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General Antifactors of Graphs

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In a superstitious company everybody has numbers that he thinks to be unlucky for himself. When they meet, everybody wants to shake hands with some of his acquaintances, but nobody wants to shake hands with an unlucky number of acquaintances. When can this be successful? This question occurred to L. Lovász (*Period. Math. Hungar.* **4**, Nos. 2–3, 1973, 121–123), where the case when everyone has one unlucky number (antifactor problem) is answered. In this paper we give a "Tutte-type good characterization" (and a simple polynomial algorithm) to decide this question when several unlucky numbers are allowed, but no one in the company has two neighboring unlucky numbers. © 1993 Academic Press, Inc.

1. INTRODUCTION

Let G be an arbitrary graph, and let a set $H(x)$ of non-negative integers be associated with every vertex $x \in V(G)$. An *H-factor* is a set $F \subseteq E(G)$ for which $d_F(x) \in H(x)$ for all $x \in V(G)$. ($V(G)$ is the vertex set, $E(G)$ the edge-set of the graph G. $\delta_F(x)$ is the set of edges of F incident to x, $d_F(x)$ is the cardinality of this set. If $F = E(G)$, then we simply write $\delta(x)$ and $d(x)$. Similarly, for $x \subseteq V(G)$, $\delta(X)$ denotes the set of edges with exactly one endpoint in X.)

The problem of deciding the existence of an H-factor is NP-complete. In fact, if we restrict G to be 3-regular, and $H(x)$ to be equal to $\{0, 3\}$ or $\{1\}$ for every $x \in V(G)$, we already get an NP-complete problem (cf. Lovász [7], Lovász and Plummer [9]). This suggests studying the existence of *general antifactors*, that is, H-factors, where H has the property

$$(*) \qquad i \notin H(x) \text{ implies } i+1 \in H(x), \text{ for every } x \in V(G).$$

The *antifactor* problem is the special case where $H(x) = \mathbb{N} \setminus g(x)$ ($g(x) \in \mathbb{N}$) for all $x \in V(G)$. The methods we use below badly need $(*)$, they are based on it, this is the limit of their use.

This fact was pointed out by Lovász [7] who generalized the structure-theory of matchings to "general factors," that is, H-factors, where $H(x)$ is a subset of an interval (which depends on x), and $(*)$ is required only for

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this interval. However, Lovász's paper does not provide a polynomial algorithm to solve this problem.

Recently, Cornuéjols [5] has given a polynomial algorithm.

Both Lovász [7] and Cornuéjols [5] generalize Tutte's theorem in some sense, but in these results the property generalizing the "oddness" of Tutte's theorem seems as difficult to check as the existence of general factors. Lovász [8] gave the following simple direct characterization for the antifactor problem, and a straightforward polynomial algorithm.

THEOREM (Lovász [8]). *Suppose* $H(x) = \mathbb{N} \setminus g(x)$ $(g(x) \in \mathbb{N})$ *for all* $x \in V(G)$. *Then G has an H-factor if and only if G is not a tree which can be oriented so that the indegree of vertex v is* $g(v)$, *for all* $v \in V(G)$.

The goal of the present paper is to give a Tutte-type answer to the general antifactor problem as well. Unfortunately, this answer turns out to be more complicated, and different in spirit. (However, Lovász's theorem can be simply deduced from the main result, see end of Section 3.)

By "Tutte type characterization" we mean an apparent reason for the non-existence of graph-factors, where "apparent" means "very easy to check," and could be defined precisely to mean a constraint checkable in linear time, or with a "cutting plane proof" (cf. Chvátal [2], see Section 4).

