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On the geodesic-structure of graphs: a polyhedral approach to metric decomposition

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In this paper we show the connection to Polyhedral Combinatorics of Graham and Winkler's powerful "Metric Decomposition Theorem" (we also provide a new proof of it via polyhedral combinatorics), we sharpen it in the bipartite special case, and exhibit the connection of the results to multiflow theory.

Distance functions of graphs, and mainly those of bipartite graphs play an important role in the theory of multiflows. Namely, the following two questions are of primary interest:

1. What are the non-negative length functions on the edges of a graph according to which all minimum cardinality paths between any two vertices are also shortest ?
2. When does this set of length functions consist only of the constant function ?

In polyhedral terms these questions are equivalent to asking for the extreme rays of the *metric cone*—the set of those non-negative weights on pairs of vertices which satisfy the triangle inequality—which lie on the same face as the distance function of the graph. These questions were initiated by earlier works of Papernov, Avis and Lomonosov, where some partial answers have also been given.

In this paper we provide a complete combinatorial answer to these questions. The main result is that in the bipartite case the only extreme rays are those provided by Graham and Winkler's decomposition, and these extreme rays also correspond to bipartite graphs.

First we prove a general answer to the polyhedral question, which, however, does not yet provide a satisfactory combinatorial structure in the interesting special cases. This is used then to prove the following results:

- To deduce Graham and Winkler's theorem about the decomposition of distance functions of graphs as a sum of *integer* metrics, and to show the relation of this decomposition to polyhedral combinatorics.
- To show that in the important *bipartite* special case the Graham-Winkler decomposition is unique also among the fractional relaxations of the problem, and *summands are metrics "induced by" bipartite graphs*.

We finally show the connection of the results to theorems and methods concerning multiflows.

Key words: Distances, decomposition, metric cone, primitive metrics, geodesics.

1. Introduction

Let $G = (V, E)$ be a graph, and $w : E(G) \longrightarrow \mathbb{R}_+$. A path P is a *geodesic* of G, if it is the shortest (minimum cardinality) path between its two endpoints. It is a *w-geodesic* , if it is a shortest path according to the weight function w. (A geodesic is a $\underline{1}$-geodesic. $\underline{1}$ denotes the identically 1 function on the edges. $P(a, b)$ denotes the subpath of P joining a and b. An

(a,b)-path is a path with endpoints a and b. $w(P)$ denotes the sum $\sum_{e\in P} w(e)$.) Let $\mathcal{P}(G,w)$ denote the set of w-geodesics of G.

A function $\mu : V \times V \longrightarrow \mathbb{R}_+$ is called a *metric* on V, if it satisfies the triangle inequality

$$\mu(x,y) + \mu(y,z) \geq \mu(x,z), \qquad \text{for all } x,y,z \in V.$$

We shall suppose in addition that μ is a symmetric function: $\mu(x,y) = \mu(y,x)$. A metric will be called *integer*, if it is an integer valued function.

If G is a graph, $E(G)$ and $V(G)$ will denote its edge-set and vertex-set respectively. Paths and circuits are considered to be sets of edges. Any weight function $w : E(G) \longrightarrow \mathbb{R}_+$ *induces* a metric $\lambda_{G,w}$ on V : $\lambda_{G,w}(x,y) :=\min\{\sum_{e\in P} w(e) : P \text{ is an } (x,y) \text{ path}\}$. $\lambda_G := \lambda_{G,\underline{1}}$ will be called the *distance function* of G.

Clearly, $\mathcal{P}(G,w) = \mathcal{P}(G,\lambda_{G,w})$, whence for studying $\mathcal{P}(G,\underline{1})$ we can restrict ourselves to metrics, and even to the projections of metrics to the edge-set of a graph:

If G is a graph, a function $\mu : E(G) \longrightarrow \mathbb{R}_+$ is called a *metric on* G, if

$$\mu(f) \leq \mu(C \setminus \{f\}) \quad \text{for every circuit } C \text{ and each} f \in C.$$

Proposition 1. *Let $\mu : E(G) \longrightarrow \mathbb{R}_+$. The following statements are equivalent:*

(i) *μ is a metric on G.*
(ii) *For every $e \in E(G)$ the one-edge path $\{e\}$ is a μ-geodesic.*
(iii) *μ is the projection (restriction) to $E(G)$ of some metric on V.*

Indeed, each statement implies trivially the one that follows cyclically. For example, if (ii) holds, then the distance function $\lambda_{G,\mu}$ (induced by the edge-weights μ) extends μ. $\lambda_{G,\mu}$ is a metric on V, and (iii) is proved.

Our main concern is to characterize the functions $w : E(G) \longrightarrow \mathbb{R}_+$ for which $\mathcal{P}(G,w) \supseteq \mathcal{P}(G,\underline{1})$. Such a w must also be a metric on G, because edges are geodesics, therefore they are w-geodesics, and consequently Proposition 1 (ii) holds for them. We would like to understand *what is the place of the distance functions of graphs among the set of all metrics.* More concretely, we would like to describe the minimal face of the metric cone containing the distance function of a given graph.

We say that a function $w' : E(G) \longrightarrow \mathbb{R}+$ *decomposes* $w : E(G) \longrightarrow \mathbb{R}+$, if $\mathcal{P}(G,w') \supseteq \mathcal{P}(G,w)$. It is easy to see that this holds if and only if $\lambda_{G,w} - \varepsilon\lambda_{G,w'}$ is a metric on V for some $\varepsilon > 0$ (see the proof of Proposition 2 below).

