Available online at www.sciencedirect.com
ScienceDirect
Journal of Combinatorial Theory, Series B 97 (2007) 518–552
Journal of Combinatorial Theory Series B
www.elsevier.com/locate/jctb

Minmax relations for cyclically ordered digraphs ☆

András Sebő

CNRS, Laboratoire G-SCOP, Grenoble, France

Received 9 March 2005

Available online 22 November 2006

Abstract

We prove a range of minmax theorems about cycle packing and covering in digraphs whose vertices are *cyclically ordered*, a notion promoted by Bessy and Thomassé in their beautiful proof of the following conjecture of Gallai: *the vertices of a strongly connected digraph can be covered by at most as many cycles as the stability number*. The results presented here provide relations between cycle packing and covering and various objects in graphs such as stable sets, their unions, or feedback vertex- and arc-sets.

They contain the results of Bessy and Thomassé with simple algorithmic proofs, including polynomial algorithms for weighted variants, classical results on posets extending Greene and Kleitman's theorem (that in turn contains Dilworth's theorem), and a common generalization of these. The most general minmax results concern the *maximum number of vertices of a k-chromatic subgraph* ($k \in \mathbb{N}$)—as a consequence, this number is *greater than or equal to the minimum of* $|X| + k|\mathcal{C}|$, *running on subsets of vertices* X *and families of cycles* $\mathcal{C}$ *covering all vertices not in* X, *in strongly connected digraphs*. This is the "circuit cover" version of a conjecture of Linial (like Gallai's conjecture is the circuit cover version of the Gallai–Milgram theorem, meaning that path partitions are replaced by circuit covers under strong connectivity); we also deduce the circuit cover version of a conjecture of Berge on path partitions; these conjectures remain open, but the proven statements also bound the maximum size of a k-chromatic subgraph, contain Gallai's conjecture, and Bessy and Thomassé's theorems.

All presented minmax relations are proved using cyclic orders and a unique elementary argument based on network flows—algorithmically only shortest paths and potentials in conservative digraphs—varying the parameters of the network flow model. In this way antiblocking and blocking relations can be established, leading to a general polyhedral phenomenon—a combination of "integer decomposition, integer rounding" and "dual integrality"—that also contains the matroid partition theorem and Dilworth's theorem. We provide

☆ This research is part of the ADONET network's program (a Marie Curie training network of the European community), "milestone 3.1." It was completed when the author was a visitor at the Caesarea Edmond Benjamin de Rothschild Foundation Institute for Interdisciplinary Applications of Computer Science at the University of Haifa. The support and the exceptional hospitality of the Institute were a decisive factor in the successful termination of the work.

E-mail address: andras.sebo@imag.fr.

0095-8956/$ – see front matter
doi:10.1016/j.jctb.2006.09.005

a common reason for all these minmax equalities to hold and some possible other ones which satisfy the same abstract properties.

Keywords: Stable set; Coloring; Orders; Covering; Packing; Polyhedron; Network flow; Minimum cost flow

0. Introduction

Let $G = (V, A)$ be a directed graph (digraph), $V = V(G)$ is its vertex-set, $A = A(G)$ its edge-set, $n := |V|$. The vertices $x, y \in V$ are said to be adjacent if either $xy \in A$, or $yx \in A$. If $a = xy \in A$, then $a^{-1} := yx$ whether or not it is in A. For $B \subseteq A$, $B^{-1} := \{b^{-1}\colon b \in B\}$. A sequence of vertices is called a *walk* if $xy \in A$ for the successor y of x; it is a *closed walk* if the last vertex of the sequence coincides with the first; if every vertex occurs at most once, then we use the term *path* and *circuit*, respectively. The circuit C as a set also means the vertex-set $V(C)$ of C; the arc-set of C is denoted by $AC = A(C)$. If A is a finite set and $x \in \mathbb{R}^A$ we use the notation $x(B) := \sum_{v \in B} x_v$. For $X \subseteq V$, $\delta(X)$ denotes the set of arcs with exactly one endpoint in X, and $\delta^{\text{in}}(X)$, $\delta^{\text{out}}(X)$ the incoming and outgoing arcs among these; if $x \in V$ we write $\delta(x)$ for $\delta(\{x\})$. A graph is called strongly connected if for every $u, v \in V$ there exists a path from u to v.

A *stable set* in G is a set of pairwise nonadjacent vertices. The *stability* of G is the cardinality $\alpha(G)$ of a maximum stable set. Bessy and Thomassé [3,5] proved in a surprisingly simple and elegant way, using Dilworth's theorem, *Gallai's conjecture* relating α with circuit-covers:

Theorem 0.1. *The vertices of a strongly connected digraph can be covered with α circuits.*

They also give three exact minmax theorems between refinements of graph parameters concerning digraphs with an ordered set of vertices. In this paper we provide somewhat different simple proofs and proceed further with an insight from network flows and polyhedral combinatorics, mainly concerning the feedback of graphs. A *feedback* (*vertex-*)*set* (FS) is a set $U \subseteq V(G)$ where $G - U$ is acyclic. A *feedback arc-set* (FAS) is a set $F \subseteq A(G)$ where $G - F$ is acyclic.

We provide a full proof of Theorem 0.1 in a few lines in 0.3 of this introduction, using basic flows and total dual integrality. The latter will be replaced later on by min cost flows.

0.1. Cyclic orders and compatibility

A *cyclic order* on V arises from a *linear order* $v_1, \ldots, v_n$ of V with the additional relation that v_n is followed by v_1. (Linear orders are considered to be finite series where all elements of V occur exactly once.) Arcs of G can be considered going forward (or clockwise) in this order, and can be associated with the "corresponding interval of the circle between the tail and the head of the arc in clockwise direction"—the formal definitions ("length" or "winding") will correspond to this image. For "arcs of the circle" we rather use the term *interval* in the future to avoid confusion with the arcs $a \in A$ of the digraph $G = (V, A)$. With the clockwise convention, an *interval* $[u, v]$ where $u, v \in V$ is the set of vertices that follow u and precede v in the clockwise order, including u and v. Similarly, for linear orders, where we require that u precedes v. The *length* of an interval I, denoted by $\text{length}(I)$ is one less than the number of vertices contained in it, and the length of an arc $a = uv$ is $\text{length}(a) := \text{length}([u, v])$.

A cyclic order has n *openings*, described by two consecutive vertices in the cyclic order: the (u, v)-*opening* is the linear order whose first element is v, and it is followed by the elements of V in clockwise order with u as last element. (We "cut" between u and v.) In a cyclic order every element has a *successor* and a *predecessor* (in a linear order all but two of the elements have this), which we also imagine to be to the right and left, respectively. An element and its successor (predecessor) are called *consecutive*. Two openings of the same cyclic order are called *cyclic shifts* of one another.

A linear order has *forward* arcs ($v_i v_j, i < j$) and *backward* arcs ($v_j v_i, i < j$). In a (u, v)-opening, the arcs xy of G with $[xy] \supseteq [uv]$ become backward arcs. Given a digraph with a fixed cyclic order, the *winding* or index $\mathrm{ind}(C)$ of a circuit C of G (with respect to the fixed cyclic order) is

$$\mathrm{ind}(C) := \frac{\sum_{a \in AC} \mathrm{length}(a)}{n},$$

which is obviously an integer, and is "the number of tours made by the circuit." It is easy, but important to check, that *the winding of a circuit is equal to its number of backward arcs in any opening,* also called the winding of the opening. In particular, *cyclic shifts of linear orders have all the same number of backward arcs in every circuit*! *The invariance of the number of backward arcs of circuits through cyclic shifts* is the first essence of Bessy and Thomassé's solution of Gallai's conjecture, and it is further exploited in the present work. The winding will be equally used for cyclic and linear orders, but backward arcs exist only in linear orders.

Let us see now some examples. If $v_1, \ldots, v_n$ is a cyclic order and $1 \leqslant i < j \leqslant n$, $v_i v_j, v_j v_i \in A$, then $\mathrm{length}(v_i v_j) = j - i$ and $\mathrm{length}(v_j v_i) = i + n - j$, $\{v_i v_j, v_j v_i\}$ is a circuit, $\mathrm{ind}(\{v_i v_j, v_j v_i\}) = \frac{\mathrm{length}(v_i v_j)+\mathrm{length}(v_j v_i)}{n} = 1$. Another example: let $1, \ldots, 8$ be a cyclic order and $1, 3, 6, 2, 4, 5, 7, 8$ be a circuit. What is the winding of this circuit? It is

$$\frac{\mathrm{length}(13) + \mathrm{length}(36) + \cdots + \mathrm{length}(78) + \mathrm{length}(81)}{8} = \frac{2+3+4+2+1+2+1+1}{8} = 2.$$

This can be as well obtained by opening anywhere: the $(8, 1)$-opening has two backward arcs, 81 and 62; the backward arcs of the $(4, 5)$-opening are 36 and 45. These two sets of backward arcs are different, even disjoint, but they indeed have the same size.

Clearly, $\mathrm{ind}(C) \geqslant 1$ for every circuit C. If $\mathcal{C}$ is a set of circuits, $\mathrm{ind}(\mathcal{C}) := \sum_{C \in \mathcal{C}} \mathrm{ind}(C) \geqslant |\mathcal{C}|$.

A *forward path* in a linear order is a path with only forward arcs. We can also define forward paths with respect to a cyclic order: P is such a path, if it is a forward path in some opening. (Equivalently, if it goes 'forward' (clockwise) in some interval $[u, v]$, $u, v \in V$, of the representing circle.)

A second, essential property used by Bessy and Thomassé (even if this one turns out not to be necessary for proving Gallai's conjecture) is the *invariance of the winding through interchanging nonadjacent vertices that are consecutive in a given cyclic order*—this is also straightforward to check—motivating the following definition:

Two cyclic orders are *equivalent*, if one arises from the other by a sequence of *elementary changes*, that is, interchanges between nonadjacent vertices that are consecutive in the cyclic order. We also say that two linear orders are equivalent if they are openings of equivalent cyclic orders, that is, if they arise from one another by a sequence of elementary changes and cyclic shifts; the mentioned linear orders are as well said to be equivalent to the mentioned cyclic orders. Clearly, for all circuits C, $\mathrm{ind}(C)$ *is the same in any equivalent cyclic order*, and as we have already noticed, this number is *equal to the number of backward arcs in any opening of any equivalent cyclic order*.

These simple facts constitute the very heart of the results and are tools for converting straightforward results about flows into surprising results concerning the invariant $\mathrm{ind}(\mathcal{C})$ ($\mathcal{C}$ is a set of circuits). For an example, interchange first vertices 2 and 3 in the above introduced example, and then interchange 3 with 1, and 8 and 7 consecutively. Interchanges make possible some new changes: two equivalent orders can be far away from one another. However, equivalence can be characterized and decided in polynomial time [7].

The documents [3,5] present some elegant notions that lead to very original new results. In the rest of this introduction we introduce these slightly differently, and provide polyhedral interpretations that are important throughout the paper. As a first application a simple proof of Gallai's conjecture follows in a few lines.

Given $G = (V, A)$ we say that a linear order of V is *compatible* with G if every arc that is contained in a circuit is also contained in a circuit with exactly one backward arc, and arcs not contained in any circuit are forward arcs. A cyclic order is said to be *compatible* with G, if it has an opening which is compatible with G.

By the invariance of the winding, in a linear order compatible with G arcs that are contained in a circuit are contained in a circuit with exactly one backward arc *in every cyclic shift*; in particular, for strongly connected digraphs this is equivalent to *the compatibility of every opening*, called "coherence" in [5] defined with yet another equivalent definition: every arc is contained in a circuit of winding 1. (The generalization is not essential, but it is convenient that it also applies to digraphs that are not necessarily strongly connected.) We call a linear order *weakly compatible* if every backward arc is contained in a circuit of winding 1. A compatible order is always weakly compatible, but the converse is not true; a cyclic order of a strongly connected graph is compatible if and only if all of its cyclic shifts are weakly compatible.

Fixing a linear order $v_1, \ldots, v_n$ of V we say $v_i <_G v_j$ ($v_i, v_j \in V$) if v_j can be reached from v_i in the acyclic digraph consisting of the forward arcs of G. (Then in particular, $i < j$.) Clearly, $<_G$ is a partial order. (Antisymmetric, antireflexive and transitive. We say two vertices to be *comparable*, if they are in relation.)

The following statement makes clear that compatible orders generalize the role linear orders play for acyclic digraphs.

(1) *Let $G = (V, A)$ be a digraph given with a weakly compatible linear order. Then vertices adjacent in G are comparable in $<_G$.*

Indeed, suppose the given linear order $v_1, \ldots, v_n$ is weakly compatible with $G = (V, A)$. Let $v_i, v_j \in V$ be adjacent, and suppose without loss of generality $i < j$. If $v_i v_j \in A$, then we have $v_i <_G v_j$ immediately; if $v_j v_i \in A$, then it is a backward arc, and according to weak compatibility it is contained in a circuit C where it is the unique backward arc. Therefore there exists a forward path between v_i and v_j, and so again $v_i <_G v_j$.

Sets of pairwise $<_G$-incomparable vertices form of course a stable set in the acyclic digraph of forward arcs (we are used to this when working with posets) but the point of (1) is that they also form a stable set in G! That is, there is also no backward arc between them.

We state now a variant (without the strong connectivity constraint) of the most important result of Bessy and Thomassé's. Surprisingly, *every digraph can be represented in a compatible way.* This is a key-fact used throughout Sections 1, 2.

(2) *Every digraph has a compatible linear order*

We include a proof for the sake of completeness. This variant will be part of an algorithm for finding a feedback arc-set (FAS) with several relevant additional properties.

It is sufficient to prove the claim for strongly connected digraphs: applying it to the strongly connected components of a graph, these strongly connected components can then be listed in an order dictated by the acyclic digraph defined by the arcs between them.

Proof of (2). Let $G = (V, A)$ be strong, and F a FAS minimizing

$$\sum\bigl\{|C \cap F|\colon\ C \text{ is a directed circuit of } G\bigr\}.$$

Fix a linear order so that $G - F$ has only forward arcs, that is, all the backward arcs are in F. Since the backward arcs form a FAS, by its minimal choice, F is exactly the set of backward arcs. So $\operatorname{ind}(C) = |C \cap F|$ for every circuit C.

Now let $e \in A$ be arbitrary, and take a cyclic shift so that e is a backward arc. Let B be the set of backward arcs after the shift, $e \in B$. We show that B is a minimal FAS (inclusionwise). A FAS $B' \subset B$ has $|C \cap B'| \leqslant |C \cap B| = \operatorname{ind}(C) = |C \cap F|$ for any circuit C. The inequality is strict for circuits meeting $B \setminus B'$, and since G is strong, if $B' \neq B$ such a circuit exists, contradicting the choice of F. So B is indeed a minimal FAS.