For the more special, better known factorization problems, the Tutte-type characterization for the existence of graph factors and the good characterization of the weighted optimum get a common, deeper explanation in the polyhedral descriptions. The former corresponds to Farkas' Lemma, the latter to the duality theorem applied to the minimal system of inequalities describing the convex hull of the investigated graph factors. To determine this system of inequalities, the usual way is to first write down a bigger polyhedron, which however does not contain new integer points. Then use non-negative combinations giving new inequalities with integer coefficients, and round the corresponding right hand sides, to get sharper valid inequalities for all the integer points of the given polyhedron. According to Chvátal [1], after a finite number of such steps one gets down to the convex hull of the integer points. A procedure that makes clear the emptiness of a polyhedron by repeatedly taking linear combinations and rounding, proving in this way an inequality that has obviously no solution, is called a *cutting plane proof* (Chvátal [2]).

We will not need to define this term more precisely, we only mention it to explain what we mean by a Tutte-type characterization. Tutte's theorem and all its generalizations can be interpreted as simple cutting-plane proofs; so can the theorem we present below for *H*-factors where *H* satisfies $(*)$. (For more details see Section 4.) However, the naturally stated optimization problems on *H*-factors, or the related linear description of their convex hull, remain open.

Let us call the pair (G, H) *dense* if H satisfies $(*)$, and G is connected.

In Section 2 we characterize the existence of H-factors for dense pairs with 2-connected G; this characterization is used in Section 3 to obtain the characterization for arbitrary graphs. In Section 4 we explain how our result can be interpreted as a cutting plane proof.

2. Two-Connected Graphs

First we prove a simple statement concerning arbitrary dense pairs. For a given dense pair (G, H) and $F \subseteq E(G)$, we call $x \in V(G)$ *feasible* (in F), if $d_F(x) \in H(x)$.

PROPOSITION 1. *If (G, H) is dense, and $a \in V(G)$, then there exists $F \subseteq E(G)$ so that x is feasible for all $x \in V(G) \setminus a$.*

Proof. Let $F \subseteq E(G)$ with $|\{x \in V(G) \setminus a : d_F(x) \notin H(x)\}|$ minimal. If indirectly, there is a $b \neq a$, $d_F(b) \notin H(b)$, then let P be a path between a and b, and let $P(x, y)$ denote the subpath between two of its vertices x and y. By $(*)$, $d_{F \Delta P}(b) \in H(b)$.

Let x be the vertex with $d_{F \Delta P}(x) \notin H(x)$ nearest to b on P (if there exists no such vertex, then define $x = a$). $d_{F \Delta P(b,x)}(x) \in H(x)$, and because of the choice of x, for all vertices y between b and x, $d_{F \Delta P(b,x)}(y) = d_{F \Delta P}(y) \in H(y)$. Thus, $F \Delta P(b, x)$ contradicts the choice of F, because b becomes feasible. Q.E.D.

For $a \in V(G)$ let

$$N_{G,H}(a) = \{d_F(a) : F \subseteq E(G),\ d_F(x) \in H(x) \text{ if } x \neq a\},$$

that is, $N_{G,H}(a)$ is the set of all possible degrees in a, under the condition that all the other vertices are feasible.

Remarks. We often use the following easy statements:

— By Proposition 1, $N_{G,H}(a) \neq \emptyset$.

— There exists an H-factor if and only if $N_{G,H}(a) \cap H(a) \neq \emptyset$.

— If, $i, i+1 \in N_{G,H}(a)$, then by $(*)$, $N_{G,H}(a) \cap H(a) \neq \emptyset$, and consequently there exists an H-factor.

Given a dense pair (G, H), we call $H(x)$ and also $x \in V(G)$ *odd*, or *even*, if $H(x)$ consists of only odd or of only even integers. A vertex will be said to have *fixed parity* if it is odd or even.