If w is a metric on G, then w' is too, because every one-edge path is a w-geodesic, consequently a w'-geodesic, and by Proposition 1 it follows that w' is a metric. Thus *the edge-weightings decomposing* 1 *are automatically metrics*. In other words, *if for some metric m on V there exists $\varepsilon > 0$ so that $\lambda_G - \varepsilon m$ is a metric, then $m = \lambda_{G,w}$ for some metric w on G.*

If w and w' are integer metrics, we say that w' *integrally decomposes* w, if $w - w'$ is also a metric. A *decomposition* of an element of a (metric) cone, is a non-negative linear combination of vectors in this cone, whose result is the given element. We will always suppose that the coefficients in this linear combination are 1. This does not restrict the generality, because a scalar multiple of a metric is a metric, whence in any decomposition we can replace the vectors by their scalar multiple defined by their coefficient. A decomposition will be called *integer*, if all the summands are integer metrics.

A metric (on V or G) is called *primitive* if the only metrics which decompose it are this metric itself, and its scalar multiples. Thus, a metric is primitive if and only if it is an extreme ray of the metric cone. *The relation "w' decomposes w" is a partial order, and primitive metrics are the minimal elements according to this partial order.* A graph is called *primitive* if its distance function is primitive. Equivalently, a graph is primitive if and only if the only functions w on the edges for which $\mathcal{P}(G,w) \supseteq \mathcal{P}(G,\underline{1})$ are the constant functions. In these terms our questions can be stated as follows. *What are the primitive graphs G ? More generally, for a given G characterize the metrics decomposing* 1. The first of these questions has been raised in Papernov (1975), Lomonosov (1978,1985) and Avis (1980), where the main tools of the subject, and some partial answers have been developed. The present paper is a direct continuation of these works.

In the answers, a binary relation on the edges of G plays a determining role, the roots of which can be found in Avis's and Lomonosov's work: let us explain the "vis-à-vis" relation defined in Lomonosov (1978), (1985) and Avis (1980) which was the starting point of the present work, and which may help understanding our motivations in Section 2. With the help of this relation, a sufficient condition was given in Lomonosov (1985) for a bipartite metric to be primitive. This sufficient condition is based on the following fact:

> *If C is a circuit of even cardinality and every subpath of C of cardinality at most $|C|/2$ is a geodesic of G, then opposite edges on C must have the same weight in every weight function which decomposes $\underline{1}$.*

Indeed, let the edges of C be (in their cyclical order on C) $e_1, \ldots, e_{2k}$. Then $\{e_1, \ldots, e_k\}$ and $\{e_{k+1}, \ldots, e_{2k}\}$ are shortest paths in G between the same vertices. Thus any metric μ decomposing $\underline{1}$ satisfies $\mu(e_1) + \ldots + \mu(e_k) = \mu(e_{k+1}) + \ldots + \mu(e_{2k})$. Similarly, $\mu(e_2) + \ldots + \mu(e_{k+1}) = \mu(e_{k+2}) + \ldots + \mu(e_1)$. Subtracting these two equations from each other, we get $\mu(e_1) = \mu(e_{k+1})$.

Note that in order to deduce the equation $\mu(e_1) = \mu(e_{k+1})$ we only used two pairs of shortest paths, in other words the "vis à vis relation" is too restrictive. This remark is behind the definition of the "oppositeness" relation in Section 2, whose classes provide the Graham-Winkler decomposition in the bipartite case, and the behind the "straight decomposition" in the proof of multiflow theorems.

Let us state now the answers we give to the questions we have put.

First we prove a simple general answer to the polyhedral question valid for arbitrary graphs (Theorem 1 below).

Since the number of triangle inequalities is a polynomial of the number of vertices, with the help of the ellipsoid method any metric can be written as the non-negative combination of extreme rays of the metric cone, in polynomial time. Theorem 1 provides a restricted system of equalities and inequalities necessary and sufficient for the *projection* of a metric to the edges of the graph, to decompose the $\underline{1}$ function. It is still true but less trivial that a polynomial number of such equalities and inequalities suffice, but this will actually be irrelevant for us. What matters is the tight structure of the equalities and inequalities which plays a crucial role in the proof of the main results (Theorem 2 and 3):

It is used to prove that in the bipartite case the same problem can be solved in a combinatorial way (Theorem 2). In fact, the number of extreme rays decomposing the distance function is also polynomial in the bipartite case: the extreme rays are actually linearly independent; in fact all the extreme rays of the minimal face of the distance function can be determined in polynomial time; they are bipartite metrics (defined below) and the distance function of the graph is the sum of these extreme rays.

An equivalent formulation of our result is that the integer decomposition provided by a celebrated theorem of Graham and Winkler (1985) provides the unique way of writing the distance function of a bipartite graph as a non-negative (not necessarily integer) combination of the extreme rays of the metric cone, and all these extreme rays are induced by bipartite metrics.

Graham and Winkler's theorem (Theorem 3 below) is equivalent to the existence of a unique integer decomposition for general graphs. The summands will then not all be primitive.

To state the results more precisely we need some more definitions.

For basic notions and terminology about polyhedra and polyhedral cones we refer to Schrijver (1986).

Metrics induced by $0-1$-functions of graphs with V as vertex set will be called *graph-induced metrics* on V. (Note that in Lomonosov (1985) "hop-metrics" are exactly the distance functions of graphs, whereas now graph-induced metrics also allow 0 values. The difference is trivial (contract 0 weight edges, or equivalently, identify vertices whose distance is 0), but this flexibility will serve our convenience.) The $0-1$ metrics on G are the projections of these metrics. It is easy to see that a function $w : E(G) \longrightarrow \{0,1\}$ is a metric if and only if $V(G)$ has a partition so that $\{e \in E(G) : w(e) = 1\}$ is the set of edges whose two end-points are in different classes. (This partition is the set of components of the graph formed by the 0-weight edges.)

