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Circuit packings on surfaces with at most three cross-caps

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We prove a minimax theorem on the minimum length of specific kinds of one-sided walks on compact surfaces with at most three cross-caps. This theorem is a common generalization of a result of Schrijver about one-sided circuits on the Klein bottle (1989b), and of Karzanov (1990) about planar paths. The special cases include, besides these, new results concerning integer packings of metrics in planar graphs, or graphs embedded on the torus, which in turn, imply some fractional multiflow theorems.

We establish a blocking relation between two classes of polyhedra. For one of these, the defining system of linear inequalities has an integer dual solution for every Eulerian objective function to minimize, whereas for the other—just for the one whose dual is a path packing problem—an easy example will show that this does not hold. However, in some vertices of this polyhedron there exists an integer dual solution implying an integer multiflow theorem.

The proof we provide here to the main result uses the framework of Schrijver (1989b), and a theorem of Karzanov (1990) on planar multiflows. Besides, the proof has to deal with some new phenomena which make necessary to originate the blocking polyhedron of our specific kinds of circuits from the geometry of the surface.

Key words: Circuit packings, cut and metric packings, compact nonorientable surfaces, multiflows, disjoint paths, blocking polyhedra.

1. Introduction

For all basic definitions of topological character and facts on surfaces we refer to Lefschetz (1949), Giblin (1981), Stillwell (1980), and Schrijver (1991): in order to fit the page-limit of this volume we state only the definitions where some ambiguity could arise.

Let us denote the nonorientable surface with three cross-caps (of genus 3, Euler characteristic -1), by Σ.

A curve is a continuous function $\gamma : [0,1] \longrightarrow \Sigma$; if $\gamma(0) = \gamma(1)$ it is called a *closed curve*, if it is a bijection, it is a *simple curve*. Closed curves will be considered to be homeomorphic images in Σ of the unit circle $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, (that is, we will not need the particular point which is both a starting and endpoint). Homotopy of closed curves will always mean *free homotopy*. If it is clear from the context, we will delete the adjective "closed".

A simple closed curve on a surface either has a neighborhood homeomorphic to the cylinder, or a neighborhood homeomorphic to the Möbius-strip. In the former case the curve is called *two-sided*, in the latter case *one-sided*. (In the literature the two-sided curves are also called orientation keeping, or orientable, and the one-sided ones orientation reversing, or non-orientable.)

One of the directions of research in the field of multiflows concerns planar disjoint path problems, where any two vertices to be joined, called *terminals*, lie on the same face. The results concern the particular cases when all pairs lie on one face (Okamura, Seymour (1981)), on two faces (Okamura (1983), Schrijver(1989b)), or on three faces (Karzanov (1990)). Some corresponding "polar" –cut and metric packing– theorems were proved by Hurkens, Schrijver, Tardos (1989), Schrijver (1989a), Karzanov (1990). The faces containing both elements of a pair of terminals are often called *holes*.

We will refer to some of these theorems in the following, but the size limit of this volume does not allow to state them. (An account can be found in Frank (1988).) However, we state a variant of Karzanov's theorem (1990), which will be explicitly needed:

A symmetric function (this means $m(u,v) = m(v,u)$) $m : V \times V \longrightarrow \{0,1,2\}$ is called a *cut-metric* if there exists a partition of V into 2 parts so that $m(u,v) = 1$ if u and v are in different parts, otherwise it is 0; it is called a *2,3-metric*, if V has a partition into 5 parts $\{A_1, A_2, A_3, B_1, B_2\}$, so that $m(u,v) = 1$ if $u \in A_1 \cup A_2 \cup A_3$, $v \in B_1 \cup B_2$; $m(u,v) = 2$ if $u \in A_i$, $v \in A_j$ or $u \in B_i$, $v \in B_j$ $(i \neq j)$; and $m(u,v) = 0$ otherwise, that is if u and v are in the same class of the partition. Given a graph G two metrics m_1 and m_2 will be called disjoint, if $m_1(e) + m_2(e) \leq 1$ for every $e \in E(G)$.

$\text{dist}_G(u,v)$ denotes the distance of $u,v \in V(G)$, that is the cardinality of a shortest path between u and v.

Karzanov's theorem *Let H_1, H_2 and H_3 be the vertex sets of three faces of a planar graph G. Then there exists a family of pair wise disjoint cut metrics and 2,3-metrics $m_1, \ldots, m_k$ so that for every $u, v \in H_i$ $(i = 1,2,3)$,*

$$dist_G(u,v) = m_1(u,v) + \ldots + m_k(u,v).$$

It is easy to see that the condition of this theorem is necessary. The proof of the sufficiency is difficult, it generalizes the proof of Schrijver (1989a) in an essential way.