PROPOSITION 2. *Suppose $a \in V(G)$ is not a cut vertex in G. Then $N_{G,H}(a)$ either contains neighboring numbers, or if not, then it consists of all the odd, or all the even integers in $[0, d_G(a)]$.*

Proof. First we show that

$$i \in N_{G,H}(a),\ i+1 \notin N_{G,H}(a),\ i+2 \leqslant d_G(a) \text{ implies } i+2 \in N_{G,H}(a).$$

Let $F \subseteq E(G)$ be such that $d_F(a) = i$, and $e_1, e_2 \in \delta(a) \backslash F$. Since a is not a cut vertex, there exists a cycle C containing both e_1 and e_2. All the vertices of $V(G) \backslash a$ are feasible in $F \Delta C$ as well: otherwise, like in the proof of Proposition 1, we have a subpath P of C beginning with a such that in $F \Delta P$ all vertices of $V(G) \backslash a$ are feasible, in contradiction with $i+1 \notin N_{G,H}(a)$. Thus $i+2 = d_{F \Delta C}(a) \in N_{G,H}(a)$. Similarly,

$$i \in N_{G,H}(a),\ i-1 \notin N_{G,H}(a),\ i-2 \geqslant 0 \text{ implies } i-2 \in N_{G,H}(a),$$

(The only difference is that now the choice $e_1, e_2 \in \delta(a) \cap F$ has to be made, or we simply apply the already proved statement to $H'(x) := \{d_G(x) - h : h \in H(x)\}$ and note that $N_{G,H'} = d_G - N_{G,H}$.) Q.E.D.

COROLLARY 1. *If (G, H) is dense, and G does not have an H-factor, then all the vertices of G except possibly the cut vertices have fixed parity.*

Proof. We show that if a is not a cut vertex but it does not have fixed parity, then $N_{G,H}(a) \cap H(a) \neq \emptyset$ which means that there exists an H-factor. This is obvious if $N_{G,H}(a)$ contains neighboring numbers. If $N_{G,H}(a)$ does not contain neighboring numbers, then by Proposition 2, it consists of all the odd or all the even numbers of the interval $[0, d_G(a)]$. At least one of these is in $H(a)$, because, by supposition, $H(a)$ does not have fixed parity. Q.E.D.

COROLLARY 2. *If (G, H) is dense, and G is 2-connected, then there exists no H-factor, if and only if x has fixed parity for all $x \in V(G)$, and the number of odd vertices is odd.*

Proof. By Corollary 1 we immediately have that every vertex has fixed parity. Choose an arbitrary $a \in V(G)$. By Proposition 1 there exists $F \subseteq E(G)$, such that $d_F(x)$ has the same parity as $H(x)$ for all $x \neq a$. Since there is no H-factor, the parity of $d_F(a)$ is different from that of $H(a)$. The proposition follows now from the fact that the number of vertices with $d_F(x)$ odd is even. Q.E.D.

We are still far from our goal: in order to see clearly when there is no H-factor, we should better understand the set $N_{G,H}(x)$ for cut vertices as well.

In Fig. 1(a), the cut vertex a does not have fixed parity, in fact any degree but 2 is allowed. Despite this, the graph does not have an H-factor. Here we still have an easy explanation: in an H-factor the degree of a should be 1 in both blocks.