If $w : E(G) \longrightarrow \{0,1\}$ is a metric and in addition every circuit has even weight, then w and $\lambda_{G,w}$ will be called *bipartite metrics*. It is easy to see that w, is a bipartite metric if and only if the classes of the underlying partition have a bipartition so that the endpoints of every $e \in E(G)$, $w(e) = 1$ are in different classes of the bipartition.

Theorem 1. *Let G be a graph. A weight function* $w : E(G) \longrightarrow \mathbb{R}_+$ *is a metric and decomposes* $\underline{1}$, *if and only if for an arbitrary geodesic P, and path Q,* $|Q| \leq |P| + 1$, *between the same pair of vertices,* $w(Q) = w(P)$ *holds if* $|Q| = |P|$ *and* $w(Q) \geq w(P)$ *holds if* $|Q| = |P| + 1$.

Theorem 2. *If G is bipartite then the extreme rays of the minimal face of the metric cone containing the distance function of G are linearly independent. Moreover, each of these extreme rays is a bipartite metric, and the distance function of G is their sum.*

Uniqueness of decompositions of metrics has been studied by M. Deza et M. Laurent (1992). They call a metric, or the graph which induces it *rigid*, if it can be uniquely written as a sum of not identically zero (not necessarily integer) metrics, that is, if and only if the extreme rays of the minimal face containing it are linearly independent. They prove that many interesting classes of metrics are rigid. One of their examples to this phenomenon is the class of "hypercube embeddable graphs", a class of bipartite graphs characterized by Djoković (1973). (This characterization follows from Theorem 2 or 3.) According to Theorem 2 bipartite metrics are always rigid.

We state now Graham and Winkler's metric decomposition theorem (Graham and Winkler (1985)).

Let G_1 and G_2 be graphs and define their (Cartesian) *product* to be the graph G, with $V(G) := \{(v_1, v_2) : v_1 \in V(G_1), v_2 \in V(G_2)\}$, and $E(G) := \{(u_1, u_2)(v_1, v_2) : u_1 = v_1, u_2v_2 \in E(G_2)$ or $u_1v_1 \in E(G_1)$ and $u_2 = v_2\}$. It can be seen immediately that the Cartesian product is associative and commutative. Products which differ only in the order of the members will be considered to be the same. A graph is called *irreducible*, if it cannot be written as the product of two graphs each of which has at least one edge.

An *isometric embedding* of G_1 into G_2 is a mapping $f : V(G_1) \longrightarrow V(G_2)$ with the property that $\lambda_{G_1,\underline{1}}(u,v) = \lambda_{G_2,\underline{1}}(f(u), f(v))$ for all $u, v \in V(G_1)$.

It is easy to check that isometrically embedding a graph into the product of G_1 and G_2 is equivalent to decomposing its distance function into a sum of graph-induced metrics λ_1 and λ_2, where shrinking the vertices at distance 0 according to λ_i we get G_i ($i = 1, 2$).

Theorem 3 (Graham, Winkler (1985)). *Every graph has a unique isometric embedding into the product of irreducible graphs. These irreducible graphs can be determined in polynomial time.*

We will prove Theorems 2 and 3 simultaneously (see Theorem 5) via polyhedral combinatorics (elementary linear algebra).

Then Theorem 2 trivially implies the bipartite special case of Theorem 3. (The converse is more difficult: Theorem 3 states the uniqueness only for integer sums of graph-induced metrics; however, looking at the relations defining the partitions implicit in Theorems 2 and 3 (see their proof in Section 2), the unicity in Theorem 2 can be easily proved. On the other hand, we do not see any simpler way of proving the bipartiteness of the summands in the Graham-Winkler decomposition of bipartite graphs, than repeating our full proof !)

Let us finally collect some basic definitions and easy facts about the metric cone that will be used as trivial, without reference throughout the paper.

The set of metrics on V form a polyhedral cone in the $|V|^2$ dimensional space, which is called the *metric cone.*

Proposition 2. *Let G be a graph, and suppose $\mu : E(G) \longrightarrow \mathbb{R}_+$ is a metric. Then the following statements are equivalent*

(i) *μ decomposes* 1.
(ii) *All triangle inequalities satisfied with equality by the distances in G are also satisfied with equality by $\lambda_{G,\mu}$.*
(iii) *$\lambda_{G,\mu}$ lies on the minimal face of the metric cone containing the metric induced by G.*
(iv) *There is an $\varepsilon > 0$ such that $\lambda_G - \varepsilon\lambda_{G,\mu}$ is still a metric.*

Proof. Let us first prove that (i) implies (ii). Suppose that μ decomposes 1. A triangle equality $\lambda_G(x,y) + \lambda_G(y,z) = \lambda_G(x,z)$ means that pasting together a shortest (x,y)-path with a shortest (y,z)-path we get a shortest (x,z)-path. Since μ decomposes $\underline{1}$, the same holds for *mu*-shortest paths, whence $\lambda_{G,\mu}(x,y) + \lambda_{G,\mu}(y,z) = \lambda_{G,\mu}(x,z)$. Thus, we proved (ii) from (i). (ii) also implies (i) trivially.

(iii) is just a restatement of (ii) with different terminology, they are obviously equivalent.

Suppose now that (ii) is true, and let us prove (iv). The triangle equalities of λ_G are also satisfied with equality by $\lambda_G - \varepsilon\lambda_{G,w}$ for arbitrary ϵ, because by (ii) they are triangle equalities for $\lambda_{G,w}$. On the other hand, one can clearly choose ε to be positive but small enough so that the strict triangle inequalities of λ are also satisfied by $\lambda_G - \varepsilon\lambda_{G,w}$. (We just have to take the minimum of a finite number of positive rationals.)