Let C_e be a circuit of $G - (B \setminus \{e\})$. All arcs of $C_e \setminus \{e\}$ are forward arcs: $e \in C_e$, $\operatorname{ind}(C_e) = 1$. □

It is easy to state an explicit polynomial algorithm from this proof: determine an inclusionwise minimal FAS F, and construct the linear order compatible with $G - F$; if for any cyclic shift of this order the set of backward arcs B is not inclusionwise minimal, take a proper subset B' which is still a FAS, and start the whole procedure anew with $F = B'$; the proof shows that this algorithm cannot cycle, and therefore we end up in finite time with a minimal FAS so that the sets of backward arcs of all cyclic shifts are also minimal FAS, a property that characterizes compatible orders according to the proof. We still have to check that the algorithm stops in polynomial time:

For $a \in A$ define $\operatorname{ind}(a) := 0$ if a is not contained in a circuit, and otherwise

$$\operatorname{ind}(a) := \min\bigl\{\operatorname{ind}(C)\colon\ C \text{ is a circuit, } a \in C\bigr\}.$$

Consider the sum $\Sigma := \sum_{a \in A} \operatorname{ind}(a)$. The proof establishes that at the end of the algorithm $\operatorname{ind}(a) = 1$ for all $a \in A$. In the beginning, $\operatorname{ind}(a) \leqslant n$ (for all $a \in A$). The main observation is that whenever we replace B by a proper subset B', $\operatorname{ind}(a)$ decreases by at least 1 for all $a \in B \setminus B' \neq \emptyset$, that is, for at least one arc by at least 1. In every iteration before the algorithm stops, we replace the actual B by a proper subset B'. Therefore the algorithm decreases from at most $n|A|$ to $|A|$, and by at least 1 in every iteration; therefore there are no more than $(n-1)|A|$ iterations.

The choice of F is inspired by the original proof [5]. A further study of these leads to a characterization and polynomial recognition of the equivalence of cyclic orders [7].

0.2. Cyclic stable sets

Given a digraph $G = (V, A)$ with a cyclic order, $S \subseteq V$ is called a *cyclic stable set* if S is a stable set, and it forms an interval in some equivalent cyclic order.

Note that the equivalence relation of cyclic orders does not require the digraph to be strongly connected or the cyclic order to be compatible! This is not a requirement in the rest of the paper; it will only be the condition of some corollaries (even if these are the most important ones).

The characteristic vector of $X \subseteq V$ is $\chi_X \in \{0,1\}^V$ with $\chi_X(x) = 1$ if and only if $x \in X$. For $x \in \mathbb{R}^V$ and $U \subseteq V$ we use the usual notation $x(U) := \sum_{u \in U} x_u$.

We are now able to state and provide a first proof of the weighted minmax theorems and the polyhedra implicit in Bessy and Thomassé's results [3,5]:

Given the digraph G and $w : V(G) \to \mathbb{N}$ a family of circuits having for all $v \in V$ at least $w(v)$ members that contain v is called a *w-cover*. If there are *at most* $w(v)$ members that contain v, then it is a w-packing. A 1-cover is called a *cover*, a 1-packing a *packing*. (That is, a packing is a set of pairwise disjoint circuits.) We define *arc-w-packings* and *arc-w-covers* similarly, for functions $w : A(G) \to \mathbb{N}$ by replacing vertices by arcs in the definition.

Theorem 0.2. *Let $G = (V, A)$ be a strongly connected digraph given with a compatible cyclic order, and $w : V(G) \to \mathbb{N}$. Then* $\max\{w(S)\colon\ S \text{ is a cyclic stable set}\} = \min\{\mathrm{ind}(\mathcal{C})\colon\ \mathcal{C} \text{ is a } w\text{-cover}\}$.

Bessy and Thomassé's theorem is the special case of Theorem 0.2 when $w(v) = 1$ for all $v \in V$. Conversely, it is also straightforward to prove it from the cardinality special case by *replication* of $w(v)$ nonadjacent vertices (that is, add $w(v) - 1$ copies, each adjacent to the same vertices as v and not joined among themselves) for each $v \in V(G)$. However, such a replication does not keep polynomial solvability of the related problems, since the size of the replicated digraph is not necessarily a polynomial of the input size.

This theorem is the key to Gallai's conjecture. It is shown in Sections 1, 2 that (1), (2) and standard network flow tools are sufficient for proving Bessy and Thomassé's results and several variants and extensions, algorithmically, and in a natural way, for arbitrary weights. (Note though that the original proof could also provide an algorithm for the weighted case with some work: that proof uses Dilworth's theorem which in turn can be reduced to maximum matchings, and these reductions do have weighted generalizations ....)

Given a graph $G = (V, A)$ with a (not necessarily compatible) cyclic order we define two polyhedra formed by set of all $x \in \mathbb{R}^V$ satisfying either of the following systems of inequalities:

$$x(C) \leqslant \mathrm{ind}(C), \quad \text{for every circuit } C,\ x \geqslant 0, \tag{BT}$$

$$x(S) \leqslant 1, \quad \text{for any cyclic stable set } S,\ x \geqslant 0. \tag{AntiBT}$$

We make no a priori assumptions for defining these polyhedra, they make sense without any condition. Note, however, that maximizing $w^T x$ over (BT), where $w : V \to \mathbb{R}$ does not have a bounded optimum if $v \in V$ is not contained in any circuit, and $w(v) > 0$. On the other hand, if such a vertex does not exist then both the primal and the dual problem are feasible, and have bounded optima. Digraphs with all vertices contained in some circuit are going to be called *simplified*. Every digraph becomes simplified by adding loops to vertices. We allow loops (their winding number is 1) and parallel edges (with inverse orientation as well).

These two polyhedra are in "antiblocking relation" (according to Theorem 0.2, see [10,15]). We do not treat here the definition or the basic results about blocking and antiblocking, since they will be used only in remarks. The results we need follow directly via network flows.

The dual of the linear program $\max w^T x$ over (BT) is:

$$\text{minimize} \sum_{C \in \mathcal{C}} y_C \,\mathrm{ind}(C) \text{ subject to } \sum_{C\colon\, v \in C \in \mathcal{C}} y_C \geqslant w(v)$$
$$\text{for every } v \in V, \text{ and } y_C \geqslant 0 \text{ for every circuit } C. \tag{BTdual}$$

Informally, abusing the notation, given a digraph with a cyclic order we also denote (BT), (AntiBT), (BTdual) the polyhedron consisting of the solutions of the corresponding system of inequalities.

For an illustration of the network flow method that is repeatedly used in the present work we include first a simple algorithmic proof of the two stated theorems.

0.3. Simple proof of Gallai's conjecture

We provide now a short proof of Theorem 0.1. The proof is assuming basic knowledge of combinatorial optimization; for a self-contained more elementary and algorithmic proof using only network flows, see next section. An even shorter non-algorithmic proof referring to total unimodularity and linear programming can be found in [7].

A system of linear inequalities is said to be *totally dual integral*, shortly *TDI*, Edmonds–Giles [8] and Schrijver [15].

Proof of Theorem 0.1. Let $G = (V, A)$ be a strongly connected digraph. According to (2) G has a compatible cyclic order, fix such an order. For arbitrary $w : V \to \mathbb{N}$ consider the linear program of maximizing $w^T x$ under (BT). This optimization problem can be reduced—by the standard gadget of splitting each vertex v into an in- and out-copy v_{in} and v_{out} (in this order), putting a (forward) arc e_v between the two with lower capacity $w(v)$, and for every arc $uv \in A$ putting an arc $u_{\text{out}}v_{\text{in}}$ of weight 1 if uv is a backward arc and of weight 0 otherwise—to minimum cost circulations. Therefore (BT) is TDI, and has integer vertices according to the Edmonds, Giles' theorem.

Let x maximize $w^T x$ under (BT). We show that x *is a* 0–1 *vector, moreover the characteristic vector of a stable set*:

If (BT) is bounded, then x exists, and—as noticed above—it is integer.

Let $v \in V$. Since G is strongly connected, there exists an arc $uv \in A$ incident to v; by compatibility, uv is contained in a circuit C_{uv} with $\text{ind}(C) = 1$, whence $x_u + x_v \leqslant x(C_{uv}) \leqslant \text{ind}(C) = 1$; in particular, since x is a nonnegative integer vector, $x_v \in \{0, 1\}$ for all $v \in V$. Similarly, for every arc $ab \in A$: $x_a + x_b \leqslant x(C_{ab}) \leqslant 1$ (where C_{ab} is a circuit of winding 1 containing ab), so x is the characteristic vector of a stable set.

So $x(V) \leqslant \alpha$. On the other hand, by the duality theorem of linear programming there exists a dual optimum of (BT) for the all 1 objective function of value $x(V)$, and since (BT) is TDI there is also an integer dual solution (circulation) with this objective value. But such an integer dual solution is a circuit cover of V with $x(V)$ ($\leqslant \alpha$) circuits. Gallai's conjecture is proved. □

In Section 1 the integer primal and dual solutions to (BT) will be explicitly constructed—in polynomial time, and actually in a more general context avoiding the Edmonds–Giles theorem using mincost flows for the dual of the minimization problem on (BT) and the "potentials" dual to this problem for the minimization problem on (BT). Substituting this solution to the proof, total dual integrality can be completely avoided).

We also noticed primal and dual integrality of (BT), and therefore most of Theorem 0.2 on the way. Our debt is that integer primal solutions are *cyclic* stable sets. We will get four different proofs of this fact in Sections 1 and 2.

The invariance of the winding can be combined with network flows in multiple ways for getting tractable variants of NP-hard problems; there are also many choices for the parameters of the flow model and some of them lead to interesting new problems. For instance putting upper capacities in the above sketched proof we have the TDI-ness of a similar blocking relation;

putting both upper and lower capacities that are equal, corresponds to equality constraints in the dual problem (for instance to partitioning into circuits), that is, to deleting the nonnegativity constraints in the primal, etc.

In this paper we wish to work out aspects concerning (BT) (Section 1) and (AntiBT) (Section 2), and exploit the beautiful integrality properties of these two (Section 3), which connect circuits, stable sets and colorings, and provide a deeper common reason for Greene–Kleitman type theorems [9,12], a minmax form of the matroid partition theorem [15], and maybe some other similar minmax relations to hold.

In this way sets of circuits and stable sets of digraphs are drawn into the validity circle of Greene and Kleitman's theorem, as well as independent sets of matroids. Theorems slightly different of well-known conjectures of Berge [2] and Linial [14] can be proved (Section 5). For an up to date survey of related results, conjectures and methods see [13].

1. Circuit covers and stable sets: Solving (BT)

In this section we provide a full and self-contained algorithmic proof of Theorem 0.2 and of Theorem 0.1 (Gallai's conjecture) via network flows, and establish the antiblocking relation between circuits and cyclic stable sets along with the related combinatorial observations and polynomial algorithms.

Given a digraph $G = (V, A)$, a *flow* (or circulation) is a function $f : A \to \mathbb{R}^+$ such that $f(\delta^{\text{in}}(x)) = f(\delta^{\text{out}}(x))$ for all $x \in V$. Note that we do not allow negative flow values.

We state the well-known (and easy to prove) optimality criteria for minimum cost flows (see for instance [15, Theorem 12.1, p. 178]):

(3) *Suppose $G = (V, A)$ is a digraph with lower and upper capacities $l, u : A \to \mathbb{N} \cup \{0\}$ and costs $c : A \to \mathbb{R}$. Then the feasible circulation $f : A \to \mathbb{N} \cup \{0\}$ ($l \leqslant f \leqslant u$) is optimal for (G, l, u, c) if and only if the following graph $G' = (V, (A \setminus A'') \cup A')$ with costs c' has no negative circuit*:

$$A' := \{e^{-1} \colon e \in A,\ f(e) > l(e)\}, \qquad A'' := \{e \in A \colon f(e) = u(e)\},$$
$$c'(e) := c(e) \quad \text{if } e \in A \setminus A'', \qquad c'(e) = -c(e) \quad \text{if } e \in A'.$$

The nonexistence of a negative circuit in a digraph $H = (V, A)$ given with costs $d : A \to \mathbb{R}$ can be certified with a *potential*, that is, a function $\pi : V(H) \to \mathbb{R}$ on the vertices with the property that

$$\text{for every arc } uv, \quad \pi(v) - \pi(u) \leqslant d(uv). \tag{pot}$$

It is easy to see that the distances from a fixed vertex of H form a potential, implying that there exists a potential if and only if there is no negative circuit (see [15, Theorem 8.2]), and that this potential *is integer-valued if all the costs are integer numbers.*

Note that finding a potential or a negative circuit can be achieved in polynomial time. If the data are small, improving along an arbitrary sequence of circuits is sufficient for finding the optimum—and there also exist (strongly) polynomial algorithms for arbitrary weights. (For details the reader is sent to [15, Subsections 12.2, 12.3].) These algorithms also find potentials, but the last sentence of (3) can replace this remark: such a potential can also be the result of an algorithm that finds shortest paths in a digraph without negative circuits.

The proofs of the theorems will all be based on network flows and potentials. Therefore we do not necessarily mention one by one the polynomial time bound on all the computations. All quantities that are mentioned in the sequel—including the optimal solutions to linear programs—can be computed in polynomial time, bounded by the complexity of the used network flow algorithms.

Let $G=(V,A)$ be a digraph with a given (not necessarily compatible) cyclic order $v_1,\dots,v_n$ fixed throughout this section. We first prove the following statement which has a weaker conclusion than Theorem 0.2 in that it does not tell the combinatorial properties of primal solutions of (BT), but the condition of it is also weaker (compatibility is not assumed).

(4) *If $G=(V,A)$ is a simplified digraph given with a cyclic order, then for any integer objective function $w: V(G)\to\mathbb{Z}_+$, the linear program*

$$\max\{w^T x:\ x \text{ satisfies the inequalities (BT)}\}$$

has integer primal and dual optima, and they are equal.

As mentioned in the introduction, from the integrality of the dual solution for all w the primal integrality follows by results of Edmonds and Giles concerning linear inequalities, and by the duality theorem the two are equal. In this sense (4) is redundant: the only essential statement is the integrality of the dual solution. However, we do not want to refer to Edmonds and Giles or the duality theorem or to any result in integer or linear programming at all. We provide a direct algorithmic and combinatorial proof of all the assertions:

Open the given cyclic order arbitrarily, fix the resulting linear order and let the cost of backward arcs e be $c(e):=1$, and the cost of all other arcs be 0.

For all $v\in V$ introduce two vertices, v_{in} and v_{out}, and let $\hat G=(\hat V,\hat A)$,

$$\hat V := \bigcup_{v\in V}\{v_{\text{in}}, v_{\text{out}}\},\qquad \hat A := \{x_{\text{out}}y_{\text{in}}:\ xy\in A\}\cup\{e_v:\ e_v := v_{\text{in}}v_{\text{out}},\ v\in V\}.$$

Define capacity- and cost-functions on $\hat G$:

- Do not define upper capacities (u is identically ∞).
- Define a lower capacity-function $l:\hat A\to\mathbb{Z}_+$ as follows: for the arcs e_v $(v\in V)$ let $l(e_v):=w(v)$, and let l be 0 otherwise.
- Define $c(e)$ for $e=x_{\text{out}}y_{\text{in}}$: $xy\in A$ to be $c(xy)$ (that is, 1 for backward arcs xy and 0 otherwise).

We now use Hoffman's circulation theorem. (See several variants of this theorem in Chapter 11 of [15], and for algorithmic aspects Section 12.) As shown in [7], for a simple proof of Gallai's conjecture it is sufficient to use the integrality of a linear program [11], a consequence of the total unimodularity of the associated matrix.

Since the network is strongly connected and the upper capacities are infinite, there exists a feasible flow, and there exists a minimum cost flow which is integer.

Let f be an integer minimum cost circulation in $(\hat G,l,u,c)$; it is well known and easy to see that f is a nonnegative integer combination of circuits (see [15, (11.3)]). Instead of nonnegative integer combinations, we speak about the multiset $\mathcal{C}$ of circuits (the elements are circuits, and each circuit can have any integer multiplicity) whose sum is the mincost flow f.