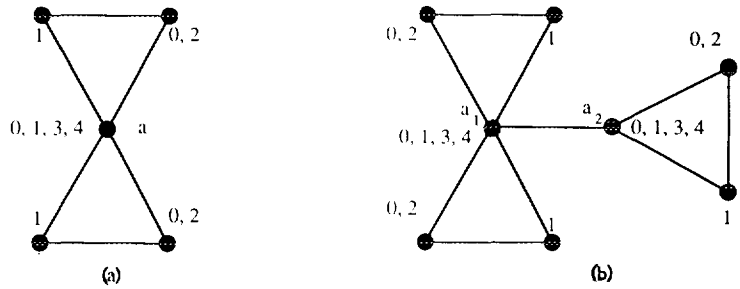

FIGURE 1

If we want to see that the graph of Fig. 1(b) does not have an H-factor, we have to find a somewhat more general reason: $G - a_1$ has two components with all vertices having fixed parity, and both have an odd number of odd vertices. Thus, if F is an H-factor, then $d_F(a_1) \geqslant 2$. Since the degree of a_1 in these components is 2, any H-factor will also miss one of the two edges, so $d_F(a_1) \leqslant 5 - 2 = 3$. Now, after having deduced this lower and upper bound, a_1 has fixed parity, and a similar argument can be applied now to a_2: we get that $d_F(a_2) \geqslant 2$, and $d_F(a_1) \leqslant 3 - 1 = 2$, but $2 \notin H(a_2)$, so F is not an H-factor, a contradiction.

The general argument we have is the direct extension of this.

3. Parity Traces

To formulate the general reason for the non-existence of general antifactors it will be helpful to have the following notion.

Let (G, H) be dense, V_0^1 denote the set of odd vertices, and V_0^2 the set of even vertices. Suppose $(a_1, ..., a_k)$ is an order of $V(G) \setminus (V_0^1 \cup V_0^2)$. $(a_1, ..., a_k)$ will be called a *parity trace* if the sets and numbers defined recursively as follows, satisfy (1) below.

0. V_0^1 and V_0^2 are defined as above. For $i = 1, ..., k$ define recursively the numbers l_i, u_i, and the sets V_i^1, V_i^2, for which (1) below holds:

1. Suppose V_{i-1}^1 and V_{i-1}^2 have already been defined. Let l_i be the number of those components C of $G - a_i$ for which $C \subseteq V_{i-1}^1 \cup V_{i-1}^2$ and $|C \cap V_{i-1}^1|$ is odd.

Let $u_i := d_G(x) - t_i$ where t_i is the number of those components C of $G - a_i$ for which $C \subseteq V_{i-1}^1 \cup V_{i-1}^2$, and the parity of $|C \cap V_{i-1}^1|$ is different from the parity of $d_G(a_i, C)$. ($d_G(a_i, C)$ is the number of edges joining a_i to some vertex of C.) Moreover, as mentioned, we suppose that (1) holds:

(1) The numbers in $[l_i, u_i] \cap H(a_i)$ have the same parity.

2. If $[l_i, u_i] \cap H(a_i)$ is odd, then let $V_i^1 := V_{i-1}^1 \cup a_i$, and $V_i^2 := V_{i-1}^2$; if it is even, then $V_i^1 := V_{i-1}^1$, and $V_i^2 := V_{i-1}^2 \cup a_i$. There is no other case according to (1). ($[l_i, u_i] \cap H(a_i) = \emptyset$ is possible, then we are free to consider it to be either odd or even.)

If $a_1, ..., a_k$ is a parity trace, then let $V^1 := V_k^1$, $V^2 := V_k^2$. Clearly, $V(G) = V^1 \cup V^2$, $V^1 \cap V^2 = \emptyset$.

EXAMPLE. In Fig. 1(a), $\{a\}$ is a parity trace: $l_1 = t_1 = 2$, $u_1 = 2$, $[l_1, u_1] \cap H(a) = \emptyset$. Let $a \in V_1^1$.

Similarly, in Fig. 1(b), (a_1, a_2) is a parity trace: $l_1 = t_1 = 2$, $u_1 = 3$, $[l_1, u_1] \cap H(a_1) = \{3\}$, thus $a_1 \in V_1^1$; $l_2 = 2$, $t_2 = 1$, $u_2 = 2$, $[l_2, u_2] \cap H(a_2) = \emptyset$. Let $a_2 \in V_2^1$.

Of course, if for some i we have $[l_i, u_i] \cap H(a_i) = \emptyset$, then we know already that there is no H-factor. This fits into the more general set of obstacles where $|V^1|$ is odd (see the Theorem below): in this case we are free to let both $a_i \in V_i^1$ or $a_i \in V_i^2$. In our examples V^1 consists of the vertices with $H(x) = \{1\}$, and of the vertices in the parity trace: $|V^1| = 3$ in the first example (Fig. 1(a)), and $|V^1| = 5$ in the second (Fig. 1(b)).