We will also suppose familiarity with basic graph theory, and clutters, their blocker, blocking polyhedra, binary clutter, etc., and basic statements about these. We use the definition of Lovász (1979) for *paths, circuits*, and *(closed) walks*; they will sometimes be considered to be edge-sets, where the edges of walks also have a multiplicity indicating how many times the walk "goes through" the given edge. (For the walks we will be working with, this multiplicity will be 0, 1 or 2.) If G is a graph, P is a walk in G, and $x \in \mathbb{R}^{E(G)}$, then $x(P)$ denotes the scalar product of x with the vector of multiplicities of P.

The results concerning one or two holes have been generalized to surfaces: Okamura and Seymour's theorem on planar multiflows and Lins' (1981) theorem on packing one-sided circuits on the projective plane are trivially equivalent; in order to generalize Lins'theorem to the Klein bottle Schrijver (1989b) worked out a proof technique which reduces the problem to the "polar" of Okamura's theorem, to the "metric packing theorem" of Schrijver (1989a).

The present paper adapts Schrijver's framework of a proof to Σ. However, this is not automatic:

Imagine Σ arising by the identification of the opposite vertices of each of three holes on the sphere: besides the closed curves that "go through one cross cap", the closed curves which "go through all the three of them" will also be one-sided. If we have in mind the goal of proving multiflow theorems, we have to exclude this latter type of closed curves. But then we don't have a binary clutter any more.

So one of the essential differences of the three cross-caps case comparing to the Klein bottle is that we have to understand how to state this exclusion in surface terms, then we have to study precisely the specific subclass of the remaining one-sided walks, and their blocking

polyhedron. These objects *do not form binary clutters.* Another difference is that we have to permit arbitrary closed walks instead of circuits, (although the multiplicity of an edge in the walks we are considering cannot exceed 2, and also, in the most interesting special cases only circuits occur). Furthermore, to deal with the blocking polyhedron of the walks we are studying we cannot restrict ourselves to purely combinatorial arguments: it is necessary to study some simple facts about how curves cross each other on Σ.

An edge-weighting of a graph is called *Eulerian*, if every weight is integer, and the sum of the weights is even on every cut. It is called *even on faces*, if every weight is integer, and the cycles bounding the faces have even weight. G is said to be Eulerian or even on faces, if the identically 1 function on the edges is so. Clearly, "Eulerian" is the "surface dual" notion of "even on faces", the former is a usual condition in multiflow theorems, whereas the latter is used for metric packings.

A first fact that will be useful to know about Σ is that it is homeomorphic to the *torus with one cross-cap.* So we shall feel free to switch between three cross-caps and one handle one cross-cap, considering them to be the same. The latter reflects better the homology group $H(\Sigma)$ of Σ.

$H(\Sigma)$ is isomorphic to $\mathbb{Z}^2 \times \mathbb{Z}_2$. This will not be explicitly used in this paper, therefore we do not define it. Some related remarks below may give some insight but can also be easily skipped.

To *cut open* along simple closed curves means to delete them from the surface. When we cut open along a walk of G homotopic to a simple closed curve, we mean cutting open along a *simple* closed curve of Σ which is "very near" to the given walk. (This means that when we go through an edge for the second times we "go" at some small distance; if we do not make this precision, a walk can contain several (two) circuits (see below), and the surface we get after cutting open can vary.)

Let us call a simple closed curve γ *orienting*, if cutting open along γ we get an oriented surface. It follows from the results of Section 2 that on Σ an orienting curve γ is also one sided. What follows exactly is in fact that $\Sigma \setminus \gamma$ is a torus with a hole, in other words γ "goes once through the crosscap" but "is 0-homotopic on the torus". (It is easy to see that this definition depends only on the homology class of γ: exactly the simple closed curves homologic to $(0,0,1)$ have this property.)

A simple closed curve is called *separating*, if cutting open along it, we get a disconnected topological space. It is easy to see that a closed walk of a graph embedded on a surface is separating, if and only if it is the symmetric difference of faces. We *extend all the definitions to not necessarily simple curves or walks*: they will be said to have some property (for example one-sided, or orienting, etc), if they are homotopic to a simple closed curve having the same property. Note that a non-orienting one-sided walk, if it is not simple, can contain an orienting circuit. (Walks consisting of an orienting one-sided circuit and a two-sided circuit going around the thickness of the torus are non-orienting !)

We can also think of orienting one-sided curves as "going through" all the three "cross-caps" once, (no matter in what order), and of the other one-sided curves as "going through one cross-cap". In Section 2 we shall prove statements which confirm this intuition, see (2). (Since the homology group arises by "making the homotopy group commutative", and every permutation of three elements is cyclic, there is only one free homotopy class of orienting curves.)

Let now G be a graph embedded on Σ. Clearly, (closed) walks of G can be considered to be (closed) curves of Σ.

We will study *non-orienting one-sided* (closed) walks.