Finally note that (i) is trivial from (iv). □

Corollary. *The following statements are equivalent*

(i) *G is primitive.*
(ii) *$\mathcal{P}(G,\underline{1})$ is (inclusion wise) maximal among all $\mathcal{P}(G,w)$, $w : E(G) \longrightarrow \mathbb{R}_+$.*

(iii) *If $\mathcal{P}(G,\mu)$ contains all the geodesics, then μ is constant on the edges.*
(iv) *$\underline{1}$ is not the sum of nonzero metrics.*
(v) *The distance function of G is an extreme ray of the metric cone.*

(vi) *If we consider the triangle inequalities satisfied with equalities as linear equations, they have no solution other than a constant function.*

2. Results and proofs

We will first find a generating system with advantageous combinatorial properties of the space of linear equalities satisfied by the metrics on G lying on the minimal face of the distance function of G. We also exhibit a restricted set of linear inequalities, which together with the equalities, make sure that every solution is a metric. The result we obtain is Theorem 1 (see Theorem 4 which is somewhat sharper variant put in a form convenient for later use).

Let C be an even circuit of G, $C = 2k$, and let a and b be two vertices of C. If both (a,b) paths on C are geodesics (shortest (a,b) paths) of G, then (a,b) will be called a *diametral* pair of C. (Then $\lambda_G(x,y) = k$ holds.)

If (a,b) is a diametral pair, the *diametral equation* (C,a,b) is the equation of the form $\sum_{e\in P} x(e) = \sum_{e\in Q} x(e)$, where P and Q are the two paths in C which join a and b.

The following theorem says that diametral equations constitute a generating system of the equalities describing the minimal face of a graph-induced metric, and also exhibits a set of essential inequalities of the face:

Theorem 4. *Let G be a graph, and $w : E(G) \longrightarrow \mathbb{R}_+$. The following statements are equivalent.*

(i) *w is a metric which decomposes $\underline{1}$.*
(ii) *w satisfies all diametral equations, and $w(Q) \geq w(P)$ holds for an arbitrary geodesic P and minimal non-geodesic Q with the same endpoints for which $|Q| = |P| + 1$.*

Q is a *minimal non-geodesic* if it is not a geodesic, but every proper subpath of it is a geodesic. ($|Q|$ is then 1 or 2 bigger than the shortest path between its endpoints.)

Proof. Suppose that w is a metric, and it decomposes $\underline{1}$. Let $a,b \in V(G)$, and let P, Q be two (a,b)-paths, where P is a geodesic. Since w decomposes $\underline{1}$, P is a w-geodesic, that is, $w(Q) \geq w(P)$. If Q is also a geodesic, we have for the same reason $w(Q) \leq w(P)$, that is, $w(Q) = w(P)$: the diametral equations and the inequalities of (ii) follow even more generally.

Conversely, suppose now that w satisfies (ii), and let us prove (i), that is $\mathcal{P}(G,w) \supseteq \mathcal{P}(G,\underline{1})$, which means the following:

> *if P is a geodesic and Q is a path between the same pair of vertices, then $w(Q) \geq w(P)$.*

Suppose indirectly that there exist $a,b \in V(G)$, a shortest (a,b) path P, and an (a,b)-path Q so that $w(Q) < w(P)$. Choose this counterexample so that $|Q|$ is minimum.

Clearly, P and Q are vertex-disjoint, because if they have a common vertex x, then both $P(a,x)$ and $P(x,b)$ are geodesics, and either $w(Q(a,x)) < w(P(a,x))$ or $w(Q(x,b)) < w(P(x,b))$, contradicting the minimality of the counterexample Q. Moreover Q is not a geodesic, because then $w(Q) = w(P)$ would be a diametral equation contradicting the indirect assumption.

Let $e = aa' \in Q$, $f = bb' \in Q$, and let Q_e, Q_f be shortest (a', b) and (a, b') paths respectively. If both $|Q \setminus \{e\}| = |Q_e|$ and $|Q \setminus \{f\}| = |Q_f|$ hold, then Q is a minimal non-geodesic and $|Q| = |P|+1$ or $|Q| = |P|+2$ follows. In the former case (1) contradicts the indirect assumption, whereas in the latter case both $Q \setminus \{e\}$ and $P \cup \{e\}$ are geodesics, and the diametral equation $w(Q \setminus \{e\}) = w(P \cup \{e\})$ contradicts the indirect assumption.

So $|Q \setminus \{e\}| > |Q_e|$ say. Then $|Q| > |Q_e \cup \{e\}|$, and by the minimality of the counterexample Q, $w(P) \leq w(Q_e \cup \{e\})$, $w(Q_e) \leq w(Q \setminus \{e\})$. Hence $w(P) \leq w(Q_e \cup \{e\}) \leq w(Q)$, contradicting the indirect assumption. □

Since for bipartite graphs the condition $|Q| = |P| + 1$ in (ii) cannot hold, the following surprising fact follows: in bipartite graphs we do not have to care about inequalities, an arbitrary function which decomposes $\underline{1}$ is automatically a metric. We will see (Theorem 5) that even more is true.

Let C be an even circuit, $|C| = 2k$, $e, f \in C$, and let P, Q be the two components of $C \setminus \{e, f\}$. We say that e, f are *opposite* on C, if $|P| = |Q| = k$, and $P \cup \{e\}$, $P \cup \{f\}$, $Q \cup \{e\}$, $Q \cup \{f\}$ are all geodesics of G.