(Note that (a_2, a_1) is also a parity trace.)

We state and prove now the main result of the paper:

THEOREM. *Let (G, H) be dense. Then there exists an H-factor, if and only if there is no parity trace with $|V^1|$ odd.*

Proof. The necessity of the condition is easy: Suppose there exists an H-factor F, and a parity trace $(a_1, ..., a_k)$. We have to show that $|V^1|$ is even. Clearly, it is enough to prove that $d_F(x)$ is odd if and only if $x \in V^1$.

We prove the following statement by induction on $j \leqslant k$:

(2) If $F \subseteq E(G)$ is feasible on $V_j^1 \cup V_j^2$, then $d_F(x)$ is odd if $x \in V_j^1$, and $d_F(x)$ is even if $x \in V_j^2$.

For $j = 0$ we know this by definition. Suppose this is true for $j = i - 1$. If $C \subseteq V_{i-1}^1 \cup V_{i-1}^2$, and $|C \cap V_{i-1}^1|$ is odd, then $d_F(C)$ is odd, and consequently $d_F(a_i, C) \geqslant 1$. $d_F(a_i) \geqslant l_i$ immediately follows.

If $C \subseteq V_{i-1}^1 \cup V_{i-1}^2$, $|C \cap V_{i-1}^1| \not\equiv d_G(a_i, C) \pmod 2$, then the parity of $d_F(C)$ is different from that of $d_G(a_i, C)$ yielding $d_F(a_i, C) \leqslant d_G(a_i, C) - 1$. This implies immediately $d_F(a_i) \leqslant d_G(a_i) - t_i = u_i$.

Consequently $d_F(a_i) \in [l_i, u_i] \cap H(a_i)$, whence $d_F(a_i)$ is odd provided $a_i \in V_i^1$, and even provided $a_i \in V_i^2$. Thus (2) is proved. Since $(a_1 ..., a_k)$ is a parity trace, $V(G) = V^1 \cup V^2$. Applying (2) to $j = k$ we have that $d_F(x)$ is odd if and only if $x \in V^1$ as we hoped.

To prove the essential, opposite implication, let us suppose that G does not have an H-factor. We construct the parity trace $(a_1, ..., a_k)$ with $|V^1|$ odd.

Suppose V^1_{i-1}, V^2_{i-1} are already defined $(1 \leqslant i \leqslant k)$.

We construct $(a_1, ..., a_k)$ so that (1) holds. This determines V^1_i, V^2_i.

Claim. There exists $a_i \in V(G) \setminus (V^1_{i-1} \cup V^2_{i-1})$, for which all components of $G - a_i$ except possibly one are subsets of $V^1_{i-1} \cup V^2_{i-1}$.

Indeed, according to Corollary 1, the elements of $V(G) \setminus (V^1_0 \cup V^2_0)$ are cut vertices. It is easy to see that for any non-empty subset A of the set of cut vertices of an arbitrary graph there exists an $a \in A$ such that all the other elements of A are in the same component of $G - a$. (This is clear, e.g., from the block-structure.) Applying this to $A := V(G) \setminus (V^1_{i-1} \cup V^2_{i-1}) \subseteq V(G) \setminus (V^1_0 \cup V^2_0)$ we get our Claim.