Let us call an edge-set of G which has a non-empty intersection with every such walk a *1-blocker*. A 0-1-2 vector, whose scalar product with every such walk is at least 2 will be called a

2-*blocker*. A 1-2-*blocker* is either a 1-blocker or a 2-blocker.

In Section 3 it will turn out that all 1-blockers and 2-blockers arise in a natural way from the geometry of Σ.

Let $\mathcal{P}$ be the set of all non-orienting one-sided walks of G. Let $\mathcal{B}_1$ be the set of minimal 1-blockers, and let $\mathcal{B}_2$ be the set of minimal 2-blockers which do not contain the union of two disjoint 1-blockers.

$$P(G,\Sigma) := \{x \in \mathbb{R}^{E(G)} : x \geq 0, x(B_1) \geq 1 \text{ for all } B_1 \in \mathcal{B}_1,\, x(B_2) \geq 2 \text{ for all } B_2 \in \mathcal{B}_2\}$$

$$Q(G,\Sigma) := \{x \in \mathbb{R}^{E(G)} : x \geq 0,\, x(P) \geq 1 \text{ for all } P \in \mathcal{P}\}.$$

Our main result is the following:

Theorem 1. *$P(G,\Sigma)$ and $Q(G,\Sigma)$ are a blocking pair of polyhedra. Moreover, for every objective function c even on faces, the linear program $\min\{c^Tx : x \in P(G,\Sigma)\}$ has an integer optimal dual solution.*

In the definition of the linear program we mean the system of linear inequalities to be the system used for defining $P(G,\Sigma)$. In this paper objective functions are always meant to be non-negative.

It is easy to embed a K_5 on Σ in such a way that the edges of a triangle "go through three different cross-caps" and from among the other edges, only the edge disjoint from this triangle "goes through" a cross-cap. This is an example where the inequalities defining $Q(G,\Sigma)$ do not form a totally dual integral system: there is no integer dual solution for the everywhere 1 objective function, even if G is Eulerian. (There exists a fractional packing of non-orienting one-sided circuits of size 3, but there do not exist 3 edge-disjoint such walks.)

However, an interesting phenomenon occurs: in some vertices of $Q(G,\Sigma)$ there exists an integer dual solution for every Eulerian objective function for which the given vertex is an optimal solution. We state a general conjecture characterizing these vertices (Section 5).

The relatively richer structure of our surface makes necessary to check some simple topological statements which will determine the combinatorial structure of the problem. This preliminary knowledge will be provided in Sections 2 and 3. In Section 4 we sketch the proof of Theorem 1. In Section 5 we show some connections with metric packings and multiflows.

Finally, let us state a corollary of Theorem 1 (proved in Section 5). It is a generalization of a result of Schrijver (1989b) for two faces, and through the Farkas Lemma implies a new (fractional) planar multiflow theorem, which may have integer solutions for Eulerian graphs (see Section 5).

Theorem 2. *Let H_1, H_2 and H_3 be the vertex sets of three distinct faces of a graph G embedded to the surface of the sphere so that it is even on faces. Suppose $\{s_1,\ldots,s_r\} \subseteq H_1$, $\{t_1,\ldots,t_r\} \subseteq H_2$, where the indices are given according to the clockwise order of the vertices on the corresponding faces. Then there exists a family of pair wise disjoint cut-metrics and 2,3-metrics $m_1,\ldots,m_k$ so that for $u = s_i, v = t_i$ $(i = 1,\ldots,r)$, or $u, v \in H_3$,*

$$dist_G(u,v) = m_1(u,v) + \ldots + m_k(u,v).$$

2. Orienting and non-orienting walks

In order to have under control the specific kinds of walks we are considering, and mainly because non-orienting one-sided walks do not form a binary clutter, we have to understand some basic properties of orienting curves.

(1) *Orienting curves cross every one-sided curve of Σ.*

Indeed, suppose $\Sigma \setminus \gamma$ an orientable surface, that is it does not contain one-sided curves. A curve which does not cross γ is in $\Sigma \setminus \gamma$, so it cannot be one-sided, as claimed.

(2) *Neither of two non-crossing one-sided curves is orienting.*

Indeed, if any of them was orienting, then, according to (1) it would cross the other.

Imagine now Σ as arising by identifying opposite vertices on each of three holes of the sphere. Then the holes become one-sided curves on Σ, and according to (2) none of them is orienting. Thus curves "going once through one of the cross-caps" are not orienting. On the other hand let γ be a curve "going through each of the three arising cross-caps exactly once". This curve is obviously one-sided, and cutting Σ open along γ, it is easy to see that we get an orientable surface (the torus with a hole). Somewhat more generally:

(3) *Let γ be a separating simple curve on Σ, and suppose that one of the connected components of $\Sigma \setminus \gamma$ is homeomorphic to the Möbius strip, the other to the torus with a hole. Then the one-sided curves of this Möbius strip are orienting.*

Indeed, cutting open along a one-sided simple curve of the Möbius strip, we get a surface which arises by the identification (along γ) of the hole of a hollow torus, and of a hole of the annulus arising by cutting open the Möbius strip along our one-sided simple circuit. The result of this identification is the torus with a hole as claimed.