In other words, $e = ab$ and $f = cd$ are opposite edges of the (even) circuit C, if and only if both (a, d) and (b, c) are diametral pairs of C. Then, subtracting from each other the equations $C(a, d)$ and $C(b, c)$ (see their definition in the beginning of this section) we get the equation $x(e) = x(f)$. We say that e and f are *opposite in* G , if they are opposite edges on some circuit of G. Let us call the classes of the reflexive and transitive closure of the oppositeness relation *primitive classes.*

For metrics occurring in integer decompositions of non-bipartite metrics, the inequalities of Theorem 4 (ii) imply some more equations related to odd circuits:

Let C be an odd circuit of G, $C = 2k + 1$, and let a, b_1, b_2 be vertices of C, $b_1b_2 \in E(C)$, where an (a, b_i) subpath P_i of $C \setminus \{b_1b_2\}$ is a geodesic of G of cardinality k for both $i = 1$ and $i = 2$. Then (a, b_1b_2) will be called a *diametral pair* of C. The *odd diametral equation* $C(a, b_1b_2)$ is defined as $x(P_1) = x(P_2)$.

Theorem 4 has the following corollary for integer decompositions:

Corollary. *A function $w : E(G) \longrightarrow \{0, 1\}$ is a metric integrally decomposing $\underline{1}$ if and only if it satisfies every diametral equation and every odd diametral equation.*

Proof. The (more difficult) "if" part is an easy consequence of Theorem 4: let $w : E(G) \longrightarrow \{0, 1\}$ satisfy the diametral and odd diametral equations; the odd diametral equations imply that the inequalities in (ii) of Theorem 4 hold for w, moreover, they imply that for w the gap between the two sides of these inequalities is at most 1; it follows that $\underline{1} - w$ also satisfies them. Thus (ii) holds for both w and $\underline{1} - w$, and then, by Theorem 4, (i) also holds, that is, both are metrics (decomposing $\underline{1}$).

Let now w decompose $\underline{1}$, and recall that w is then a graph-induced metric. According to Theorem 4 w satisfies the diametral equations. Let us prove now that it also satisfies an arbitrary odd diametral equation (C, a, b_1b_2). Let P_1 and P_2 be as in the definition above.

Suppose $\underline{1} = w_1 + \ldots + w_k$ is an integer decomposition. The summands are obviously graph-induced metrics, let the set of edges of weight 1 in w_i be E_i, $(i = 1, \ldots, k)$. Since w_i decomposes w, $w_i(P_1 \cup \{b_1b_2\}) \geq w_i(P_2)$, that is $|(P_1 \cup \{b_1b_2\}) \cap E_i| \geq |(P_2 \cap E_i|$, and because of $\sum_{i=1}^{k} |P_1 \cup \{b_1b_2\} \cap E_i| = |P_1 \cup \{b_1b_2\}| = |P_2| + 1$, $\sum_{i=1}^{k} |P_2 \cap E_i| = P_2$, we deduce that $|(P_1 \cup \{b_1b_2\}) \cap E_i| = |P_2 \cap E_i|$ for all indices with the exception of one, say $i = 1$, for which $|(P_1 \cup \{b_1b_2\}) \cap E_1| = |P_2 \cap E_1| + 1$.

In exactly the same way $|(P_2 \cup \{b_1b_2\}) \cap E_i| = |P_1 \cap E_i|$ for all indices with the exception of one, and clearly, it must be $i = 1$, for which $|(P_2 \cup \{b_1b_2\}) \cap E_1| = |P_1 \cap E_1| + 1$. But then $E_1 \supseteq \{b_1b_2\}$, and the odd diametral equation $|P_1 \cap E_i| = |P_2 \cap E_i|$ follows, that is $w_i(P_1) = w_i(P_2)$ holds for every $i = 1, \ldots, k$ (including $i = 1$). This is just the equation (C, a, b_1b_2). □

We will see (Theorem 5) that every (odd) diametral equation can be generated by equations between the edges. Oppositeness was a way of deducing such simple equations from the diametral equations. We first exhibit two more ways of doing this, which also use odd diametral equations. Then we show that the three kinds of equations generate already every (odd) diametral equation.

Let C be an odd circuit, $|C| = 2k + 1$, $e, f \in C$, and let P, Q be the two components of $C \setminus e, f$. We say that e, f are *odd opposite* on C, if $|P| = k$, $|Q| = k - 1$, and P, $Q \cup \{e\}$, $Q \cup \{f\}$ are all geodesics of G.

In other words, $e = ab \in C$ and $f = cd \in C$, (where we choose the notation so that P in the above definition is an (a, c) path and Q is a (b, d) path), are odd opposite edges on the circuit C, if and only if (a, f) and (c, e) are diametral pairs. Then, subtracting from each other (C, a, f) –which is $x(P) = x(Q \cup \{e\})$–, and (C, c, e), –which is $x(P) = x(Q \cup \{f\})$–, we get the equation $x(Q \cup \{e\}) = x(Q \cup \{f\})$, that is $x(e) = x(f)$.

A third and last situation where an equation of the type "$x(e) = x(f)$" can be deduced, this time using both diametral and odd diametral equations: let C be an even circuit, $C = 2k$, and let (a, b) be a diametral pair on C; suppose $e_a := aa', e_b := bb' \in C$, where b and b' are joined by two paths of cardinality k on C, but they are not diametral, because $\lambda_G(a', b') = k - 1$. We will say then that e_a, e_b are *mixed opposite* on C.

If P_1 denotes the (a', b') path of cardinality $k - 1$ and P_2, P_3 are the (a', b), (a, b') paths of cardinality $k - 1$ on C, then $x(P_1) = x(P_2)$ and $x(P_1) = x(P_3)$ are odd diametral equations (plus some number of diametral equations if P_1 is not disjoint from P_2 or P_3). The equation $x(P_2) = x(P_3)$ follows. Subtracting from this the equation $C(a, b)$, we get that $x(e_a) = x(e_b)$.

We say that e and f are *odd (mixed) opposite in* G, if they are odd (mixed) opposite edges on some circuit of G.