Clearly, a circulation with lower capacities given by l provides *a dual solution of* (BT), *and by the correspondence of the winding and the number of backward arcs in openings, the cost of this flow is equal to the value of the dual solution*:

$$\sum_{e\in A} f(e)c(e) = \sum_{e\in A}\biggl(\sum_{C\in\mathcal{C}}\chi_C\biggr)(e)c(e) = \sum_{C\in\mathcal{C}}\sum_{e\in AC} c(e) = \operatorname{ind}(\mathcal{C}).$$

On the other hand, any potential $\pi : \hat{V} \to \mathbb{Z}$ (see (pot)), yields directly a primal solution for (BT) of the same value as the dual solution:

Now associate to the circulation f in the digraph $\hat{G}$ with lower capacities l and costs c the auxiliary digraph $H = (U, F)$ (depending on f, l, c) in the way defined by (3):

$U := \hat{V}$, $F = \hat{E} \cup \hat{E}'$, where

$$\hat{E}' := \{e^{-1}\colon e \in \hat{E} \setminus \{e_v\colon v \in V\},\ f(e) > 0\} \cup \{e_v^{-1}\colon v \in V,\ f(e_v) > w_v\}.$$

Define $c'(e) := c(e)$ if $e \in \hat{E}$, $c'(e) := -c(e)$ if $e \in \hat{E}'$, that is, -1 on inverted backward arcs provided there is more than 1 circuit containing the backward arc, and 1 on the backward arcs themselves; 0 on all other arcs of H.

By (3) there is no negative circuit in this auxiliary digraph, therefore there exists an integer potential π: define

$$x_v := \pi(v_{\text{in}}) - \pi(v_{\text{out}}) \quad \text{for all } v \in V.$$

We show that $x := (x_v)_{v \in V}$ satisfies (BT), and $\sum_{v \in V} w_v x_v$ is equal to the cost of f.

First, let us check $x_v \geqslant 0$. Indeed, by (pot) $\pi(v_{\text{out}}) - \pi(v_{\text{in}}) \leqslant c'(e_v) = c(e_v) = 0$.

In order to prove the other inequalities of (BT), let C be an arbitrary circuit in G.

$$\begin{aligned}\sum_{u \in C} x_u &= \sum_{u \in C} \big(\pi(u_{\text{in}}) - \pi(u_{\text{out}})\big) = \sum_{uv \in AC} \big(\pi(v_{\text{in}}) - \pi(u_{\text{out}})\big) \leqslant \sum_{uv \in AC} c'(u_{\text{out}} v_{\text{in}}) \\ &= c(C) = \text{ind}(C),\end{aligned} \tag{$*$}$$

and for $C \in \mathcal{C}$ there is equality here, because $C \in \mathcal{C}$ implies $f(u_{\text{out}} v_{\text{in}}) > 0$ for all $uv \in AC$, and then $\pi(v_{\text{in}}) - \pi(u_{\text{out}}) = c'(u_{\text{out}} v_{\text{in}})$ for all $uv \in AC$.

Last, let us check that the defined primal objective value $\sum_{v \in V} w_v x_v$ is the same as the dual objective value $\text{ind}(\mathcal{C})$ (see (BTdual)). According to $(*)$:

$$\text{ind}(\mathcal{C}) = \sum_{C \in \mathcal{C}} \text{ind}(C) \geqslant \sum_{C \in \mathcal{C}} \sum_{u \in C} x_u = \sum_{v \in V} \bigg(\sum_{C \in \mathcal{C}} \chi_C\bigg)(v) x_v.$$

Since $\sum_{C \in \mathcal{C}} \chi_C(v) \geqslant w_v$ $(v \in V)$—see (BTdual)—it is now sufficient to note that for all $v \in V$ with strict inequality we have $x_v = 0$. Indeed, if the strict inequality holds than both e_v and e_v^{-1} are present in $\hat{G}$ and both with weight 0, implying $\pi(v_{\text{in}}) = \pi(v_{\text{out}})$.

Let us also note that we did not use the duality theorem, nor anything we know about duality in flows, only the existence of an integer circulation! (Feasibility is obvious here.) The difference between (4) and Theorem 0.2 is exactly the following assertion:

Theorem 1.1. *Suppose $G = (V, A)$ is strongly connected and is given with a compatible cyclic order. Then x is an integer solution of* (BT) *if and only if $x = \chi_S$, where S is a cyclic stable set.*

Proof of Theorem 0.2 using Theorem 1.1. Let $w\colon V(G) \to \mathbb{N}$. By (4) the dual optimum of (BT) is the minimum of $\text{ind}(\mathcal{C})$ among w-covers $\mathcal{C}$. Again by (4) the primal optimum is also integer. Theorem 0.2 follows now if we also prove that the primal solutions are the cyclic stable sets, which is exactly Theorem 1.1. □

In the proof of Theorem 1.1 we use only a very restricted and easy part of the following lemma that is intended to clarify what (BT) means (it shows how it can be generalized and specialized) but we will need all the equivalences later.

The *arc-multiplicity* vector of a walk is the vector in $\mathbb{N}^A$ that tells how many times the walk goes through the arcs. The *vertex-multiplicity* vector of a closed walk can be defined as $z(v) := m(\delta(v))/2$ $(= m(\delta^{\text{in}}(v)) = m(\delta^{\text{out}}(v)))$ $(v \in V)$, where m is the arc-multiplicity vector. A closed walk can be decomposed into circuits, and the arc- and vertex-multiplicity vectors are the sums of the multiplicity vectors of these circuits (and for these the multiplicity vector and the characteristic vector coincide).

Lemma 1.1. *Let $G = (V, A)$ be a simplified digraph given with a compatible cyclic order. Then the following statements are equivalent*:

(i) $x \in \mathbb{Z}^n$ *is a solution of* (BT).
(ii) $x = \chi_S$, *where* $|S \cap C| \leqslant \operatorname{ind}(C)$ *for every circuit C of G.*
(iii) $x = \chi_S$, *and for some* (*equivalently, for any*) *opening of the cyclic order, every closed walk of G meets at most as many vertices in S* (*with multiplicity*) *as it meets of backward arcs* (*also with multiplicity*).

Proof. Suppose $x \in \mathbb{Z}^n$ is a solution of (BT). Since G is simplified every $v \in V$ is contained in a circuit, and an arc incident to v which is contained in a circuit is also contained in a circuit winding once. Therefore, v is contained in a circuit winding once and it follows that $x \in \{0, 1\}^n$ (as before). Let $S := \{v \in V\colon\ x(v) = 1\}$. Now assertion (ii) is just an obvious rewriting of (BT).

To prove (iii) from (ii) let W be a closed walk and note that the difference on W between the multiplicities of its backward arcs and the multiplicities of its vertices that are in S is equal to the sum of these differences taken on the circuits decomposing W. However, (ii) states that this difference is nonnegative on circuits and therefore it is so on W too, proving (iii).

Finally, (iii) implies (i) because (i) is the specialization of (iii) to circuits. (Recall that the number of backward arcs of a circuit in an opening of a cyclic order is equal to the winding of the circuit.) □

Proof of Theorem 1.1. We have three different proofs of this theorem:

- The statement follows for free from the results of Section 2. Indeed, x is an integer solution of (BT) if and only if it is an integer vector in the antiblocking polyhedron of the set $\{\chi_C/\operatorname{ind}(C)\colon\ C \text{ is a circuit}\}$, and Corollary 1.1 states that such vectors are characteristic vectors of the coefficient vectors in (AntiBT), that is, of cyclic stable sets. (For this proof you need to know the main facts about antiblocking [10] which we do not want to count on here.)
- The statement is also implicit in Bessy and Thomassé's theorems since with replication one gets the weighted versions that establish the antiblocking relation between circuits normalized by their winding, and cyclic stable sets. So the original proofs of any of the two main results of [5] also lead to a proof of Theorem 1.1.
- According to (i) ⇒ (ii) of Lemma 1.1 the statement is a consequence of the combinatorial reformulation (5) below that we will check directly. □

This theorem is less evident than analogous claims for other integer linear programs arising from graphs. For instance a vertex-set of a graph is a stable set if and only if it is an integer vector satisfying the edge inequalities—and this is an obvious statement—the questions concern only integrality of the relaxation (which are easy for us through network flows). Similar obvious statements hold for most formulations of problems in NP as integer linear programs: such a formulation is usually easy for NP-hard problems as well (for instance the travelling salesman problem), so we are surprised to need here a longer proof. The optimization problem on (BT) or on its antiblocker are polynomially solvable, yet the usually obvious statement, equivalent to Theorem 1.1, is not straightforward here:

(5) *Suppose $G = (V, E)$ is a strongly connected digraph given with a compatible cyclic order, and $S \subseteq V$. Then S is a cyclic stable set if and only if for every circuit C:*

$$|S \cap C| \leqslant \operatorname{ind}(C).$$

We repeat: the reader can skip the proof of this statement, it will anyway follow from results of the next section (see the above proof). However, a direct combinatorial proof might be a luxury that clarifies this important fact:

Let first $S \subseteq V$ be a cyclic stable set and check the easy 'only if' part of the statement like in [5]: S satisfies the condition. Indeed, it is sufficient to prove the claim for the equivalent cyclic order where S is an interval, since the winding of every circuit remains unchanged through equivalent orders. Consider the opening of the cyclic order where the first elements are those of S. We have to prove that any circuit C has at least $|S \cap C|$ backward arcs. Indeed, for each $v \in S \cap C$ the arc of C that enters v is a backward arc, and for any two different vertices the two entering arcs are also different, proving the only if part of the statement.

To prove the 'if' part of (5), note that (i) of Lemma 1.1 is the condition of (5). So according to Lemma 1.1 we can suppose that S satisfies (iii).

We prove by induction on $|S|$ that S is a cyclic stable set. For $|S| = 1$ this is obvious. Let $t \in S$ be arbitrary, and apply the induction hypothesis to $S \setminus \{t\}$: it is a cyclic stable set, and therefore there is an opening of the given cyclic order starting with $S \setminus \{t\}$. Denote by T_1 the part of this linear order between $S \setminus \{t\}$ and t, and T_2 the part that follows t.

Define the following auxiliary digraph $H = (S, B)$: we join $x, y \in S$ with an arc $(x, y) \in B$ if and only if in G with linear order (S, T_1, t, T_2)

- either there exists an (x, y)-path with exactly one backward arc and not containing t,
- or $t \in \{x, y\}$: either $x = t$ and there exists a (t, y)-path with exactly one backward arc; $y = t$ and there exists an (x, t) forward path:

$$S_1 := \bigl\{s \in S \setminus \{t\}\colon t \text{ can be reached from } s \text{ in } H\bigr\}, \qquad S_2 := S \setminus S_1.$$

Claim. *The linear order (S_2, T_1, t, T_2, S_1) is equivalent to the given one, and there is no forward path in G with this order between any two elements of S (that is, they are incomparable in $<_G$).*

This claim finishes the proof by Lemma 1.2 below.

Clearly, (S_1, S_2, T_1, t, T_2) is also an opening of a cyclic order equivalent to the given one. (Any permutation of a cyclic stable set yields equivalent cyclic orders. With an abuse of notation, S_1 and S_2 also denote these sets in any order.) Now (S_2, T_1, t, T_2, S_1) is just a shift of this order (another opening of the given cyclic order) and therefore its equivalence with the given cyclic order is clear.

We first prove that *there exists no circuit in H that contains t*. Indeed, suppose C is such a circuit of size p, and let the walk W consist of the concatenation of the paths $P_{(x,y)}$, $(x, y) \in AC$. The walk W goes through the p vertices of S that are in C, on the other hand, each $P_{(x,y)}$, $(x, y) \in C$, contains one backward arc except the one with $y = t$, which has no backward arc:

W contains exactly $p-1$ backward arcs, contradicting (iii), and proving that H has no circuit containing t.

Therefore there exists no path from t to S_1 in H, which means that there is no forward path in G with the order (S_2, T_1, t, T_2, S_1) from t to S_1. In this order there is clearly no forward path from S_1 to S_1 or from S_2 to S_2 either; furthermore there is no forward path from S_2 to $S_1 \cup \{t\}$ in G, because that would imply an arc from S_2 to $S_1 \cup \{t\}$ in H, contradicting the definition of S_2. We checked all possibilities for the starting and end-point of a forward path in G with the linear order (S_2, T_1, t, T_2, S_1), and conclude that there is no such path, finishing the proof of the claim and of (5).

We used the following simple but basic lemma of Bessy and Thomassé's which concerns only linear orders. We include a proof for completeness:

Lemma 1.2. *Given $G = (V, A)$ and a compatible linear order of V, the elements of a set $S \subseteq V$ are pairwise incomparable by $<_G$ if and only if S is stable and with elementary changes (without cyclic shift) it can be brought to an interval.*

Indeed, if S is stable and there exists an equivalent linear order where S is an interval, then the elements of S are pairwise incomparable, because the order of the two endpoints of any arc remains the same through elementary changes (and therefore the set of forward arcs does not change), on the other hand, any path between two points of S must contain a backward arc.

Conversely (and this is the essential part that we will use several times) if the elements of S are pairwise incomparable, then first by (1) they form a stable set. Second, take an equivalent linear order where the first and last element of S are closest possible. If, for a contradiction, $v \in V \setminus S$ is between the first and last element of S, let S_1 be the set of elements of S that are before v, and S_2 the set of those that are after. Since S is incomparable, either there is no forward path from S_1 to v, or no forward path from v to S_2.

Suppose for instance the first holds, let S_1' be the set of vertices reachable from S_1 with a forward path. Then v is nonadjacent with all elements of S_1' that are preceding it, because they clearly do not send an arc to v, and if v would send an arc to them, then this arc would be a backward arc and by compatibility there would be a forward path from S_1' to v and therefore also from S_1 to v, contradicting the fact that there is no such path.

So $v \notin S$, $v \notin S_1'$, and v is to the right of S_1'. Supposing that v is the first (leftmost) such element of V, it can be moved to the left of all elements of S_1, whence to the left of all elements of S, contradicting the minimality of the number of vertices between the first and last element of S.

We restate Theorem 1.1 for readers who know about antiblocking pairs of sets of vectors:

Corollary 1.1. *Let G be strongly connected and endowed with a compatible cyclic order. The set of vectors $\{\chi_C / \operatorname{ind}(C)\colon\ C$ is a circuit$\}$ and $\{\chi_S\colon\ S$ is a cyclic stable set$\}$ is an antiblocking pair.*

We finish this section with two additional remarks that we will summarize in a new theorem. Again, the intuition is provided by linear programming, but in the proofs we can get rid of it:

First, note that deleting the nonnegativity constraints the only change in the dual solution is that we have to write equalities instead of inequalities. Therefore, just by a single modification in the proof of (4), we get that the minimum of $\operatorname{ind}(\mathcal{C})$ among partitions into circuits can also be determined: indeed, on the arcs e_v $(v \in V)$ put both lower and upper capacities equal to w_v. All the proof works, except that $x_v \geqslant 0$ will not hold any more which corresponds exactly to the deletion of the nonnegativity constraints as we know it from linear programming.

The proof of (4) also informs us exactly of what we have to do to encode lower or upper bounds on the variables: these will correspond to costs on the arcs parallel or inverse to e_v. And actually we can multiply the polytope by an arbitrary constant $k \in \mathbb{N}$:

$$x(C) \leqslant k\,\mathrm{ind}(C), \quad \text{for every circuit } C \text{ of } G. \tag{kBT}$$

The following theorem enables us to handle all this at the same time, and with slight modifications also blocking or partitioning variants of the results:

Theorem 1.2. *If $G = (V, A)$ is a simplified digraph given with a cyclic order, then for any integer objective function $w : V(G) \to \mathbb{Z}_+$, $a, b : V(G) \to \mathbb{N}$ $(a \leqslant b)$, and $k \in \mathbb{N}$ the linear program*

$$\max\bigl\{w^T x\colon\ x \text{ satisfies the inequalities (kBT)},\ a \leqslant x \leqslant b\bigr\}$$

has integer primal and dual optima, and they are equal.