Choose now a_i to be a point with the property stated in the Claim. Let $\mathscr{B}$ be the family of graphs induced by the sets of the form $C \cup \{a_i\}$, where C is a component of $G - a_i$. a_i is not a cut vertex in $B \in \mathscr{B}$, and

$$N_{G,H}(a_i) = \sum_{B \in \mathscr{B}} N_{B,H}(a_i). \tag{3}$$

(If $X, Y, ..., Z$ are sets of numbers, then $X + Y + \cdots + Z := \{n : n = x + y + \cdots + z,\ x \in X,\ y \in Y, ..., z \in Z\}$.) Since G has no H-factor, $N_{G,H}(a_i)$ does not contain neighboring integers, whence for all $B \in \mathscr{B}$, $N_{B,H}(a_i)$ does not contain neighboring integers either. Applying Proposition 2, for each $B \in \mathscr{B}$, $N_{B,H}(a_i)$ consists of all the odd or of all the even integers of the interval $[0, d_B(a_i)]$. The parity here must be equal to the parity of $|B \cap V_{i-1}|$ for all $B \in \mathscr{B}$, $B \subseteq V^1_{i-1} \cup V^2_{i-1}$. Comparing this with (3), and with the definition of l_i, u_i we see that $N_{G,H}(a_i) \subseteq [l_i, u_i]$.

On the other hand, by the choice of a_i, for all $B \in \mathscr{B}$, $V(B) \subseteq V^1_{i-1} \cup V^2_{i-1}$ holds, except for possibly one $B_0 \in \mathscr{B}$. But also $N_{B_0,H}(a_i)$ consists of all the odd or all the even integers in $[0, d_{B_0}(a_i)]$. Thus, according to (3), $N_{G,H}(a_i)$ consists of each second number starting with either l_i or $l_i + 1$ and until either u_i or $u_i - 1$, depending on the parity of $N_{B_0,H}(a_i)$. In other words $N_{G,H}(a_i)$ *consists of all the odd or all the even integers in* $[l_i, u_i]$. Since there is no H-factor, and using $(*)$,

$$[l_i, u_i] \cap H(a_i) = [l_i, u_i] \setminus N_{G,H}(a_i), \tag{4}$$

whence (1) holds.

Until now we proved that if there is no H-factor, then there exists a parity trace $(a_1, ..., a_k)$. We show now that for this parity trace $|V^1_k|$ is odd.

Let $F \subseteq E(G)$ be such that every $x \in V(G) \setminus a_k = V^1_{k-1} \cup V^2_{k-1}$ is feasible. Since $(a_1, ..., a_{k-1})$ is a parity trace, (2) can be applied: $d_F(x)$ is odd if $x \in V^1_{k-1}$, and even, if $x \in V^2_{k-1}$. Hence, the parity of $d_F(a_k)$ $(\in [l_k, u_k] \cap$

$N_{G,H}(a_k)$) is the same as that of $|V^1_{k-1}|$, and according to (4) different from the parity of $[l_k, u_k] \cap H(a_k)$. It follows that $|V^1_{k-1}|$ and $[l_k, u_k] \cap H(a_k)$ have different parities. Q.E.D.

This proof can easily be turned into a $O(|V(G)||E(G)|)$ algorithm which either determines an H-factor, or provides a parity trace proving the non-existence of such an H-factor.

For the sake of another example let us conclude this section by checking how parity traces specialize in the antifactor case. Note first that in this special case $V^1_0 \cup V^2_0$ consists of vertices of degree 1 and 2, and that a_1 has one neighbor in each component of $G - a_1$. If $C \subseteq V^1_0 \cup V^2_0$ is a component of $G - a_1$, and v is the neighbor of a_1 in C, then clearly, $d_C(v) \leqslant 1$. Thus, either $d_C(v) = 0$ ($V(C) = \{v\}$), or there exists another $u \in V(C) \backslash v$, with $d_C(u)$ odd, that is, $d_C(u) = 1$. In the former case $d_G(v) = 1$, whereas in the latter case $d_G(u) = 1$. We have shown that the existence of a parity trace implies that there exists a vertex of degree 1 in G. Lovász's antifactor theorem follows now by a straightforward induction.

4. Cutting Plane Proofs

Finally, let us sketch how parity traces can be interpreted as cutting-plane proofs.