Let $e, f \in E(G)$. We define the θ relation as the transitive closure of the relation "e and f are opposite or odd or mixed opposite". The classes of this relation will be called *irreducible classes*. This relation will turn out to be the same as Graham and Winkler's "θ" relation (see Proposition 3). For bipartite graphs and only for these the θ relation coincides with oppositeness.

For later reference we collect into one statement the trivial part of Theorem 4 and of its corollary, and the remarks we had on equations between the weights of opposite and odd opposite edges:

Proposition 3. *If w decomposes $\underline{1}$, then it satisfies every diametral equation, and it is constant on the primitive classes; if w integrally decomposes $\underline{1}$, then it also satisfies the odd diametral equations, and it is constant on the irreducible classes.*

Before proceeding further, let us note that the relation of being opposite, odd opposite, or in θ relation can be characterized directly in terms of λ_G, and the set of related edges can be determined in polynomial time:

Proposition 4.

- *ab and cd are opposite edges in G if and only if $\lambda_G(a, c) = \lambda_G(b, d)$ and $\lambda_G(a, d) = \lambda_G(b, c)$, and these two numbers are different;*

- *ab and cd are odd or mixed opposite edges in G if and only if three of the numbers* $\lambda_G(a,c)$, $\lambda_G(b,d)$, $\lambda_G(a,d)$, $\lambda_G(b,c)$ *are equal, and the fourth is one smaller or one bigger respectively;*
- $e\theta f$ *for* $e = ab$, $f = cd$, *if and only if* $\lambda_G(a,c) + \lambda_G(b,d) \neq \lambda_G(a,d) + \lambda_G(b,c)$.

Indeed, the "only if" parts are trivial from the definitions, and the "if" parts are also easy: to prove the first statement we may assume $\lambda_G(a,d) = \lambda_G(b,c) < \lambda_G(a,c) = \lambda_G(b,d)$ say. Then a shortest (a,d) path is disjoint from any shortest (b,c) path, because otherwise we could switch in the intersection point from one path to the other, proving $\lambda_G(a,d)+\lambda_G(b,c) < \lambda_G(a,c)+\lambda_G(b,d)$, contradicting the inequality we assumed. For the second statement the proof is similar, and the third is just a synthesis of the first two.

This does not characterize (non-integer) decompositions and primitivity in arbitrary graphs though. However, since there is a polynomial number of triangle inequalities, this can be done with the ellipsoid method, and with some work a small basis of diametral equations can also be provided.

Graham and Winkler define their "θ" relation with the statement in Proposition 3. This form is useful for computing the irreducible classes, but the content of this relation for us remains oppositeness and odd oppositeness.

Let us call a metric *irreducible*, if it is not the nontrivial sum of integer metrics; a graph is irreducible if the $\underline{1}$ function is irreducible.

We focus now on proving Theorems 2 and 3. We know already the unicity: according to Proposition 3 the summands of an (integer) decomposition must be constant on the primitive (irreducible) classes. However, we do not see yet that the irreducible classes are metrics decomposing $\underline{1}$. The first half of the following theorem is obviously equivalent to Graham and Winkler's theorem (Theorem 3), and the second to Theorem 2:

Theorem 5. *Let G be arbitrary. The characteristic vectors of the irreducible classes of G are metrics which integrally decompose* $\underline{1}$. *If G is bipartite, then these classes are all primitive and bipartite.*

Proof. According to Theorem 4 and its corollary, it is enough to prove that

> *every diametral equation and odd diametral equation can be written as the sum of equations of the form* $x(e) = x(f)$, *where e and f are opposite, odd opposite or mixed opposite edges of G.*

Suppose not, and suppose first that the minimum of $|C|$ among the counterexamples C is even. Let (C,a,b) be this minimum counterexample, $|C| = 2k$, and let (a,b) be a diametral pair on C. Let $a_0 := b_k := a$, $a_k := b_0 = b$, and let the two (a,b) paths on C be $A := (a_0a_1, \ldots, a_{k-1}a_k)$ and $B := (b_0b_1, \ldots, b_{k-1}b_k)$.

If (a_i, b_i) is either diametral or $\lambda_G(a_i,b_i) = k-1$ for every $i = 0, \ldots, k$, then $a_{i-1}a_i$ and $b_{i-1}b_i$ are opposite or mixed opposite respectively ($i = 1, \ldots, k$), and the equation (C,a,b) is the sum of the corresponding equalities.

Otherwise let t be the smallest index i for which this does not hold. Clearly, $t > 0$.

Suppose furthermore, that a and b have been chosen so that t is minimum. Then $t = 1$, because if not, then a_0a_1 and b_0b_1 are opposite edges, and it follows that (C, a_1, b_1) is also a counterexample with $t' = t - 1$. (If it was not a counterexample, then adding $x(a_0a_1) - x(b_0b_1)$ to the left hand side of (C, a_1, b_1) and the negative of this to the right hand side of it, we get

(C, a, b).)

Thus (a_1, b_1) is not diametral, and $\lambda_G(a_1, b_1) \leq k-2$. We show that

the diametral equation (C, a_0, b_0) is the sum of diametral equations involving strictly smaller circuits,

a contradiction.

Let P be a minimum cardinality (a_1, b_1) path, and suppose the last vertex of P in A is a_p, and the first vertex of it in B is b_q. Since a subpath of a shortest path is also shortest, we can suppose that the subpath of P joining a_1 and a_p is $P(a_1, a_p) = \{a_1a_2, \ldots, a_{p-1}a_p\}$, and $P(b_q, b_1) = \{b_qb_{q-1}, \ldots, b_2b_1\}$. Let $r := |P(a_p, b_q)|$.