This theorem extends (4) and its content means exactly that (kBT) is "box TDI" for any $k \in \mathbb{N}$. Our longer but elementary formulation is better adapted to our problem and is sharper in this case. We give simple separate proofs to primal and dual integrality, without reference to linear programming:

Proof. The proof is a straightforward generalization of the proof of (4) but we repeat it for the sake of clarity, and in the beginning with full details.

Open the given cyclic order arbitrarily, fix the resulting linear order and let the cost of backward arcs e be $c(e) := k$, and the cost of all other arcs be 0.

For all $v \in V$ introduce two vertices, v_{in} and v_{out}, and let $\hat{G} = (\hat{V}, \hat{A})$,

$$\hat{V} := \bigcup_{v \in V} \{v_{\mathrm{in}}, v_{\mathrm{out}}\},$$

$$\hat{A} := \{x_{\mathrm{out}} y_{\mathrm{in}}\colon\ xy \in A\} \cup \{e_v\colon\ e_v := v_{\mathrm{in}} v_{\mathrm{out}},\ v \in V\} \cup \{\overrightarrow{e_v} := v_{\mathrm{in}} v_{\mathrm{out}},\ v \in V\}$$
$$\cup \{\overleftarrow{e_v} := v_{\mathrm{out}} v_{\mathrm{in}},\ v \in V\},$$

that is, e_v has a parallel and an inverted copy.

Define capacity- and cost-functions on $\hat{G}$:

- Define a lower and upper capacity-function $l, u : \hat{A} \to \mathbb{Z}_+$ as follows: for the arcs e_v $(v \in V)$ let $l(e_v) := u(e_v) := w(v)$, and let otherwise $l := 0$, $u := \infty$.
- Define $c(e)$ for $e \in \hat{A}$, $e = x_{\mathrm{out}} y_{\mathrm{in}}$ $(xy \in A)$ to be $c(xy)$ (that is, k if xy is a backward arc and 0 otherwise).
- Define $c(e_v) := 0$, $c(\overrightarrow{e_v}) = -a_v$, $c(\overleftarrow{e_v}) = b_v$.

Now we use that there exists an integer flow (circulation) which can again be deduced from basic flow theory (for instance from Hoffman's circulations). Let f be an integer minimum cost flow with this data which again is a multiset $\mathcal{C}$ of circuits (the elements are circuits, and each circuit can have any integer multiplicity) providing a dual optimum in exactly the same way as for (4).

Again, the auxiliary digraph associated with the optimal flow f can be determined and (3) can be applied: a potential π exists in the auxiliary digraph. Define again

$$x_v := \pi(v_{\mathrm{in}}) - \pi(v_{\mathrm{out}}) \quad \text{for all } v \in V.$$

This satisfies (kBT) for the same reason as the corresponding part of the proof of (4), and it is straightforward to check that it satisfies the upper and lower bounds. □

2. Colorings and maximum circuits: Solving (AntiBT)

According to Theorem 1.1 the antiblocking polyhedron of (BT) is (AntiBT). (As already mentioned, blocking and antiblocking are not defined here, their role is only motivation. Some references for the interested reader for the references and basic results: [10,15].) However, we proved much more for (BT): total dual integrality (see Theorem 0.2). A similar surplus holds for (AntiBT). Despite the fact that the vertices of (AntiBT) are the noninteger vectors $\{\chi_C/\operatorname{ind}(C)\colon\ C$ is a circuit$\}$ and therefore no true integrality result can be hoped for, we will prove the most that can be expected: *for any objective function there exists an integer dual solution of* (AntiBT) *with an objective value equal to the optimum rounded up*. This is called the *integer rounding property* [1,15] of (AntiBT). It is shown in these that this property is equivalent to the following *integer decomposition property* of the antiblocking set.

Let us define the integer decomposition property for (BT): for any integer vector x in k times (BT), that is, in (kBT)—the convex hull of the sums of k cyclic stable sets—there exist k *integer* vectors $x_1, \dots, x_k$ in (BT) such that $x = x_1 + \cdots + x_k$. We also say that cyclic stable sets have the integer decomposition property. Let us check that the integer rounding property of (AntiBT) implies the integer decomposition property of cyclic stable sets: if $w \in$ (kBT), then w can be written as the (fractional) sum of stable sets where the sum of coefficients is at most k, that is, with objective function w, (AntiBT) has a dual solution of value at most k. So the dual optimum of (AntiBT) rounded up is at most k, and therefore, by the integer rounding property, there exists also an integer dual solution of size at most k. If strictly less than k, than add 0 the right number of times to get the integer decomposition property.

These properties match Theorem 1.2 in a very lucky way resulting in a generalization of Greene and Kleitman's theorem of Bessy and Thomassé's theorems and of a variant of Berge's conjecture concerning strongly connected digraphs.

We deduce the integer decomposition property by providing a new self-contained algorithmic proof of Bessy and Thomassé's theorem on the "cyclic circular chromatic number," using again shortest paths in digraphs. On the way we get the polynomial solvability of the cyclic coloration problems of digraphs given with a cyclic order, that was left open by [5].

Then we note that cyclic stable sets have the integer decomposition property. After some general (but seemingly new) observations about the 0–1 solutions of multiples of such polyhedra we deduce the new minmax theorems from these polyhedral results: these seem to be the only properties that are necessary for deducing "Greene–Kleitman type" results from "Dilworth-type" results. We get a general theorem in this way containing both theorems of Bessy and Thomassé's and Greene's.

2.1. Cyclic coloration

The (undirected or directed) graph G is said to be q-colorable ($q \in \mathbb{N}$), if one can assign to its vertices integers in the interval $(0, q]$ so that adjacent vertices have different colors, that is, there exists $f : V(G) \to \mathbb{Z} \cap [1, q]$ so that

$$\bigl|f(x) - f(y)\bigr| \geqslant 1, \quad \text{if } xy \in A(G),$$

called a *q-coloration*. A slight relaxation of this definition arises by deleting the integrality constraint in this formulation:

According to [16] the graph G has *circular chromatic number* q ($q \in \mathbb{Q}$, $q \geqslant 1$) if there exists $f : V(G) \to \mathbb{Q} \cap (0, q]$ which is a *circular coloration*, that is,

$$\operatorname*{dist}_q\bigl(f(x), f(y)\bigr) \geqslant 1, \quad \text{if } xy \in A(G),$$

where dist_c denotes the distance on the circle of length c, that is, supposing $0 < a \leqslant b \leqslant c$,

$$\operatorname*{dist}_c(a, b) = \min\{b - a, a + c - b\}.$$

If q is an integer, then clearly, G has a q-coloration if and only if it has a circular q-coloration: a q-coloration is also a circular q-coloration, and it is also easy to construct a $\lceil q \rceil$-coloration from a circular q-coloration. (The proof of Corollary 2.1 is a sharpening of this argument.)

Given a cyclic order on $V(G)$ and a circular q-coloration f of G ($q \in \mathbb{Q}$) we say that f is a *cyclic coloration* of the given order (or of G with this order), if in addition for some linear order $\prec$ equivalent to the given one: $x \prec y$ implies $f(x) \leqslant f(y)$ $(x, y \in V(G))$. (Informally, the condition means that there is a cyclic order equivalent to the given one which is cyclically the same as the order defined by f.) The relation "$\prec$" is going to be referred to as a *linear order defined by f and equivalent to the given cyclic order*.

We denote the minimum of c for which there is a cyclic c-coloration of G by $\xi_{\mathrm{cycl}}(G)$, the minimum number of cyclic stable sets that partition $V(G)$ by $\xi(G)$, and the fractional relaxation of this number, that is, the (dual) optimum of (AntiBT) for the all 1 objective function by $\xi^*(G)$. That is, ξ^* is the minimum of $\sum_{i=1}^m y_i$ such that $y_1, \ldots, y_m \geqslant 0$, and $\sum_{i=1}^m y_i \chi_{S_i} \geqslant 1$, where $S_1, \ldots, S_m$ are cyclic stable sets, and ξ is the minimum if the y_i $(i = 1, \ldots, m)$ are in addition constrained to be integer. This dual solution $\{(y_i, S_i)\colon i = 1, \ldots, n\}$ is called a (fractional) coloration, and the S_i with $y_i > 0$ are the color classes; if the y_i $(i = 1, \ldots, n)$ are integer (consequently 0–1), then we indeed get back the definition of colorations.

It turns out (see Corollary 2.1) ξ (and ξ^*) can be realized by fixing one equivalent cyclic order, that is, all the color classes in an optimal coloration can be chosen to be intervals in one and the same cyclic order.

Bessy and Thomassé [5] provided a simple non-algorithmic proof of the following minmax relation. A short algorithmic proof using some recent results can be found in [7]. For a self-contained proof—in view of the extensive use we make of it—we sketch a variant of this proof where the potential values are all in the interval $[0, q]$:

Theorem 2.1. *Given a strongly connected digraph G with a compatible cyclic order, $\xi_{\mathrm{cycl}}(G)$ is equal to the maximum of $|C|/\operatorname{ind}(C)$ over all circuits C.*

Proof. First we check the trivial inequality $\xi_{\mathrm{cycl}}(G) \geqslant |C|/\operatorname{ind}(C)$ for every circuit C: let $c \in \mathbb{Q}$, and suppose $f : V \to \mathbb{Q}_+$ is a cyclic c-coloration and C an arbitrary circuit; we prove $c \geqslant |C|/\operatorname{ind}(C)$. Indeed, for every arc $xy \in A$ we have $\mathrm{dist}_c(f(x), f(y)) \geqslant 1$. Summing this for the arcs of C, the left-hand side sums up to at most $\operatorname{ind}(C)$ times the length c of the circle, that is, $\operatorname{ind}(C)c$, and the right-hand side sums up to $|C|$. We get therefore $\operatorname{ind}(C)c \geqslant |C|$ as claimed.

To prove the equality, we will again use (3): we define arc-weights so that there is no negative circuit and the corresponding potentials provide an optimal coloring. Let $q \in \mathbb{Q}_+$ be the maximum of $|C|/\operatorname{ind}(C)$ over all directed circuits, and let Q be a circuit for which the maximum is reached. (By the duality theorem $q = \xi^*$ but we will not use this.) Let $a \in V(Q)$ be arbitrary.

For openings of equivalent cyclic orders which have a as first element, define the arc-weight-function $w : A(G) \to \mathbb{Z}$ to be -1 on every forward arc, and $q-1$ on every backward arc. Define $\lambda : V(G) \to \mathbb{Q}$ by

$$\lambda(v) := \min\{w(P):\ P \text{ is a path in } G \text{ from } a \text{ to } v\}.$$

Claim 1. *There is no negative closed walk in (G, w), and $w(Q) = 0$.*

Indeed, for any circuit C, $w(C) = -|C| + \operatorname{ind}(C)q \geqslant 0$, because $q \geqslant |C|/\operatorname{ind}(C)$, and the equality holds for Q. Closed walks can be decomposed to circuits, proving the claim.

According to Claim 1 the function λ is well defined, and does not depend on whether we restrict ourselves to simple paths or not. Clearly, $\lambda(a) = 0$. Among linear orders that have a as their first element we choose an opening of an equivalent cyclic order that minimizes

$$U = U(\lambda) = \max\{\lambda(v):\ v \in V\},$$

and among these we maximize

$$L := L(\lambda) = \min\{\lambda(v):\ v \in V\}.$$

Suppose without loss of generality that this linear order is the given one.

Claim 2. *For all $v \in V$: $-q < \lambda(v) \leqslant 0$.*

We prove the lower bound first. Suppose, for a contradiction, that

$$X := \{v \in V:\ \lambda(v) \leqslant -q\} \neq \emptyset.$$

Consider the opening where a is the last element (shift a "from the first to the last place"): *there is no forward arc in this order from X to $V \setminus X$*, because such an arc $xy \in A$, $x \in X$, $y \in V \setminus X$ would imply, if $y \neq a$, $\lambda(y) \leqslant \lambda(x) - 1 \leqslant -q - 1 < -q$, that is, $y \in X$; if $y = a$, then let P be a w-shortest path between y and x, and note that $P \cup \{xy\}$ is a circuit, and

$$w(P) + w(xy) \leqslant -q + (q - 1) = -1,$$

a contradiction with Claim 1.

Since the given order is compatible with G, by (1) there is also no backward arc to X from $V \setminus X$ (in this same opening where a is the last element). So the order where X comes last, preceded by a is equivalent to the given one; with a shift we see that the order starting with a followed by X and then by the rest of the elements (in the original order), is an opening of the same cyclic order. Denote by w', λ' the weight and potential functions respectively, defined by this order.

We have $w(P) = w'(P)$ for any path with both endpoints in $V \setminus X$ (or with both in X), since they have the same number of arcs e entering X, that is with $w(e) = -1$, as arcs f leaving X, that is with $w(f) = q - 1$; since then $w'(e) = q - 1$, $w'(f) = -1$, we have indeed $w(P) = w'(P)$. Therefore (using that $a \notin X$) $\lambda(v) = \lambda'(v)$ for all $v \in V \setminus X$. On the other hand, the weight of all arcs entering X increases from -1 to $q - 1$ therefore proving

$$\lambda'(x) = \lambda(x) + q \leqslant 0 = \lambda(a) \leqslant U$$

for all $x \in X$, and contradicting the maximum choice of L (U is unchanged). (We used that paths to vertices in X use one more arc entering X than arc leaving X and X contains every vertex v with $\lambda(v) = L$.) So $X = \emptyset$.

The proof of the upper bound is similar.

Claim 3. $|\lambda|$ *is a cyclic q-coloration of G (with respect to the given cyclic order if the color $|\lambda(v)| = 0$ is redefined to be q).*

We first prove the second part, that is, the linear order by $|\lambda|$ $(= -\lambda)$ is equivalent to an opening of a cyclic order equivalent to the given one. Let $\prec$ be a linear order starting with a, equivalent to the given cyclic order, minimizing

$$\iota := \left|\left\{(x, y)\colon\ x, y \in V,\ x \prec y,\ \left|\lambda(x)\right| > \left|\lambda(y)\right|\right\}\right|.$$

We show that $\iota = 0$. If not, there exist two consecutive elements $x \prec y$, such that $|\lambda(x)| > |\lambda(y)|$. It is not possible that $xy \in A(G)$, because then it is a forward arc and so $\lambda(y) \leqslant \lambda(x) - 1 < \lambda(x) \leqslant 0$. Since y is the immediate successor of x, by compatibility $yx \in A(G)$ would also imply $xy \in A(G)$. So x and y are nonadjacent, and therefore we get an equivalent order by interchanging them and decreasing ι, a contradiction with the choice of $\prec$.

Now the reader can easily check that λ is a circular q-coloration, more precisely let $f := \lambda$ unless $\lambda(v) = 0$, when we define $f(v) := q$, and prove that f is a q-coloration.

Hint: $\mathrm{dist}_q(f(x), f(y)) = \mathrm{dist}_q(\lambda(x), \lambda(y)) \geqslant 1$. □

The proof is clearly algorithmic once the number $q = \xi_{\mathrm{cycl}}$ is determined. A first, brute force approach to determine q: a logarithmic search already providing a polynomial algorithm: try different values following logarithmic search and use any algorithm to detect a negative circuit [15, Chapter 8], and when there is no negative circuit a potential can also be determined in this way, moreover with strongly polynomial algorithms.

As mentioned before, Theorem 2.1 provides another proof of the antiblocking relation (Corollary 1.1) between (BT) and (AntiBT).

