliques, then there exists a (unique) order $v_{-(\omega-1)}, ..., v_{-1}, v_0, v_1, ..., v_{\omega-1}$ of $\{v_0\} \cup N(v_0)$ so that $\{v_i, v_{i+1}, ..., v_{i+\omega-1}\}$ is an ω-clique in G for all $i = -(\omega-1), ..., -1, 0$.*

Proof. We can suppose A, $B \subseteq N(v_0)$. Because of (0), $A \cup v_0$ and $v_0 \cup B$ are ω-cliques with only v_0 in common.

Claim. The critical edges induced by $\Omega(v_0)$ in $I(G)$ form a spanning tree of $\Omega(v_0)$.

Indeed, the ω-clique $\Omega(v_0)$ of $I(G)$ satisfies (i) of Lemma 3.1. Hence it also satisfies (iv), and the claim is proved.

It follows from the claim that there exists a path P of critical edges in $I(G)$ between $\phi(A \cup v_0)$ and $\phi(v_0 \cup B)$, $|P| \leqslant \omega - 1$. Since $A \cup v_0$ and $v_0 \cup B$ have only one common element, $|P| = \omega - 1$. Now Lemma 3.2 immediately implies our theorem ($k = \omega - 1$). Q.E.D.

The heart of this proof of Theorem 3 is Lemma 3.1; the rest is a straightforward translation of the results about $I(G)$ to results about G, where the translator is Lemma 3.2. The original proof of this theorem works directly on G, whereas here, the main part of the work is made in terms of $I(G)$. Surprisingly, this seems to make an essential difference even between the main lines of the two proofs.

THEOREM 4. *If G is partitionable, $I(G)$ has a vertex of degree $2\omega - 2$ and $I(\bar{G})$ has a vertex of degree $2\alpha - 2$ then G is an odd hole or antihole or has a small transversal.*

Proof. According to Lemma 3.1, the condition is equivalent to the existence of an ω-clique Q and an α-stable set S so that the critical edges induced by Q and the cocritical nonedges induced by S form spanning trees. Let us denote these by H_Q and H_S, respectively. We would like to show that either G or $\bar{G}$ satisfies the condition of Theorem 2. Let x be a vertex of degree 1 in H_Q, and let $xq \in E(H_Q)$ be the edge incident to it.

Claim. There exists a vertex $y \in V(G)$ for which $x(Q \backslash x)y$ is forcing.

Indeed, let Q' be the ω-clique which contains q but does not contain x (see (1)). Because of (7) $Q' \supseteq Q \backslash x$ must hold. But then $Q' \backslash Q$ consists of one vertex; let us denote it by y: it obviously has the claimed property.

Case 1. $Q \cap S = \emptyset$. Then $Q' = (Q \backslash x) \cup y$ meets S; thus $y \in S$: since xy is a cocritical nonedge, $H_S \cup xy$ is a connected graph, that is, for $S \cup x$ (6)(ii) holds. Consequently (6)(i) also holds. Apply now Corollary 1.1 to $\bar{G}$.

Case 2. $Q \cap S = v$, where the degree of v is at least 2 in at least one of H_Q and H_S. Then Theorem 2 immediately implies the statement. The only case that remains is:

Case 3. $Q \cap S = x$, where the degree of x is 1 in both H_Q and H_S. The vertex adjacent to x in H_Q is denoted by q as before, and the unique vertex adjacent to it in H_S is denoted by s.

Then, from the claim we get that there exist vertices y_Q and y_S so that $x(Q \backslash x)y_Q$ is forcing and $x(S \backslash x)y_S$ is coforcing. Both $y_Q = s$ and $y_S = q$ cannot hold, because that would contradict (10). But if $y_Q \neq s$ then $\bar{G}$, and if $y_S \neq q$ then G satisfies the condition of Theorem 2. Q.E.D.

Remark. At the very end of the above proof we could have applied Theorem 1 or its corollaries instead of Theorem 2. (If, say, $y_Q \neq s$, then by, say, (10) $y_Q \notin S$, and because of $\alpha(G) = |S|$, y_Q is adjacent to some vertex s' of S. y_Q and s' are adjacent forced vertices, and Corollary 1.2 can be applied.) This fact has some significance: Case 1 and Case 2 of the proof of Theorem 4 can be handled for $I(G)$ too, that is, in a tentative proof of Conjecture 1 as well. Indeed, according to Corollary 2.3 (see in the Introduction or in Section 2) we can replace G by $I(G)$ in the condition of the utilized corollaries of Theorem 1. Only that subcase of Case 2 does not go through trivially, when a vertex of $I(G)$ is adjacent to the two critical edges and two critical nonedges of $I(G)$. It can be easily checked that this means exactly that the condition of the following conjecture is satisfied.

Conjecture 3. If G is minimal imperfect and it contains four vertices a, b, c, d such that ab, cd are critical edges, and bc, da are cocritical nonedges, then G is an odd hole or antihole.

According to the remark, Conjecture 3 implies Conjecture 1. (Again, odd holes and antiholes satisfy the condition of this conjecture.) Some other corollaries of Theorem 4 follow.

COROLLARY 4.1. *If G is partitionable and G has an ω-clique Q and an α-stable set S such that $G-Q$ and $\bar{G}-S$ are uniquely colorable, then G is an odd hole, antihole, or has a small transversal.*

COROLLARY 4.2. *If G is partitionable and G has an ω-clique Q and an α-stable set S such that the critical edges induced by Q and the cocritical nonedges induced by S, respectively, form a connected graph, then G is an odd hole, antihole, or has a small transversal.*

Corollary 4.1 is an immediate consequence of Theorem 4 using Lemma 3.1(ii) $\Rightarrow$ (i), and then Corollary 4.2 follows using Lemma 3.1(iii) $\Rightarrow$ (i). Here we state the corresponding reformulations of the strong perfect graph conjecture and some of their variants.

The following statements are equivalent to the strong perfect graph conjecture. For every minimal imperfect graph G:

G has an ω-clique which meets at most $2\omega-2$ other ω-cliques.

G has an ω-clique Q such that $G-Q$ is uniquely colorable.

G has an ω-clique Q whose induced critical edges form a connected graph (or equivalently, $\bar{G}$ has a forced class).

G contains a chain consisting of $\omega-1$ consecutive critical edges.

Either $H=G$ or $H=\bar{G}$ contains a "local web structure," that is, $2\omega(H)$ vertices which can be ordered so that any consecutive $\omega(H)$ of them form a clique in H (or equivalently, either $I(\bar{G})$ or $I(G)$ have adjacent forced vertices).

In either $H=G$ or $H=\bar{G}$ there exist vertices $v_0 \neq a$, $v_0 a \notin E(H)$, such that $H(v_0 \cup N(v_0) \cup a)$ is uniquely colorable.

The third of the above reformulations meets an assertion on uniquely colorable perfect graphs (see the remark at the end of the Introduction). Compare the last reformulation with Conjecture 1, noting that $|N(v_0)| = 2\omega - 2$ means exactly that $N(v_0)$ induces a uniquely colorable graph. (This follows easily from the characterization of uniquely colorable graphs by Fonlupt and Sebő [8]. The proof of Lemma 3.1 may give a hint; see also Sebő [24, (*), (i)]. These results, used together with (0) and (1), imply the last reformulation as well.)

Acknowledgments

I am indebted to Myriam Preissmann and Grigor Gasparian for many essential suggestions, inducing important rearrangements and shortcuts; to Myriam Preissmann and Frédéric Maffray for detectinig several flaws in a previous version.

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