We have: $(p-1)+r+(q-1) = d(a_1, b_1) \leq k-2$. Let us add to P the edges a_0a_1 and b_0b_1 to get an (a, b)-path, from now on this path will be denoted by P. whence $p+r+q = |P| \geq d(a_0, b_0) = k$. Thus equality holds in all these inequalities, which can be read as $r+p = k-q$ or $r+q = k-p$.

But $r+p = |P(a, b_q)|$, and $k-q = |B \setminus P(b_q, b)|$ $(b_k = a)$. Since $B \setminus P(b_q, b)$ is a shortest path between a and b_q, a and b_q are diametral points of the circuit $C_1 := P(a, b_q) \cup B \setminus P(b_q, b)$, and (C_1, a, b_q) is a diametral equation. Similarly, (C_2, b, a_p) is a diametral equation, where $C_2 := P(b, a_p) \cup A \setminus P(a_p, a)$. Clearly, $|C_1| = 2(r+p) < 2k$, $|C_2| = 2(r+q) < 2k$, and adding up (C_1, a, b_q) and (C_2, a, b_q) we get:

$$x(P(a, b_q)) + x(B \setminus P(b_q, b)) = x(A \setminus P(a_p, a)) + x(P(b, a_p))$$

Subtracting $x(P(a_p, b_q))$ from both sides, we get the diametral equation (C, a, b), which is thus the sum of diametral and odd diametral equations of smaller circuits, as claimed.

If the minimum counterexample is the odd diametral equation (C, a, b_0b_1), where $|C| = 2k+1$, $(k \in \mathbb{Z}_+)$, we proceed similarly:

Let the numbering be now $a_0 := b_{k+1} := a$, $a_k := b_0$, and let the disjoint (a, b_0) and (a, b_1) paths on C be $A := (a_0a_1, \ldots, a_{k-1}a_k)$ and $B := (b_{k+1}b_k, \ldots, b_2b_1)$. (C, a, b_0b_1) is then the equation $x(A) = x(B)$.

If now a_0a_1 is odd opposite to b_0b_1, b_0b_1 is odd opposite to b_kb_{k+1}, b_kb_{k+1} to $a_{k-1}a_k$, …, and b_1b_2 to a_0a_1, then every edge of C is related by the transitive closure of the odd oppositeness relation, implying all odd diametral equations of the circuit (and much more). If not, we can suppose like in the first part of the proof that already a_0a_1 and b_0b_1 are not odd opposite, that is, $\lambda_G(a_1, b_1) \leq k-1$. Since $\lambda_G(a_0, b_1) = k$, $\lambda_G(a_1, b_1) \geq k-1$ also follows.

Thus $\lambda_G(a_1, b_1) = k-1$. Let P be a shortest (a_1, b_1) path.

From the minimality of our counterexample we have the odd diametral equation

$$x(A \setminus \{a_0a_1\}) = x(P),$$

and the diametral equation

$$x(P \cup \{a_0a_1\}) = x(B).$$

(If P is not disjoint from A or B we must add several diametral equations to one (odd) diametral equation to get these equations.)

Adding up these two equations $x(A) = x(B)$ follows, as claimed.

Now we prove the bipartiteness of the primitive classes if G is bipartite.

Suppose the statement is not true. Fix an arbitrary primitive class D, and let C be a minimum cardinality circuit, $|C| = 2k$, for which $|D \cap C|$ is odd.

Let us use the same method as in the first part of the above proof. It cannot be that all paths of length k on C are shortest, because then C can be partitioned into pairs of opposite edges, which belong to the same primitive class: there exist vertices a and b whose distance on C is k, but there is an (a,b) path $|P| < k$ joining them. We can suppose that only the first and last vertex of P is on C, otherwise we can take the appropriate subpath of P.

The endpoints of P partition C into two paths, which form with P two circuits C_1 and C_2, both of which have smaller cardinality than C, and $C = (C_1 \setminus C_2) \cup C_2 \setminus C_1$. Thus, by the choice of C, both $|D \cap C_1|$ and $|D \cap C_2|$ are even. But $|D \cap C| = |D \cap C_1| + |D \cap C_2| - 2\,|D \cap P|$ is also even, a contradiction which finishes the proof of the theorem. □

Corollary 1. *Let G be a bipartite graph. Then*

a. *the characteristic function χ_D of any primitive class D of G induces a primitive bipartite metric.*
b. *$w : E(G) \longrightarrow \mathbb{R}_+$ decomposes $\underline{1}$ if and only if it is constant on the primitive classes.*
c. *G is primitive if and only if it has one primitive class.*

It follows that for bipartite graphs the rank of the face containing $\underline{1}$ is the number of primitive classes.

Theorem 2 is just a variant of this corollary. Let us now state an analogous version of Graham and Winkler's theorem:

Corollary 2. *Let G be an arbitrary graph. Then*

a. *the characteristic function χ_D of any irreducible class D of G induces an irreducible metric.*
b. *$w : E(G) \longrightarrow \mathbb{Z}_+$ integrally decomposes $\underline{1}$ if and only if it is the characteristic vector of the union of some irreducible classes.*
c. *G is irreducible if and only if it has one irreducible class.*

The necessity in the corollaries follows from Proposition 3 and the sufficiency from Theorem 5.

3. Applications to multiflows

We first exhibit the relation of the results to "straight decompositions", a tool used for proving multiflow theorems. We illustrate our point on a simple proof of Okamura and Seymour's theorem (1981) on multiflows. In this application the surplus of Theorem 2 (comparing to Theorem 3) of providing a decomposition into *bipartite* metrics plays an essential role.