2.2. Rounding

We deduce now from Theorem 2.1 the promised corollaries:

Corollary 2.1. *If $G = (V, A)$ is an arbitrary strongly connected digraph given with a compatible cyclic order, then $\xi_{\mathrm{cycl}} = \xi^* = \max\{|C|/\mathrm{ind}(C)\colon\ C$ is a circuit$\}$, and $\xi = \lceil \xi^* \rceil$, moreover there exists an optimal ξ-coloration and an optimal ξ^*-coloration where all the color classes in both are intervals in the same cyclic order (equivalent to the given one).*

Proof. If f is a cyclic ξ_{cycl}-coloration, then $f'(v) := \lceil f(v) \rceil$ is a coloration where the color classes are cyclic stable sets, so

$$\xi \leqslant \lceil \xi_{\mathrm{cycl}} \rceil. \tag{round}$$

Moreover, the color classes in the defined $\lceil \xi_{\mathrm{cycl}} \rceil$-coloration are all intervals in the equivalent linear order defined by f. Since $\lceil \xi_{\mathrm{cycl}} \rceil$-colorations will turn out to be optimal colorations, f' is suitable to be the optimal ξ-coloration in the last assertion.

We prove now

$$\xi_{\mathrm{cycl}} = \xi^* = \max\left\{|C|/\mathrm{ind}(C)\colon\ C \text{ is a circuit}\right\}. \tag{fract}$$

This follows if we check $\xi_{\mathrm{cycl}} \geqslant \xi^* \geqslant \max\{|C|/\mathrm{ind}(C)\colon\ C$ is a circuit$\}$, because then according to Theorem 2.1 we have equality between the first and the last terms, and therefore throughout.

The inequality $\xi_{\text{cycl}} \geqslant \xi^*$ follows from the fact that a cyclic coloration f of value ξ_{cycl} provides a fractional coloration of the same value: take for any $0 < y \leqslant \xi_{\text{cycl}}$ the cyclic stable set $Y := \{v \in V\colon\ y < f(v) \leqslant y+1\}$, where $y+1$ is understood cyclically, modulo ξ_{cycl}; let $\mathcal{Y}$ be the family of the different stable sets defined in this way; let λ_Y $(Y \in \mathcal{Y})$ be the length of the interval $I = I_Y$ for which $y \in I$ defines Y. Clearly, the sum of the coefficients λ_Y for stable sets Y that contain v is indeed at least 1 for every vertex v, and $\{I_Y\colon\ Y \in \mathcal{Y}\}$ is a partition of the circle of length ξ_{cycl}. So indeed, $\sum_{Y\in\mathcal{Y}} \lambda_Y = \xi_{\text{cycl}}$. (Informally: we took half-open "windows" of length 1, with the straightforwardly defined multiplicities.)

To check $\xi^* \geqslant \max\{|C|/\operatorname{ind}(C)\colon\ C$ is a circuit$\}$ (which is the trivial part of the duality theorem for (AntiBT), a sharpening of the trivial part of Theorem 2.1) let $S_1, \ldots, S_m$ be cyclic stable sets and $y_1, \ldots, y_m \geqslant 0$ so that $\sum_{i=1}^m y_i \chi_{S_i} \geqslant 1$ and $\sum_{i=1}^m y_i = \xi^*$, and C an arbitrary circuit. Using these inequalities and $|S_i \cap C| \leqslant \operatorname{ind}(C)$ (which is the trivial 'if' part of (5)), that is, $\chi_{S_i}^T \chi_C / \operatorname{ind}(C) \leqslant 1$, we have:

$$\xi^* \geqslant \sum_{i=1}^m y_i \left(\chi_{S_i}^T \chi_C / \operatorname{ind}(C)\right) = \left(\sum_{i=1}^m y_i \chi_{S_i}\right)^T \chi_C / \operatorname{ind}(C) \geqslant \underline{1}^T \chi_C / \operatorname{ind}(C) = |C| / \operatorname{ind}(C).$$

Finally, from (round) and (fract) we get:

$$\xi \leqslant \lceil \xi_{\text{cycl}} \rceil = \lceil \xi^* \rceil \leqslant \xi,$$

and therefore there is equality throughout, finishing the proof of the statement. □

Corollary 2.2. *In a strongly connected digraph given with a compatible cyclic order,* (BT) *has the integer decomposition property* (*that is, cyclic stable sets have this property*) *and* (AntiBT) *has the integer rounding property.*

Proof. We have already seen that the latter statement implies the former. We have to prove that for any objective function $w : V(G) \to \mathbb{N}$ we have $\xi_w^* = \lceil \xi_w^* \rceil$, where ξ_w and ξ_w^* are the optimal integer, respectively fractional dual solutions of (AntiBT) with objective function w. This follows from Corollary 2.1 through the replication of a clique:

Let $v \in V$, and introduce $w_v - 1$ new copies of v (joined to the same vertices as v, with arcs of the same orientation), to have w_v copies altogether, and order them $v = v_1, \ldots, v_{w_v}$. Let any $v_i v_j$ $(i < j = 1, \ldots, w_v)$ be an arc. Now for any circuit C of winding 1 through v, replace the arcs xv, vy by the path $x, v = v_1, \ldots, v_{w_v}, y$. We get a circuit of winding 1 and appropriate shortcuts show that each of the new arcs is contained in a circuit of winding 1. Applying this operation and the same argument for all $v \in V$ the defined directed graph $\hat{G}$ is clearly compatible.

An optimal fractional dual solution for (AntiBT) defined by G for the objective function w implies a fractional cover of the vertices of $\hat{G}$, with sum of multiplicities ξ_w^*. Now apply Corollary 2.1 to $\hat{G}$: $\xi_w = \xi(\hat{G}) = \lceil \xi^*(\hat{G}) \rceil = \lceil \xi_w^* \rceil$. □

3. A synthesis

In this section we prove a common generalization of the results of the previous sections and of several classical results on posets—namely of the theorems of Bessy and Thomassé and of Greene and Kleitman [9,12]. The latter contain Dilworth's theorem. We show how to maximize the cyclically k-colorable subgraphs of a graph, and we also determine the convex hull of the unions of k disjoint cyclic stable sets. We do this using only the general polyhedral properties we proved, and state the polyhedral results separately.

3.1. Maximizing k-colorable subgraphs

Theorem 3.1. *Given a strongly connected digraph with a compatible cyclic order and a number $k \in \mathbb{N}$, the maximum cardinality of a set that can be partitioned into k cyclic stable sets is equal to the minimum of $|X| + k\,\mathrm{ind}(\mathcal{C})$ over all $X \subseteq V$ and all covers $\mathcal{C}$ of $V \setminus X$ with circuits.*

Proof. On the one hand, by Theorem 1.2 one can maximize $\sum_{i=1}^{n} x_i$ over (kBT) and the additional condition $0 \leqslant x \leqslant 1$ with an integer vector x_0. On the other hand, according to Corollary 2.2, any integer solution of (kBT) and $x \geqslant 0$ is the sum of k cyclic stable sets; since $x_0 \geqslant 0$ satisfies (kBT) and all of its coordinates are 0 or 1, this sum is in fact a (disjoint) union. Conversely, the union of at most k cyclic stable sets satisfies (kBT) and $0 \leqslant x \leqslant 1$. Therefore the maximum of $\sum_{i=1}^{n} x_i$ subject to (kBT) and $0 \leqslant x \leqslant 1$ is the maximum cardinality of the union of k (disjoint) cyclic stable sets.

In order to prove that this number is equal to the minimum stated in the theorem we use the integrality of the dual solution in Theorem 1.2: we have a dual variable y_C for each circuit C, and also a dual variable y_v associated to each upper bound constraint $x_v \leqslant 1$, that is to each vertex $v \in V$. Denoting by X the set of vertices v for which $y_v > 0$ (that is, $y_v = 1$), and using the integrality of the dual solution, we get exactly the claim. □

Unlike paths or chains of posets, one vertex alone is not always a circuit. Therefore the set X cannot be replaced by $|X|$ 1-element circuits to provide a nicer formula (like those known for posets). However, one can add loops (considered as circuits with $\mathrm{ind}(C) = 1$) to some or all of the vertices without changing the definition of stable sets. (The inequalities $x_v \leqslant 1$ $(v \in V)$, if v is a loop are satisfied even if $x(v) = 1$.)

Corollary 3.1. *Given $k \in \mathbb{N}$, and a strongly connected digraph with a compatible cyclic order and with a loop at every vertex, the maximum cardinality of the union of k cyclic stable sets is equal to the minimum of $\sum_{C \in \mathcal{C}} \min\{k\,\mathrm{ind}(C), |C|\}$, where $\mathcal{C}$ is a covering of V with circuits.*

This innocent looking statement contains both results of Bessy and Thomassé's including Gallai's conjecture, and also Greene and Kleitman's theorem on posets, which itself generalizes Dilworth's theorem. Let us show here how Bessy and Thomassé's results are contained in it—the connection to posets is treated in the next section:

- For $k = 1$ the loops x of $\mathcal{C}$ in the corollary (the elements of X of the theorem) can be replaced by any circuit C_x of winding 1, and at least one such circuit exists by compatibility. So the loops can be deleted, and on the minimum side we have: $\min\{k\,\mathrm{ind}(C), |C|\} = \min\{\mathrm{ind}(C), |C|\} = \mathrm{ind}(C)$; substituting this and $k = 1$ into the corollary, we get back Bessy and Thomassé's first theorem, Theorem 0.2.
- For $k = \lceil \xi^*(G) \rceil$ we have $k = \lceil \xi^*(G) \rceil \geqslant |C| / \mathrm{ind}(C)$ for every circuit C, that is, $k\,\mathrm{ind}(C) \geqslant |C|$, $\min\{k\,\mathrm{ind}(C), |C|\} = |C|$. It follows that
$$\sum_{C \in \mathcal{C}} \min\bigl\{k\,\mathrm{ind}(C), |C|\bigr\} \geqslant n$$
for any cover $\mathcal{C}$, and therefore, because of the loops, the minimum of this sum is equal to n. By Theorem 3.1 then the maximum union of $k = \lceil \xi^*(G) \rceil$ cyclic stable sets is also n, proving the essential part $\xi^* = \max\{|C| / \mathrm{ind}(C)\colon C \text{ is a directed circuit}\}$, $\xi = \lceil \xi^* \rceil$ of Corollary 2.1, which is the main result of Section 2, equivalent to the integer decomposition property.

Of course, we could have used Theorem 3.1 or the equivalent Corollaries 3.1, 2.1 or 2.2 to deduce the results of the preceding sections.

Let us now deduce corollaries where the assertion is not concerned with cyclic orders (but in the proof cyclic orders do help) in the same way as Gallai's conjecture is a corollary of Bessy and Thomassé's theorem. The following corollaries generalize Gallai's conjecture Theorem 0.1, and this can be put either following the analogy of Theorem 3.1 or that of Corollary 3.1:

Corollary 3.2. *Given a strongly connected digraph and a number $k \in \mathbb{N}$, there exist k stable sets $S_1, \dots, S_k$, a set $X \subseteq V$, and a circuit collection $\mathcal{C}$ covering $V \setminus X$ such that*:

$$|S_1 \cup \dots \cup S_k| \geqslant |X| + k|\mathcal{C}|.$$

Indeed—like in the proof of Gallai's conjecture—use (2): D is strongly connected, so it has a compatible order, and for this order Theorem 3.1 can be applied. The assertion follows then from Theorem 3.1 after the obvious substitution $\operatorname{ind}(C) \geqslant 1$ for all $C \in \mathcal{C}$.

Corollary 3.3. *Let $D = (V, A)$ be a strongly connected digraph with a loop in every vertex, and $k \in \mathbb{Z}_+$. Then there exist k stable sets $S_1, \dots, S_k$, and there exists a circuit collection $\mathcal{C}$ covering V such that*

$$|S_1 \cup \dots \cup S_k| \geqslant \sum_{C \in \mathcal{C}} \min\{k, |C|\}.$$

Let X and $\mathcal{C}$ be those of Corollary 3.2. We can suppose without loss of generality that $|C| \geqslant k$ for all $C \in \mathcal{C}$, since otherwise we delete C from $\mathcal{C}$, and add its elements to X without increasing the right-hand side of Corollary 3.2. Now let $\mathcal{C}' := \mathcal{C} \cup \{vv\colon\ v \in X\}$ and show the statement for $\mathcal{C}'$. It is clearly a cover, and

$$\sum_{C \in \mathcal{C}'} \min\{k, |C|\} = \sum_{C \in \mathcal{C}', |C|=1} 1 + \sum_{C \in \mathcal{C}', |C| \geqslant k} k = |X| + k \sum_{C \in \mathcal{C}} 1 = |X| + k|\mathcal{C}|,$$

whence the right-hand side of Corollary 3.3 coincides with that of Corollary 3.2, completing the proof of the latter.

For $k = 1$ we get back Gallai's conjecture Theorem 0.1 as promised.

The inequalities used for the easy part of Theorem 3.1 must be satisfied with equality because of the assertion of the theorem and this forces the following (complementary slackness) statement as corollary:

Corollary 3.4. *Let $D = (V, A)$ be a strongly connected digraph with a loop in every vertex and $k \in \mathbb{Z}_+$. Then there exists a circuit collection $\mathcal{C}$ covering V and there exist k disjoint stable sets such that each $C \in \mathcal{C}$ intersects* $\min\{k, |C|\}$ *of them.*

Indeed—like in the two previous assertions that do not involve orders (Theorem 0.1 and Corollary 3.3)—use first (2): D is strongly connected, so it has a compatible order, and for this order Corollary 3.1 can be applied.

Fix this order, let $S_1, \dots, S_k$ be cyclic stable sets and $\mathcal{C}$ a circuit cover. Let us denote $S := S_1 \cup \dots \cup S_k$, and check the easy part of Corollary 3.1 directly:

$$|S| \leqslant \sum_{C \in \mathcal{C}} \left|S \cap V(C)\right| \leqslant \sum_{C \in \mathcal{C}} \min\{k \operatorname{ind}(C), |C|\}.$$

The first inequality holds with equality if and only if $\{S \cap V(C)\colon\ C \in \mathcal{C}\}$ is a partition of S; the second inequality holds with equality if and only if for every $C \in \mathcal{C}$ for which $k\,\mathrm{ind}(C) < |C|$, $|S_i \cap V(C)| = \mathrm{ind}(C)$ $(i = 1, \dots, k)$. We can suppose without loss of generality that circuits C with $|C| < k\,\mathrm{ind}(C)$ are loops.

Let $S_1, \dots, S_k$ be k disjoint stable sets, and $\mathcal{C}$ a circuit cover for which the equality holds in Corollary 3.1, that is, both inequalities of the above formula hold with equality. If $C \in \mathcal{C}$, $\mathrm{ind}(C) > 1$, then C is not a loop, and $|S_i \cap V(C)| = \mathrm{ind}(C) > 1$ $(i = 1, \dots, k)$. Delete from S_i all but one of the elements of $S_i \cap V(C)$ (that is, for every i, $\mathrm{ind}(C) - 1$ vertices of C are deleted from S_i). The resulting sets S_i' $(i = 1, \dots, k)$ and the unchanged circuit cover have the claimed properties, and the corollary is proved.