First we have to give a system of linear inequalities whose integer solutions are the characteristic vectors of H-factors. This is an easy excercise, which can be solved in many different ways. Here is a possible solution that has the advantage of being applicable in the case when we also permit bounds in H, and the number of constraints in it is not big.

Suppose $H(v) \subseteq [l(v), u(v)]$, $l(v), u(v) \in H(v)$, and suppose (∗) holds for arbitrary $i \in [l(v), u(v)]$.

Let us associate to each edge $e \in E$ a variable $x(e)$. The feasible values of x will be the characteristic vectors of H-factors.

For all $v \in V(G)$, $H(v)$ determines a unique partition $\mathscr{I}(v)$ of the interval $[l(v), u(v)]$ into maximal intervals of the form $I = [l(I), u(I)]$ $l(I), u(I) \in H(v)$ with the following two properties (see Fig. 2):

— either $I \subseteq H(v)$,

— or $I \cap H(v) = \{i \in I : i \equiv l(I) \equiv u(I) \pmod 2\}$.

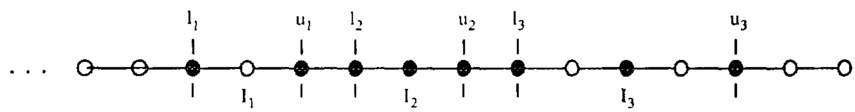

Figure 2

Let us define now the coefficients $p(v, I) : p(v, I) = 1$ if $I \subseteq H(v)$, and $p(v, I) = 2$ if only each second element of I is in $H(v)$. For all $v \in V(G)$ and $I \in \mathscr{I}(v)$ we define in addition variables $y(v, I)$, $z(v, I)$, and for all $v \in V(G)$ a variable x_v.

(E0) $0 \leqslant x(e) \leqslant 1 \ \forall e \in E(G); \qquad y(v, I), z(v, I) \geqslant 0 \ \forall v \in V(G), I \in \mathscr{I}(v)$

(E1) $$\sum_{I \in \mathscr{I}(v)} y(v, I) = 1$$

(E2) $$\sum_{I \in \mathscr{I}(v)} y(v, I)\, l(I) \leqslant x(\delta(v)) \leqslant \sum_{I \in \mathscr{I}(v)} y(v, I)\, u(I) \qquad \forall v \in V(G)$$

(E3) $$l(I) - (1 - y(v, I)) \leqslant x(\delta(v)) - p(v, I)\, z(v, I) \leqslant l(I) + (1 - y(v, I)) \qquad \forall v \in V(G), \forall I \in \mathscr{I}(v).$$

The integer solutions of this system of inequalities clearly correspond to H-factors: (E1) chooses an interval I, (E2) makes sure that the degree will be in this interval, and (E3) does not say anything if $y(v, I) = 0$, or if $p(v, I) = 1$ (it can be easily satisfied in these cases with an appropriate $z(v, I)$). On the other hand if $y(v, I) = 1$, and $p(v, I) = 2$, then it implies an equality ensuring that the parity of $x(\delta(v))$, that is, of $d_F(v)$, equals $l(v)$.

Let us see an example of how (E0), (E1), and (E2) imply new inequalities with a cutting plane proof. (Our example is the cutting plane proof of the so-called T-cut constraints, cf. Edmonds and Johnson [6].)

If for some $v \in V(G)$, $\mathscr{I}(v)$ consists of only one interval $I(v)$, and $p(v, I) = 2$ (meaning exactly that v has fixed parity), then by (E1), $y(v, I(v)) = 1$, and applying (E3) for the pair $(v, I(v))$ we get

(5) $$x(\delta(v)) - 2z(v, I(v)) = l(I(v)).$$

If T consists only of such vertices, and $\sum_{v \in T} l(I(v))$ is odd, then adding up (5) for all $v \in T$,

(6) $$x(\delta(T)) + 2x(E(T)) - 2 \sum_{v \in T} z(v, I(v)) = \sum_{v \in T} l(I(v)).$$