We only state here a special case of Okamura and Seymour's theorem, which can be easily proved to be in fact equivalent to it (see for example Frank (1990)). This special case was proved independently by Lins (1981), who stated it as a theorem on path packings on the projective plane. (The theorem we state is actually the "planar dual" of the well-known multiflow theorem, instead of path packings it packs cuts.) We try first to provide a self-dependent proof, and explain the relation to Theorem 2 (Corollary 1 of Theorem 5) afterwards:

Theorem (Lins (1981). *Let G be a bipartite graph embedded in the plane, and $C = (a_0, e_1, a_1, \ldots, a_{k-1}, e_k, a_k = b_0, f_1, b_1, \ldots, a_k, f_k, b_k = a_0)$, where $a_i, b_i \in V(G)$, $e_i, f_i \in E(G)$, $(i = 1, \ldots, k)$ is a circuit in G which bounds a face. Then there exist edge-disjoint cuts $C_1, \ldots, C_k$ so that $C \cap C_i = \{e_i, f_i\}$, if and only if all (a_i, b_i)-paths of C $(i = 1, \ldots, k)$ are shortest paths in G.*

Proof. The only if part is straightforward.

Let us suppose that C bounds the infinite face, and in the following let us call *face* only bounded faces. It can be supposed by a simple and standard trick that every face of G is a 4-cycle, and for every pair of edges e, f consecutive on some face there exists an $i \in \mathbb{Z}_+$ $(0 \le i \le k)$ so that $\{e, f\}$ is a subset of some shortest (a_i, b_i) path. (If for example the latter property does not hold for e and f, contract their non-common end-points: the distances do not change. For the former property the gadget "dual" to the "perturbation" of the edges has to be applied, check for example in Schrijver (1989a).)

Let ω be the transitive closure of the oppositeness relation on faces inside C. (That is, every face is considered except C. The classes of ω are "dual paths".) The key of this proof is the following claim.

Claim: If Q is a circuit, $a, b \in V(Q)$, and both (a, b) paths P_1, P_2 on Q are shortest paths, then the restriction of the ω relation to Q has only two element classes, each of which has one element in P_1 and one in P_2.

We prove the claim by induction on the number of faces contained in Q. If it is 1, the statement is obvious. If it is more than 1, then clearly, there exist two edges $e = ax$ and $f = xy$ adjacent on some face so that $f \notin Q$ is inside the region bounded by Q. Let P be a shortest path containing $\{e, f\}$ between two "diametral" points of C; by assumption such a path P exists. P intersects Q in a point z.

Now, z divides one of the (a, b) paths of Q into two paths, $Q(a, z)$ and $Q(z, b)$. $Q_1 := Q(a, z) \cup P(a, z)$ and $Q_2 := P(a, z) \cup Q(z, b) \cup Q(a, b)$ are circuits, and it is obvious that in Q_1 both (a, z) paths, in Q_2 both (a, b) paths are shortest paths. Since both Q_1 and Q_2 have a smaller number of faces than Q, the induction hypothesis can be applied to both. Let ω_i be the transitive closure of the oppositeness on faces inside Q_i $(i = 1, 2)$. Let $T_i(e)$ $(i = 1, 2)$ be the equivalence class containing e according to ω_i. The equivalence classes of ω are: $\{T_1(e) \cup T_2(e) : e \in P(a, z)\} \cup \{T_2(e) : e \in Q(z, b)\}$, and the claim is proved.

Apply now the claim to the circuit $Q = C$ and all pairs a_i, b_i $(i = 1, \ldots, k)$: we get that ω has no class containing two edges in the same (a_i, b_i) path on C. It follows that the classes of ω intersect C in the sets $\{e_i, f_i\}$ $(i = 1, \ldots, k)$. Each class is clearly a cut. $\square$

We described the above proof without making reference to definitions and results of the paper, but the Claim is a translation to the combinatorial language of a proof based on the following statement:

> "*every diametral equation is generated by equations of the form $x(e) = x(f)$, where e and f are opposite on faces.*"

This implies that the oppositeness relation ($= \theta$ relation because of bipartiteness) is generated by the oppositeness relation of faces. Since e_i and f_i are opposite on C, according to the claim, $e_i \omega f_i$, $(i = 1, \ldots, k)$. But the classes of ω intersect C in 0 or 2 edges, and are cuts, proving Lins's theorem.

More generally, in several other multiflow theorems, and theorems about cycle packings on compact surfaces (for example Schrijver (1989a, 1989b, 1991)) the "straight decomposition" (of the surface-dual graph) coincides with the Graham-Winkler decomposition (sometimes of an infinite graph embedded in the plane as universal covering surface).

Note that the bipartiteness of the extreme rays exhibited in Theorem 2 plays a role in the above proof of Okamura and Seymour's theorem: all irreducible classes of the graph in the proof are bipartite, moreover they are cuts. So, in this example, only Djoković's special case occurs, and the same is true for analogous presentations of theorems where the cut condition is sufficient for the existence of multiflows. However, more general bipartite metrics occur in Lomonosov (1978, 1985), Karzanov (1987, 1990), Sebő (1990), Sebő, Schwärzler (1993).

Let us mention another application, in fact of Graham and Winkler's theorem, which is related to multiflows. L. Székely has drawn our attention to a possible connection of our results to Shahrokhi and Székely's algebraic approach to multiflow problems, in which the following question came up:

Given a graph G and a subgroup Γ of its automorphisms, let $\{E_1, E_2, \ldots, E_k\}$ be the set of orbits of Γ on E. Is G Γ-*orbit-proportional*, that is, is it true for an arbitrary pair of vertices a, b, that the intersection of any shortest (a,b)-path with each orbit is minimum among all (a,b)-paths.

Clearly, G is Γ-orbit-proportional if and only if the characteristic vectors of the orbits integrally decompose the distance function of the graph. Then Graham and Winkler's theorem (Corollary 2 of Theorem 5) trivially implies that

> *G is Γ orbit proportional if and only if each orbit is the union of some irreducible classes, whence orbit-proportionality can be checked in polynomial time.*

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