Compare these two corollaries with the following conjectures of Linial [14] and Berge [2] respectively, see the latter as Problem 5 in [15]:

Conjecture 1. *Let $D = (V, A)$ be a digraph and $k \in \mathbb{Z}_+$. Then there exists a family of paths $\mathcal{P}$ partitioning V and there exist k stable sets $S_1, \dots, S_k$ such that*

$$|S_1 \cup \dots \cup S_k| \geqslant \sum_{P \in \mathcal{P}} \min\bigl\{k, \bigl|V(P)\bigr|\bigr\}.$$

Conjecture 2. *Let $D = (V, A)$ be a digraph and $k \in \mathbb{Z}_+$. Then for each path collection $\mathcal{P}$ partitioning V and minimizing*

$$\sum_{P \in \mathcal{P}} \min\bigl\{\bigl|V(P)\bigr|, k\bigr\},$$

there exist k disjoint stable sets such that each $P \in \mathcal{P}$ intersects $\min\{|V(P)|, k\}$ *of them.*

These conjectures have the same relation to the preceding two corollaries, as Gallai and Milgram's theorem (the $k = 1$ special case of the conjectures) has to Bessy and Thomassé's Theorem 0.1 (Gallai's conjecture). Surprisingly, while Gallai and Milgram's theorem was proved much earlier than Bessy and Thomassé's, for the generalizations the path partitioning version is still resisting. Yet both Gallai's conjecture and the Greene–Kleitman theorem are generalized by the corollaries, like by the conjectures.

Note that all the ingredients we used (the solution of (kBT), or rounding (see Section 2)) can be executed in polynomial time.

3.2. Polyhedra of k-unions

We state separately a general polyhedral phenomenon that was behind the results.

The polyhedron $P := \{x \in \mathbb{R}^n\colon\ Ax \leqslant b\}$ (where A and b are matrices of appropriate size) is said to have the *integer decomposition property* (IDP) if any integer vector in $kP := \{x \in \mathbb{R}^n\colon\ Ax \leqslant kb\}$ is the sum of k integer vectors each in P.

The system of inequalities $Ax \leqslant b$ is said to be *box-TDI* if for arbitrary $l, u \in (\mathbb{Z} \cup \{\infty, -\infty\})^n$ and arbitrary objective function for which the system $Ax \leqslant b$, $l \leqslant x \leqslant u$, has a finite dual optimum, it also has an integer one. We will exploit this property only for $l = \underline{0} \in \mathbb{R}^n$ and $u = \underline{1} \in \mathbb{R}^n$.

For the coordinates of our linear programs we adopt the notation $V := \{1, \dots, n\}$. The polyhedral results we are stating are true for arbitrary matrices and arbitrary objective functions, but we focus on the more pleasant form for 0–1 matrices and all 1 objective functions: we want to keep the most interesting and simple formulas we are already familiar with in the special cases.

With an abuse of terminology, vectors in $\{0,1\}^n$ are going to be considered also as (characteristic vectors of) subsets of V. Given an $m \times n$ 0–1-matrix A and $X \subseteq V$ a *covering of X with the rows of A* is a set of rows of A whose union contains X.

Given a polyhedron P, we call the union of k 0–1-vectors in P a *k-union*.

Theorem 3.2. *Let A be an $m \times n$ matrix with* 0–1 *entries, $b \in \mathbb{N}^m$, $P := \{x \in \mathbb{R}^n\colon\ Ax \leqslant b,\ x \geqslant 0\}$ has the integer decomposition property, and $Ax \leqslant kb$ is box-TDI for all $k \in \mathbb{N}$. Then for all $k \in \mathbb{N}$: $Ax \leqslant kb$, $0 \leqslant x \leqslant 1$, has also all its vertices in $\{0,1\}^n$, these are exactly the k-unions, and the maximum cardinality of the k-unions is equal to the minimum over all $X \subset V$ of $|X| + \sum_{Y \in \mathcal{A}} kb(Y)$ where $\mathcal{A}$ is a covering of $V \setminus X$ with the rows of A, and $b(Y)$ is the right-hand side corresponding to the row Y.*

It is straightforward to check with the help of the ellipsoid method (see [15]) that the solvability (by separation and optimization) of P implies that kP intersected with boxes is also solvable, and therefore the minima and maxima in the theorem can be computed in polynomial time.

Proof. We first show that the vertices of $P(k) := \{x \in \mathbb{R}^n\colon\ Ax \leqslant kb,\ 0 \leqslant x \leqslant 1\}$ are exactly the k-unions. It is evident that the k-unions are vertices of $P(k)$.

Let x_0 be a vertex of $P(k)$. Since $Ax \leqslant kb$ is box-TDI, $P(k)$ has integer vertices (by Edmonds and Giles, or it follows directly like in the proofs of Section 1), and since $0 \leqslant x \leqslant 1$ is among the defining inequalities of $P(k)$, $x_0 \in \{0,1\}^n$. Because of the integer decomposition property x_0 is the sum of k vertices of P, and since these and x_0 are in $\{0,1\}^n$ the summands are disjoint. So x_0 is a k-union.

Now the minmax equality follows simply by applying the duality theorem and total dual integrality to the all 1 objective function constrained to $P(k)$: the primal optimum is the maximum k-union as argued before, and the dual problem is a covering problem with the rows of A and unit vectors. If X denotes the participating unit vectors, we get exactly the claimed minimum value. $\square$

Corollary 3.5. *Under the conditions of the preceding theorem, adding the unit vectors to the rows of A with right-hand side* 1*, the maximum cardinality of a k-union is equal to the minimum of $\sum_{X \in \mathcal{A}} \min\{kb(X), |X|\}$, where $\mathcal{A}$ is a covering of V with the rows of A.*

Indeed, it is again easy to see that the maximum is not greater than the minimum. Now consider the k-union, the X and $\mathcal{A}$ for which the equality holds in the preceding theorem: defining $\mathcal{A}' := \mathcal{A} \cup \{\{x\}\colon\ x \in X\}$ we get that the same k-union is at least as big as $\sum_{A \in \mathcal{A}'} \min\{kb(A), |A|\}$. Therefore the equality holds for $\mathcal{A}'$, and the corollary is checked.

Note that Theorem 3.1, and Corollary 3.1 are immediate consequences of the above theorem and corollary. These latter have as well corollaries independent of b analogous to Corollary 3.3, or Corollary 3.4, but in lack of interesting applications—besides the well-known minmax theorems concerning independent sets in matroids—we omit these here. We also note that both conditions of Theorem 3.2 also hold for matroid polyhedra $P := \{x \in \mathbb{R}^n\colon\ x(A) \leqslant r(A),\ x \geqslant 0\}$, immediately yielding a minmax equality for the maximum union of k independent sets, a form of Edmonds' matroid partition theorem, see [15]. (However, the integer decomposition property of matroid polyhedra is itself equivalent to the matroid partition theorem.)

4. Feedback

Given a digraph G with a cyclic order (that is not necessarily compatible) we say that $U \subseteq V(G)$ is a *cyclic feedback* (*vertex-*)*set* (cyclic FS) if $|U \cap V(C)| \geqslant \mathrm{ind}(C)$ for every circuit C; $F \subseteq A(G)$ is a *cyclic feedback arc-set* (cyclic FAS) if $|F \cap AC| \geqslant \mathrm{ind}(C)$ for every circuit C. Recall that we get the definition of feedback (arc-)sets if we replace here $\mathrm{ind}(C)$ uniformly by 1. While the minimum feedback vertex- or arc-set problem is NP-hard, we are able to determine the minimum weight of the cyclic version, and state minmax theorems about them.

More generally, we call a multiset of vertices a cyclic feedback (vertex-) or arc-multiset if its sum on the vertices or respectively edges of C is at least $\mathrm{ind}(C)$. Equivalently, a feedback vertex-multiset is represented by a multiplicity vector $x \in \mathbb{N}^V$ satisfying the following inequalities:

$$x(C) \geqslant \mathrm{ind}(C), \quad \text{for every circuit } C \subseteq V(G), \tag{CycFeed}$$

$$x(AC) \geqslant \mathrm{ind}(C), \quad \text{for every circuit } C \subseteq V(G). \tag{ArcCycFeed}$$

In this section we work out the facts relevant to optimization on these sets, both vertex and arc versions. The *cardinality* of a multiset of vertices represented by x is $\underline{1}^T x = \sum_{i=1}^n x_i$.

Note first, that 0–1 *solutions to* (CycFeed) *have a very simple relation to* 0–1 *solutions of* (BT):

Let us say that the cyclic or linear order defined by $(x_n, \ldots, x_1)$ is the *inverse* of the one defined by $(x_1, \ldots, x_n)$. Given a digraph $G = (V, A)$ with a cyclic order and a circuit C, denote by $\mathrm{ind}^{-1}(C)$ the winding of the inverse cyclic order. Clearly,

$$\mathrm{ind}^{-1}(C) = |C| - \mathrm{ind}(C).$$

Indeed, as noticed before, the winding is the number of backward arcs in any opening. Fix an opening and note that the set of backward arcs and that of forward arcs partition $E(C)$, moreover that they are interchanged when the order is inverted, proving the formula.

Let now $U \subseteq V$ be a cyclic feedback vertex-set, that is, $|U \cap V(C)| \geqslant \mathrm{ind}(C)$ for every circuit C. Then

$$\bigl|(V \setminus U) \cap V(C)\bigr| \leqslant |C| - \mathrm{ind}(C) = \mathrm{ind}^{-1}(C)$$

for every circuit C, moreover, similarly,

(6) $x \in [0,1]^V$ *satisfies* (CycFeed) *if and only if* $\underline{1} - x$ *satisfies* (BT) *of the inverse order.*

We denote by $\underline{k}$ the all k vector of appropriate dimension.

In particular, according to (5) if the inverse order is compatible—we say then that the original order is *inverse-compatible*—we have proved the following:

(7) *Let $G = (V, A)$ be a digraph given with an inverse-compatible order. Then $U \subseteq V$ is a cyclic FS if and only if $V \setminus U$ is a cyclic stable set.*

Indeed, we have already checked that under the condition, $V \setminus U$ is a cyclic stable set of the inverse order. In order to finish the proof of this statement we only have to note the following—these simple assertions are used without reference in the sequel:

- The elementary changes for an inverse pair of orders are the same. Therefore the *equivalence class of a cyclic order is equal to the set of inverse orders of the equivalence class of the inverse order*. (Taking the equivalence class and the inverse commute.)

– Intervals are the same according to a cyclic (linear) order or its inverse.
– The stability of a set does not depend on the order.

It follows that *a set of vertices is a cyclic stable set of the inverse order if and only if it is a cyclic stable set*, finishing the proof of (7).

It follows that *cyclic feedback sets form an interval in an equivalent order, provided the given order is inverse-compatible*. However, complements of stable sets are very particular feedback sets in any digraph, so they are not very interesting ones. In fact, cyclic feedback sets (or arc-sets) are stronger (closer to general feedback sets) in cyclic orders that are not inverse-compatible:

Let $V := \{1, 2, 3, 4, 5, 6\}$, $A := \{12, 23, 34, 45, 56, 61, 31, 64\}$, and the given cyclic order is $1, 2, 3, 4, 5, 6$. This is a compatible order, but it is not inverse-compatible. The set $\{2, 5\}$ is a cyclic FS but $2, 5$ cannot be consecutive in an equivalent cyclic order. Putting weights on the other elements (or replicating), this cyclic FS will also be minimum!

4.1. Vertex-feedback

Theorem 4.1. *Given a digraph with a cyclic order, the minimum cardinality of a cyclic feedback multiset is equal to the maximum winding* $\operatorname{ind}(\mathcal{C})$ *of a set* $\mathcal{C}$ *of pairwise vertex-disjoint circuits.*

This minmax theorem and a polynomial algorithm for finding the common value of the minimum and the maximum, is an immediate corollary of the following polyhedral result which shows full analogy to (4) (the theorem does not show full analogy with Theorem 0.2, since the definition of feedback sets uses directly linear inequalities, whence the combinatorial version analogous to cyclic stable sets is shortcut, as well as Theorem 1.1, that is, (5)):

(8) *If* $G = (V, A)$ *is a digraph given with a cyclic order, then for any integer objective function* $w : V(G) \to \mathbb{Z}_+$, $a, b : V(G) \to \mathbb{N}$ $(a \leqslant b)$, *the linear program*

$$\min\{w^T x:\ x \text{ satisfies the inequalities (CycFeed) and } a \leqslant x \leqslant b\}$$

has integer primal and dual optima, and they are equal.

We prove the more general Theorem 4.2 with the proof method we are already familiarized with in the proof of (4) and of Theorem 1.2, and which gives rise to a polynomial algorithm:

Define for arbitrary $k \in \mathbb{N}$ the following system without nonnegativity constraints:

$$x(C) \geqslant k \operatorname{ind}(C), \quad \text{for every circuit } C \text{ of } G. \tag{kCycFeed}$$

If a vertex is not contained in a circuit, then the corresponding variable can be chosen to be arbitrarily small, the dual problem may be infeasible: for the feasibility or boundedness of (kCycFeed) and its duals we can assume again, without loss of generality that the input digraph is simplified.

The following theorem includes all the variations of parameters that look useful for the applications we have in mind. It is the blocking counterpart of Theorem 1.2; we invite the reader to recall the explanations preceding it.

Theorem 4.2. *If* $G = (V, A)$ *is a simplified digraph given with a cyclic order, then for any integer objective function* $w : V(G) \to \mathbb{Z}_+$, $a, b : V(G) \to \mathbb{Z}$ $(a \leqslant b)$, *and* $k \in \mathbb{N}$ *the linear program*

$$\max\{w^T x:\ x \text{ satisfies the inequalities (kCycFeed)}, a \leqslant x \leqslant b\}$$

has integer primal and dual optima, they are equal, and can be found in polynomial time.

This theorem extends (8) and its content means exactly that (kCycFeed) is "box-TDI" for any $k \in \mathbb{N}$. Our longer but elementary formulation is better adapted to our problem and is sharper in this case. We give simple separate proofs to primal and dual integrality, without reference to linear programming.

It is the third time (after (4) and Theorem 1.2) that we apply the same proof method based on circulations and potentials, so we refer to these previous proofs for explanations that we do not repeat.

Proof of Theorem 4.2. Open the given cyclic order arbitrarily, fix the resulting linear order and let the cost of backward arcs e be $c(e) := -k$, and the cost of all other arcs be 0.

For all $v \in V$ introduce two vertices, v_{in} and v_{out}, and let $\hat{G} = (\hat{V}, \hat{A})$,

$$\hat{V} := \bigcup_{v \in V} \{v_{\text{in}}, v_{\text{out}}\},$$

$$\hat{A} := \{x_{\text{out}} y_{\text{in}}\colon\ xy \in A\} \cup \{e_v\colon\ e_v := v_{\text{in}} v_{\text{out}},\ v \in V\} \cup \{\overrightarrow{e_v} := v_{\text{in}} v_{\text{out}},\ v \in V\}$$
$$\cup \{\overleftarrow{e_v} := v_{\text{out}} v_{\text{in}},\ v \in V\},$$

that is, e_v has a parallel and an inverted copy.

Define capacity- and cost-functions on $\hat{G}$:

- Define a lower and upper capacity-function $l : \hat{A} \to \mathbb{Z}_+$ as follows: for the arcs e_v $(v \in V)$ let $l(e_v) := u(e_v) := w(v)$, and let otherwise $l := 0$, $u := \infty$.
- Define $c(e)$ for $e \in \hat{A}$, $e = x_{\text{out}} y_{\text{in}}$ $(xy \in A)$ to be $c(xy)$ (that is, $-k$ if xy is a backward arc and 0 otherwise).
- Define $c(e_v) := 0$, $c(\overrightarrow{e_v}) = b_v$, $c(\overleftarrow{e_v}) = -a_v$.

Now we use the fact that there exists an integer minimum cost flow (circulation) which can again be deduced from basic flow theory—and can be found in polynomial time. Let f be an integer minimum cost flow with this data which again is a multiset $\mathcal{C}$ of circuits (the elements are circuits, and each circuit can have any integer multiplicity) providing a dual optimum in exactly the same way as for Theorem 1.2.