Add to this $-x(\delta(T)) \leqslant 0$, then divide by 2, and round,

(7) $$x(E(T)) - \sum_{v \in T} z(v, I(v)) \leqslant \left\lfloor \frac{\sum_{v \in T} l(I(v))}{2} \right\rfloor .$$

Multiplying (7) by 2, then subtracting it from (6), since the right hand side of (6) is odd, we get

(8) $$x(\delta(T)) \geqslant 1.$$

We described this well-known procedure (besides the goal of giving an example) because we need it later. For (f, g)-factors (possibly with parity constraints), and their special cases, matchings and T-joins, one can prove in this way similar inequalities to (7) and (8). In these cases, for all $v \in V(G)$, $\mathscr{I}(v)$ consists of one interval, and according to the result of Edmonds and Johnson [6], some inequalities proved similarly to (7) describe the convex hull of all graph factors. (For all of these "classical" problems the minimal description and the minimal TDI description are also well known, cf. Cook and Pulleyblank [3, 4, 10].)

If $(a_1, ..., a_k)$ is a parity trace, then after the deletion of a_1 we remain with a graph having l_1 components $C_1, ..., C_{l_1}$ for which we can show with a cutting plane proof $x(\delta(C_i)) \geqslant 1$ $(i = 1, ..., l_1)$.

Adding up these inequalities (and some inequalities of the form $x(e) \geqslant 0$) we arrive at the inequality

(9) $$x(\delta(a_1)) \geqslant l_1.$$

Similarly, for t_1 components C of $G - a_1$ it can be proved that $x(\delta(C)) \leqslant d_G(C, a_1) - 1$, and adding up these inequalities, we get

(10) $$x(\delta(a_1)) \leqslant d_G - t_1 = u_1.$$

Equation (5) is now deduced for $v = a_1$ and (6), (7), (8) follow. Similarly, continuing to do the same for $a_2, ..., a_k$, if $(a_1, ..., a_k)$ is a parity trace, and $|V^1|$ is odd, then we finally get $x(\delta(V(G))) \geqslant 1$, which obviously does not have any solution. Figure 3 represents a graph having $2k + 1$ vertices, and all the vertices are labelled with one of the letters A, B, C. Let $H_A := \{0, 2\}$, $H_B := \{1\}$, $H_C = \{0, 1, 3, 4\}$.

It is easy to see that the sequence of vertices denoted by C with the order from the left to the right is a parity trace, and $|V^1| = 2k - 1$ is odd, so this graph does not contain an H-factor.

Starting with the inequality system (E0), (E1), (E2), this parity trace determines a cutting plane proof of length $k - 1$ (and in fact of depth $k/2$)

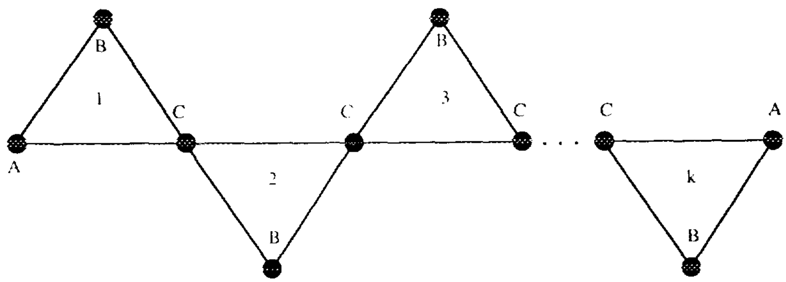

FIGURE 3

proving the emptiness of the H-factor polyhedron. If the reader would like to see a non-empty example to a "general factor polyhedron" with a long Chvátal procedure (probably with a large Chvátal rank), then he only has to modify to B the label A of one of the two vertices labeled with A. The more complicated algorithmic behavior of general factors compared to the classical factorization problems can probably be explained by their larger Chvátal rank.

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