Again the auxiliary digraph associated with the optimal flow f can be determined and (3) can be applied according to which a potential π exists in the auxiliary digraph. Define again

$$x_v := \pi(v_{\text{out}}) - \pi(v_{\text{in}}) \quad \text{for all } v \in V.$$

This satisfies (kCycFeed) with a similar sequence of equalities as Theorem 1.2:

$$\sum_{u \in C} x_u = \sum_{u \in C} \bigl(\pi(u_{\text{out}}) - \pi(u_{\text{in}})\bigr) = \sum_{uv \in AC} \bigl(\pi(v_{\text{out}}) - \pi(u_{\text{in}})\bigr) = -\sum_{uv \in AC} \bigl(\pi(u_{\text{in}}) - \pi(v_{\text{out}})\bigr),$$

where

$$\sum_{uv \in AC} \pi(u_{\text{in}}) - \pi(v_{\text{out}}) \leqslant \sum_{uv \in AC} c(u_{\text{in}} v_{\text{out}}) \leqslant c(C) = -k\,\text{ind}(C),$$

that is, finally

$$\sum_{u \in C} x_u \geqslant k\,\text{ind}(C),$$

and again for $C \in \mathcal{C}$ there is equality here.

Let us finally check that the upper and lower bounds are also satisfied for x_v. Since there is no upper capacity on $\vec{e_v}$ and $\overleftarrow{e_v}$, their costs in the auxiliary digraph are b_v and $-a_v$, respectively. Therefore $x_v = \pi(v_{\text{out}}) - \pi(v_{\text{in}}) \leqslant b_v$, and $-x_v = \pi(v_{\text{in}}) - \pi(v_{\text{out}}) \leqslant -a_v$. □

In Theorem 4.1 it is not possible to replace "multiset" by "set": consider a digraph G consisting of two circuits, each of winding 2, and intersecting in one vertex v. Then in Theorem 4.1 the vector $2e_v$ (v with multiplicity 2) is a cyclic feedback multiset; with the definition $\mathcal{C} := \{C\}$, where C is any of the two circuits of G, we have $\text{ind}(\mathcal{C}) = 2$, checking the minmax equality. However, the minimum cyclic FS is of size 3. This gap can be helped: applying the theorem to $a := \underline{0} \in \mathbb{R}^n$, and $b := \underline{1} \in \mathbb{R}^n$, we get a minmax theorem for cyclic feedback sets:

Let $\text{sur}(\mathcal{C})$ be the surplus of $\mathcal{C}$ comparing to a family of pairwise disjoint sets, that is,

$$\text{sur}(\mathcal{C}) := \sum_{v \in V} \bigl(\max\{s_v, 1\} - 1\bigr) \quad \Bigl(s := \sum_{C \in \mathcal{C}} \chi_C\Bigr), \ \chi_C \in \{0,1\}^V.$$

Theorem 4.3. *Given a digraph with a cyclic order, the minimum cardinality of a cyclic FS is equal to the maximum of* $\text{ind}(\mathcal{C}) - \text{sur}(\mathcal{C})$, *where $\mathcal{C}$ is a set of circuits.*

Indeed, since we added the inequalities $-x \geqslant -1$ to the inequalities (CycFeed), $x \geqslant 0$, a dual solution for the objective function $w := \underline{1}$ in (8) consists of a multiset $\mathcal{C}$ of circuits and a (nonnegative) dual variable y_v for each inequality $-x_v \geqslant -1$ in such a way that the corresponding combination of the coefficient vectors is at most 1 in each variable. Clearly, in an optimal solution y_v is exactly the surplus of $\mathcal{C}$ comparing to the all 1 function, that is, $y_v = \max\{s_v, 1\} - 1$ ($v \in V$) as claimed.

Note that in the above example if $\mathcal{C}$ consists of the two circuits of G, we have $s_v = 2$, and $s_u = 1$ for all the other vertices u. (Some readers may find it useful to check the easy inequality min $\geqslant$ max in Theorem 4.3 directly.)

Most of the proof of Theorem 2.1 can also be generalized to the blocking case: a "best" distance function can be defined, and is worthwhile to be studied. However, its combinatorial interpretation is missing, and no appropriate modification of compatibility is known.

As for now, it is more interesting to deduce corollaries for cyclic feedback arc-sets.

4.2. Arc-feedback

Theorem 4.4. *Given a digraph with a cyclic order, the minimum cardinality of a cyclic feedback arc-multiset is equal to the maximum winding* $\text{ind}(\mathcal{C})$ *of a set $\mathcal{C}$ of pairwise arc-disjoint circuits.*

As in the vertex-case, this minmax theorem—and the polynomial algorithm that finds the common value of the maximum and the minimum—is an immediate corollary of a more general theorem—and of its proof:

(9) *If $G = (V, A)$ is a digraph given with a cyclic order, then for any integer objective function $w : A(G) \to \mathbb{Z}_+$, $a, b : A(G) \to \mathbb{N}$ ($a \leqslant b$) the linear program*

$$\min\bigl\{w^T x\colon\ x \text{ satisfies the inequalities (ArcCycFeed) and } a \leqslant x \leqslant b\bigr\}$$

has integer primal and dual optima, and they are equal.

Similarly to the previous theorems of the kind (for instance the vertex-version), "(kArcCycFeed)" can also be defined and proved to be TDI; however, we prefer to focus here on some other phenomena.

The proof method is by now probably more than clear. Yet we repeat it slightly differently, since in the arc-case the reduction to flows is more natural—even if we still have to introduce a simple auxiliary digraph to encode the bound-constraints:

Proof of Theorem 4.4. Open the given cyclic order arbitrarily, fix the resulting linear order. Furthermore, subdivide each $e \in A$ with a new vertex v_e into a path of length two (serial extension), and denote these two arcs by e_1, e_2. Let $A_1 := \{e_1\colon e \in A\}$, $A_2 := \{e_2\colon e \in A\}$. Add now a parallel copy $\overrightarrow{e}$ of e_1, and an inverse parallel copy $\overleftarrow{e}$ of e_1, and let $\overleftrightarrow{A} := \{\overrightarrow{e}\colon e \in A\} \cup \{\overleftarrow{e}\colon e \in A\}$. We thus defined the digraph $\bar{G} := (\bar{V} := V \cup \{v_e\colon e \in A\}, \bar{A} := A_1 \cup A_2 \cup \overleftrightarrow{A})$.

Define capacity- and cost-functions on $\bar{G}$:

- Define a lower and upper capacity-function $l, u : \bar{A} \to \mathbb{Z}_+$ as follows: for all $e \in A$, $l(e_1) := u(e_1) := w(e)$, and $l(e_2) := 0$, $u(e_2) := \infty$ for all $e_2 \in A_2 \cup \overleftrightarrow{A}$.
- Define the cost function $c(e_2) := -1$ if $e \in A$ and e is a backward arc, and the cost of every other arc of $A_1 \cup A_2$ is 0.
- $c(\overrightarrow{e}) := b(e)$, $c(\overleftarrow{e}) = -a(e)$ for all $e \in A$.

Now we use the fact that there exists an integer minimum cost flow (circulation) which can again be deduced from basic flow theory, and can be found in polynomial time. Let f be an integer minimum cost flow with this data which again is a multiset $\mathcal{C}$ of circuits (the elements are circuits, and each circuit has an integer multiplicity) providing a dual optimum in exactly the same way as for Theorem 1.2.

Again the auxiliary digraph associated with the optimal flow f can be determined and (3) can be applied: a potential π exists in the auxiliary digraph, and for $e = vw \in A$ define

$$x_e := \pi(v_e) - \pi(v).$$

Let us check that x satisfies (ArcCycFeed):

$$\sum_{e \in AC} x_e = \sum_{e=vw \in AC} \bigl(\pi(v_e) - \pi(v)\bigr) = -\sum_{vw \in AC} \bigl(\pi(w) - \pi(v_e)\bigr) \leqslant \sum_{uv \in AC} c(v_e w) = \operatorname{ind}(C).$$

Let us finally check that the upper and lower bound constraints are satisfied for x_e ($e = vw \in A$). Since there is no upper capacity on $\overrightarrow{e}$ and $\overleftarrow{e}$, their costs in the auxiliary digraph are also $b(e)$ and $-a(e)$, respectively. Therefore $x_e = \pi(v_e) - \pi(v) \leqslant b(e)$, and $-x_e = \pi(v) - \pi(v_e) \leqslant -a(e)$. $\square$

Again, two circuits of winding 2 each, with one common arc provide a counterexample to writing "set" instead of "multiset" in Theorem 4.4. However, applying the theorem to a identically 0, and b identically 1 we get a minmax theorem for cyclic feedback arc-sets:

We define now $\operatorname{surarc}(\mathcal{C})$ to be the surplus of $\mathcal{C}$ comparing to a family of pairwise arc-disjoint sets, that is, for the digraph $G = (V, A)$,

$$\operatorname{surarc}(\mathcal{C}) := \sum_{e \in A} \bigl(\max\{s_e, 1\} - 1\bigr) \quad \left(s := \sum_{C \in \mathcal{C}} \chi_C\right), \ \chi_C \in \{0, 1\}^A.$$

Theorem 4.5. *Given a digraph with a cyclic order, the minimum cardinality of a cyclic FAS is equal to the maximum of* $\mathrm{ind}(\mathcal{C}) - \mathrm{surarc}(\mathcal{C})$*, where* $\mathcal{C}$ *is a set of circuits.*

The proof is easy from (9) and fully analogous to that of Theorem 4.3, we omit it.

These are the facts that can be handled by analogy with vertex-feedback sets. However, in the arc-case the *complementation* mentioned in the introductory part of this section leads to new algorithmic relations, to a relation with a stronger definition of feedback, and to the antiblocking case treated in the previous sections.

4.3. Sets of backward arcs

Feedback arc-sets, cyclic feedback arc-sets and arc-sets that are backward (or forward) arcs in an equivalent cyclic order, are three families of sets that turn out to form a chain of inclusions where equality does not necessarily hold. In this subsection we wish to make clear how to optimize on the latter two sets, that are particular feedback arc-sets.

We first state the arc-variant of the antiblocking relation of the first sections:

$$x(AC) \leqslant \mathrm{ind}(C), \quad \text{for every circuit } C \subseteq V(G),\ x \geqslant 0. \tag{arcBT}$$

Statements analogous to those of the previous sections and mainly Theorem 0.2, (4), (5) and Theorem 1.2 can be straightforwardly adapted to the arc-versions. The adapted versions can be reduced to the vertex-versions, but since the proofs work with arc-capacities such an approach is doubly-twisted. We sketch a direct proof instead:

Let us reformulate to arcs only the two last mentioned, most general statements (5) and Theorem 1.2, we use only these.

(10) *Suppose* $G = (V, A)$ *is a strongly connected digraph given with a compatible cyclic order, and* $F \subseteq A$*. Then there exists an equivalent cyclic order in which all arcs of* F *are backward arcs if and only if for every circuit* C:

$$|F \cap AC| \leqslant \mathrm{ind}(C).$$

Indeed, a subset of backward arcs in an opening of an equivalent cyclic order intersects every circuit in at most as many arcs as its winding. The converse can be proved like (5), or reduced to it as follows:

Let $F \subseteq A$, $|F \cap AC| \leqslant \mathrm{ind}(C)$ for every circuit C. Subdivide every arc $a \in A$ with a new vertex v_a, and for all $x \in V$ insert the set $\{v_a\colon a = xy \in A\}$ immediately succeeding x cyclically, to get a compatible cyclic order of the extended vertex-set $\hat{V}$. Because of compatibility, in any equivalent cyclic order of $\hat{V}$, *the vertices* x, v_a, y *follow one another in this order* (*cyclically*). (Indeed, if not, then a circuit containing x, v_a, y would wind at least twice.)

Clearly, the winding of circuits remains unchanged, and therefore, according to (5) the set $V_F := \{v_f\colon f \in F\}$ is a cyclic stable set.

The restriction to V of any equivalent cyclic order of $\hat{V}$ is clearly an equivalent cyclic order of V. Since V_F is a cyclic stable set, there exists an equivalent cyclic order of $\hat{V}$ starting with V_F. But as emphasized at the introduction of the new vertices, in any equivalent cyclic order, and for any $a = xy \in A$, the order of the vertices x, v_a, y remains unchanged, that is, v_a is followed by y and y is followed by x. Therefore, if we restrict the cyclic order starting with V_F to V, all arcs of F are backward arcs, as claimed.

Theorem 4.6. *Given a strongly connected digraph arc-weighted with nonnegative weights and endowed with a compatible cyclic order, the maximum weight of backward arcs among openings of equivalent cyclic orders is equal to the minimum of* $\operatorname{ind}(\mathcal{C})$ *among arc-w-covers $\mathcal{C}$.*

This statement is trivially equivalent to its cardinality special case through replication, which is proved in [5] as a third minmax theorem on the "maximum sets of backward arcs." The proof in [5] reduces the problem to Theorem 0.2—itself referring to Dilworth's theorem, whereas we insist on the additional information and simplicity provided by (10) and network flows, leading also to the solution of the minimization problems, which is our ultimate goal.

Proof of Theorem 4.6. Consider an arbitrary opening of an arbitrary equivalent cyclic order. If $\mathcal{C}$ is an arbitrary arc-w-cover, then $\operatorname{ind}(\mathcal{C})$ counts every backward arc a at least $w(a)$ times, so the total weight of backward arcs is at most $\operatorname{ind}(\mathcal{C})$.

In order to prove equality for some cyclic order, use that the given order is compatible, and use (10): incidence vectors of sets of backward arcs of equivalent cyclic orders are exactly the (inclusionwise maximal) 0–1 solutions of (arcBT): indeed, characteristic vectors of sets of backward arcs obviously satisfy (arcBT) (easy part of (10)); conversely, if a 0–1 vector x satisfies (arcBT), then according to (10) it is the subset of the backward arcs in some equivalent cyclic order.

For nonnegative weight functions the maximum weight integer solutions of (arcBT) can be supposed to be inclusionwise maximal, and because of compatibility to be 0–1; therefore, they are equal to the set of backward arcs in some equivalent cyclic order. On the other hand, according to (11) below (arcBT) has also an integer dual solution of the same value, finishing the proof. □

Arcs that are not contained in any circuit do not play a role in (arcBT), and it is clear that in the absence of such arcs (arcBT) is bounded, and feasible. If every arc of a graph is contained in a circuit, then all (undirected) components of it are strongly connected, therefore we can restrict ourselves to strongly connected graphs.

(11) *If $G = (V, A)$ is a strongly connected digraph given with a cyclic order, then for any integer objective function $w : A(G) \to \mathbb{Z}_+$, the linear program*

$$\max\{w^T x:\ x \text{ satisfies the inequalities (arcBT)}\}$$

has integer primal and dual optima, and they are equal.

Again, from the integrality of the dual solution for all w the primal integrality follows by results of Edmonds and Giles; on the other hand, since we handle arcs, a combinatorial proof of all the assertions is more direct than ever, it does not even contain the usual gadgets:

Open the given cyclic order arbitrarily, fix the resulting linear order and let the cost of backward arcs e be $c(e) := 1$, and the cost of all other arcs be 0.

Define capacity- and cost-functions on G:

- Do not define upper capacities (u is identically ∞).
- Define a lower capacity-function $l : A \to \mathbb{Z}_+$ to be $l(e) := w(e)$ for every $e \in A$.

Now use Hoffman's circulation theorem to get an integer optimal dual solution, and (3) to get a potential, and from the potential differences define an integer primal solution, proving the assertion.

We turn now to the optimization problem on the family of backward arcs in equivalent cyclic orders. We first translate to the arc-case the "complementation–inversion" idea of (6) and (7). The arc-version of (6) is immediate again:

(12) $x \in [0,1]^A$ *satisfies* (ArcCycFeed) *if and only if* $1 - x$ *satisfies* (arcBT) *of the inverse order.*

We now translate (7), and the translator is (12):

(13) *Let $G = (V, A)$ be a digraph given with an inverse-compatible order. Then $F \subseteq A$ is a cyclic FAS if and only if F contains all backward arcs of an equivalent cyclic order.*

Indeed, because of (12) and using (10) for the compatible inverse order: F is a cyclic FAS if and only if the arcs in $A \setminus F$ are backward arcs in an order equivalent to the inverse order, that is, F contains the set of forward arcs in this order, as claimed. (We also used that the operations of taking an equivalent order and taking the inverse commute.)

There are three notions—FAS, cyclic FAS, and the family of backward arcs in equivalent orders—whose simultaneous presence in this section might be confusing. To compare these three notions precisely, let us consider an example of Bessy [4]:

Define $G = (V, A)$ with $V = \{1, 2, 3, 4\}$, $A = \{12, 21, 21, 23, 34, 43, 43, 41\}$, and cyclic order $1, 2, 3, 4$. (21 and 43 have two parallel copies—we avoid weights for the simplicity of the description.) This order is compatible.

This example was given by Bessy in order to show that minimum number of backward arcs among openings of equivalent cyclic orders is not necessarily equal to the maximum winding of arc-disjoint circuits. Indeed, no elementary change can be realized, that is, the equivalence class of cyclic orders contains only the given cyclic order; in all the four openings of this cyclic order either 21 or 43 (or both) are backward arcs, and the circuit 1234 has also a backward arc. Therefore, the minimum in question is 3. On the other hand, all circuits are of winding 1; there are two arc-disjoint digons, but all other pairs of circuits intersect. Therefore the maximum winding of arc-disjoint circuits is only 2.

Note that by Theorem 4.4 there must exist a feedback arc multiset of size 2. Indeed, the set $\{12, 34\}$ is actually a cyclic FAS of size 2: it meets every circuit, so it is a FAS, and since the windings are 1 it follows that it is also a cyclic FAS. However, this set is not the set of backward arcs of an equivalent linear order. According to (13), cyclic feedback arc-sets are such sets provided the order is inverse-compatible, but the order $(4, 3, 2, 1)$ is not compatible: arc 23 is not contained in a circuit of winding 1. Never mind: the expected minmax theorem holds for a family of sets *containing* the family of backward arcs, and still contained in the family of feedback arc-sets.

Let us denote the family of arc-sets that are the backward arcs in an equivalent cyclic order by $\mathcal{B}$, the family of cyclic feedback arc-sets by $\mathcal{Q}$, and the family of all feedback arc-sets by $\mathcal{F}$. Clearly, $\mathcal{B} \subseteq \mathcal{Q} \subseteq \mathcal{F}$. In the above example $\mathcal{B} \subset \mathcal{Q} = \mathcal{F}$. For reverse compatible orders $\mathcal{B} = \mathcal{Q} \subseteq \mathcal{F}$. As a conclusion $\min\{|Q|\colon\ Q \in \mathcal{Q}\}$ is a better approximation for the minimum FAS than $\min\{|B|\colon\ B \in \mathcal{B}\}$, its elements are still feedback arc-sets, and the minimum can be found in polynomial time.

Minimizing $\min\{|B|\colon\ B \in \mathcal{B}\}$ is still interesting, especially in the context of [5], where the maximum of this quantity is determined only for compatible orders, and therefore the minimum, only for reverse compatible orders, by complementation.

However, as Attila Bernáth noticed, besides the two types of inequalities—"$\leqslant$" or "$\geqslant$" in (BT) or (CycFeed) respectively—equalities can also be required and the same results will hold. Let (AB) denote the system $x(C) = \mathrm{ind}(C)$ for every circuit C, $x \geqslant 0$. According to [15] a TDI system remains TDI by replacing any subset of inequalities by equalities: it follows that (AB) is also box-TDI. Putting together this observation and the result of [7] stating that two cyclic orders are equivalent if and only if the winding of every circuit is the same in the two, we get the following: (AB) intersected with the box $0 \leqslant x \leqslant 1$ has integer vertices, and these are exactly the sets $\{|B|\colon\ B \in \mathcal{B}\}$; therefore we can minimize on this family as well (with min cost flows, combinatorially), in polynomial time, without requiring compatibility.

If F is a FAS, then $G - F$ is acyclic, and the corresponding linear order defines a cyclic order for which $\min\{|Q|\colon\ Q \in \mathcal{Q}\} = \min\{|F|\colon\ F \in \mathcal{F}\}$. Therefore, finding a cyclic order with a minimum number of backward arcs is NP-hard. On the other hand, it remains a challenge to find a cyclic order for which $\min\{|Q|\colon\ Q \in \mathcal{Q}\}$ is a good approximation of the *arc-feedback* $\min\{|F|\colon\ F \in \mathcal{F}\}$ of the digraph.

For the moment we should be satisfied with a feedback arc-set F which has the following advantageous properties in the compatible linear orders of $G - F$ (which are all equivalent): it is an inclusionwise minimal FAS and so are all the sets of backward arcs of all cyclic shifts; the number (or weight, which again is not essentially more general) of backward arcs is minimum in F; the cardinality (respectively weight) of F is minimum among cyclic feedback arc-sets.

(0) Choose an arbitrary inclusionwise minimal FAS F.
(1) Run the algorithm resulting from the proof of (2)—see thereafter—starting with F.
(2) Solve (CycFeed) for the resulting cyclic order, that is, determine a minimum cardinality cyclic FAS F' with the algorithmic proof of (9). Since F is also a cyclic FAS, $|F'| \leqslant |F|$.
(3) If $|F'| < |F|$ GOTO 1 with $F := F'$, otherwise STOP.
Several variants of the stopping rule can be imagined: continuation might be conditioned by the occurrence of a proper subset in "(1)," or by whether or not F' is the set of backward arcs in an equivalent cyclic order (which can be tested according to [7]).

We cannot prove any approximation guarantee.

4.4. Integer decomposition for the arc-feedback

Completely analogously to (kBT) and to Theorem 1.2 the box-TDI-ness of the analogously defined (karc-BT) can be established. Again, this can be proved either directly, or reduced to Theorem 1.2. The same holds about Theorem 2.1. This is one way of settling statement (b) of the following theorem. We follow, another, shorter way through the following assertion that provides also (c) and (d):

For basics about polyhedra, we recommend [15]. Recall informally that we get the *faces* of a polyhedron by replacing an arbitrary subset of inequalities by equalities.

(14) *Suppose $P \subseteq \mathbb{R}^n$ has the integer decomposition property. Then $CP := \{\underline{1} - x\colon\ x \in P\}$, and every face of P also has the integer decomposition property.*

Indeed, let $k \in \mathbb{N}$, and $z \in kCP$ an integer vector. Then by definition, $z = z(1) + \cdots + z(k)$, where $z(i) \in CP$ (not necessarily integer). So $\underline{k} - z = \underline{1} - z(1) + \underline{1} - z(2) + \cdots + \underline{1} - z(k) \in kP$, and $\underline{k} - z$ is an integer vector. Since P has the integer decomposition property, $\underline{k} - z$ is the sum of

k integer vectors in P, let these be $p(1), \dots, p(k) \in P$. But then $1 - p(i) \in CP$ $(i = 1, \dots, k)$, and $z = \underline{k} - (\underline{k} - z) = \underline{k} - (p(1) + \dots + p(k)) = \sum_{i=1}^{k}(\underline{1} - p(i))$, that is, z is the sum of k integer vectors in CP, as claimed.

In order to prove the statement about the faces, let FP be a face of P, and $z \in kFP \subseteq kP$ an integer vector. Since P has the integer decomposition property, $z = z(1) + \dots + z(k)$, where $z(i) \in P \cap \mathbb{Z}^n$; if $a^T x \leqslant b$ is a valid inequality for P satisfied with equality by all vertices of FP, then

$$kb = a^T z = a^T z(1) + \dots + a^T z(k) \leqslant kb,$$

and the equality follows throughout, that is, $z(i) \in FP \cap \mathbb{Z}^n$. $\square$

We are now ready to summarize the different cases of the IDP that follow:

Theorem 4.7. *Let $G = (V, A)$ be a strongly connected digraph given with a cyclic order.*

(a) *The polytope of feasible solutions of* (BT) *has the integer decomposition property provided the given order is compatible*; *in other words, under the condition of compatibility the family of cyclic stable sets of a fixed equivalence class of cyclic orders has the integer decomposition property.*
(b) *The polytope of feasible solutions of* (arcBT) *has the integer decomposition property provided the given order is compatible*; *in other words, under the condition of compatibility the family of backward arcs of a fixed equivalence class of cyclic orders has the integer decomposition property.*
(c) *The polyhedron* (CycFeed) *has the integer decomposition property, provided the given order is inverse-compatible.*
(d) *The polyhedron* (ArcCycFeed) *has the integer decomposition property, provided the given order is inverse-compatible.*

Proof. Statement (a) is the same as Corollary 2.2. In order to prove (b) subdivide every arc with a new vertex and insert the new vertices into the order like in the proof of (10). As argued there backward arcs are exactly the cyclic stable sets of the original digraph that do not contain any vertex from V (only from $\{v_a\colon a \in A\}$). This means exactly that (arcBT) is a face of (BT) in the defined auxiliary digraph (defined by the equalities $x_v = 0$ $(v \in V)$), and then it has the IDP by (14). Now (c) follows from (a) and (d) from (b) from (6), (12) respectively, using (14). $\square$

Combinatorial corollaries could be stated for all these on the traces of Section 3.

5. An account of the corollaries

In the preceding sections we proved various minmax theorems—whose number could be completed to approximately 2^5 by making five possible mostly independent "yes–no choices":

(1) Choice between the two elements of an antiblocking or of a blocking pair.
The minmax theorems of Sections 1 and 2 correspond to the existence of integer primal and dual solutions of an antiblocking pair, or in Section 4 of a blocking pair.
(2) Choice between "$k = 1$" (max stable set problem) or "$k > 1$":
In Section 3 a common generalization of the results on maximum cyclic stable sets and optimal coloring has been proved: a minmax theorem for the max union of k stable sets.

(3) Choice between antiblocking and blocking:
In Section 4 the antiblocking relation is replaced by the blocking relation.
(4) Choice between vertex- and arc-versions.
(5) Choice between arbitrary digraphs and transitive acyclic digraphs.

While the previous sections show at least one example of each of the two choices in (1)–(4), transitive acyclic digraphs (see "(5)") have not yet been treated:

Dilworth's theorem is the assertion that in transitive acyclic digraphs the maximum size of a stable set is equal to the minimum size of a clique-cover. For a proof through matchings see for instance [9], or [15].

There are two ways for deducing Dilworth's or Greene and Kleitman's theorems from results in this paper—it is worth to know both, since their application to more general questions provides different answers.

Let $G = (V, A)$ be an acyclic digraph with a connected underlying digraph.

First, let S be the set of sources (vertices of indegree 0) in G, and T the set of sinks (outdegree 0), $S \cap T = \emptyset$. Add the arcs $\{(t, s)\colon\ t \in T,\ s \in S\}$ to the digraph.

Here is the second reduction: add two vertices, s and t to V. Let ss' be an arc for all sources s' of G, and let $t't$ be an arc if t' is a sink. Finally, add the arc ts.

Applying any of these to a compatible order of the original acyclic digraph, we get a compatible order, the graph becomes strongly connected and the results can be applied. One gets corollaries for posets in this way (including Greene and Kleitman's theorem). However, these belong to a previous generation of results: they do not need the invariance of cyclic orders under elementary operations. Cameron and Edmonds' work [6]—as I learnt recently from Irith Hartman—uses circulations in a similar way, generalizing results on posets in another direction: they establish the total dual integrality of systems of inequalities where the rows of the coefficient matrix are circuits, and the right-hand side is the value of a linear function on characteristic vectors of circuits. They call these "co-flow polyhedra." Our results use the invariance properties of cyclic orders that "lift" network flows to the level of cyclic orders, where they can be used to answer questions related to other combinatorial objects like stable sets and feedback. The results establish that (kBT), (kCycFeed) and their arc-versions are all co-flow polyhedra through the operation of opening cyclic orders!

Why do not we state all consequences? Why did not we deduce the arc-variants of the results in Section 3? Or the minmax corollary of these—for instance the one concerning the minimum union of k feedback arc-sets .... The number of minmax theorems would have grown too high: "(1)–(5)" provide 2 choices each, which can be made almost independently of one another—this represents about 2^5 choices, and approximately the same number of minmax theorems. Not all of these are interesting, but some of them might be, and to explore these with their possible applications can be the subject of further work.

Acknowledgments

Many thanks are due to Jean Fonlupt and Kathie Cameron for recalling Greene and Kleitman's theorems including their polyhedral interpretation in a moment of need, to Kathie also for communicating me Berge's conjecture. I got further information about Berge's and Linial's conjectures from Irith Hartman, updating the related references. I am indebted to Attila Bernáth for a very careful proof-reading, and to Zoli Szigeti for reading and re-reading parts that stubbornly resisted to become clear.

References

[1] S. Baum, L.E. Trotter, Integer rounding and polyhedral decomposition for totally unimodular systems, in: Lecture Notes in Econom. and Math. Systems, vol. 157, Springer, Berlin, 1978, pp. 15–23.
[2] C. Berge, k-Optimal partitions of a directed graph, European J. Combin. 3 (1982) 97–101.
[3] S. Bessy, Stabilité et décomposition en circuits d'un digraphe, thèse pour l'obtention du diplôme de doctorat, Université Claude Bernard, Lyon I, December 2003.
[4] S. Bessy, personal communication.
[5] S. Bessy, S. Thomassé, Spanning a strong digraph by α circuits: A proof of Gallai's conjecture, in: Bienstock, Nemhauser (Eds.), Integer Programming and Combinatorial Optimization, vol. 10, New York, 2004.
[6] K. Cameron, J. Edmonds, Coflow polyhedra, Discrete Math. 101 (1992) 1–21.
[7] P. Charbit, A. Sebő, Cyclic orders: Equivalence and duality, Combinatorica, in press.
[8] J. Edmonds, R. Giles, A min–max relation for submodular functions on graphs, in: Studies in Integer Programming, in: Ann. Discrete Math., vol. 1, 1977, pp. 185–204.
[9] A. Frank, On chain and antichain families of a partially ordered set, J. Combin. Theory Ser. B 29 (1980) 251–261.
[10] D.R. Fulkerson, Antiblocking polyhedra, J. Combin. Theory Ser. B 12 (1972) 50–71.
[11] T. Gallai, Problem 15, in: M. Fiedler (Ed.), Theory of Graphs and Its Applications, Czech Acad. Sci., Prague, 1964, p. 161.
[12] C. Greene, D.J. Kleitman, The structure of Sperner k-families, J. Combin. Theory Ser. A 20 (1976) 41–68.
[13] I.B.-A. Hartman, Berge's conjecture on directed path partitions—a survey, The Volume in Memory of Claude Berge, in press.
[14] N. Linial, Extending the Greene–Kleitman theorem to directed graphs, J. Combin. Theory Ser. A 30 (1981) 331–334.
[15] A. Schrijver, Combinatorial Optimization, Springer, 2003.
[16] X. Zhu, Circular chromatic number: A survey, Discrete Math. 229 (1–3) (2001) 371–410.