This statement characterizes orienting one sided simple curves: a neighborhood of such a curve is a Möbius strip containing the curve, and its border divides Σ to this Möbius strip and to a hollow torus. (We will see below, that if a simple curve is orienting, then it is one-sided, so the use of both adjectives was redundant.)

The following easy statements show the place of the orienting curves of Σ, and we will need the picture they provide:

(4) *Let γ be a two-sided simple curve on Σ. Then exactly one of the following statements is true for $\Sigma' = \Sigma \setminus \gamma$:*

(i) *Σ' is not connected, and one connected component is homeomorphic to the Möbius strip, the other to the torus with one hole.*
(ii) *Σ' is not connected, and one connected component is homeomorphic to the Klein bottle with one hole, the other to the Möbius strip.*
(iii) *Σ' is not connected, and one component is homeomorphic to the disc, the other is Σ with a hole.*
(iv) *Σ' is connected, and it is homeomorphic to the projective plane with two holes.*

Indeed, we know that Σ' either has two components with one hole each, or is connected, and has two holes. Consider now every surface with one or two holes, from which Σ arises by identifying opposite vertices of the two holes. (4) is a complete enumeration of all possible cases.

It is case (i) which is very interesting for us: according to (3) the one-sided circuits on this Möbius strip are orienting. Hence *non-orienting one-sided curves cross γ at least twice* (see (2) Section 3).

(5) *Let γ be a one-sided simple curve on Σ. Then exactly one of the following statements is true for $\Sigma' = \Sigma \setminus \gamma$:*

(i) *Σ' is homeomorphic to the Klein bottle with one hole.*
(ii) *Σ' is homeomorphic to the torus with one hole.*

Indeed, we know that Σ' has one hole. Consider all the surfaces with one hole from which Σ can arise by "putting in a cross cap".

We see from (4) and (5) that a simple curve γ is orienting if and only if (5) (ii) holds, in particular, we see from (4) that a separating curve cannot be orienting.

3. Geometric 1-2-blockers

The goal of this section is to raise 1-2-blockers in a geometric way.

Let G be a graph embedded on Σ. The 1-2 blockers we show below will all be edge-sets $B = B(\gamma)$ crossed by a curve γ of Σ or two curves γ_1, γ_2 or three curves γ_1, γ_2, γ_3 so that the curves do not go through vertices.

Geometric 1-blockers

Let us call an edge-set *binary* if it has a non-empty intersection with every one-sided walk of G. Every binary set has a non-empty intersection in particular with all non-orienting one-sided walks, so they are 1-blockers. However, we shall see examples of non-binary 1-blockers.

In other words, minimal binary 1-blockers constitute the blocker-set of one-sided walks. Since one-sided walks form a binary clutter, their blocker also.

(1) *If G is Eulerian, then every minimal binary 1-blocker has the same parity.*

Indeed, let B_1 and B_2 be two minimal binary 1-blockers. They both intersect every one-sided walk in an odd number of edges and every two-sided walk in an even number of edges. Thus the symmetric difference $B_1 \triangle B_2$ intersects every walk in an even number of edges, whence it is a cocycle. Thus $|B_1 \triangle B_2|$ is even, and the statement follows.

We exhibit now all the three examples of binary 1-blockers.

Single 1-blockers

Let γ be an orienting simple curve of Σ. Then, according to (1) (Section 2) γ intersects every one-sided curve of Σ, and so it is binary. B will then be called *single* 1-blocker.

As we have already mentioned, in the representation of Σ as a sphere with three "cross-caps", γ corresponds to a curve "going through" all the three cross-caps.

Double 1-blockers

Let γ_1 and γ_2 be two disjoint simple curves of Σ with the property that *cutting open along them we get an orientable surface*, but this is not true for either of them alone. B is then called a *double* 1-blocker. Clearly, every one-sided curve of Σ is crossed by at least one of γ_1 and γ_2. Consequently, B has then non-empty intersection with every one-sided walk of G.

Imagine Σ to be a torus with one cross cap. γ_1, γ_2 correspond then to two simple curves "homologic" on the torus (both go around the "thickness" say) and one of them is one-sided (goes through the cross-cap).

In the representation of Σ as a sphere with three identified holes these correspond to simple curves one of which goes through one of the holes, and the other goes through the other two holes.

Triple 1-blockers

Let γ_1, γ_2 and γ_3 be three disjoint simple curves of Σ with the property that *cutting open along them we get an orientable surface*, but the same is not true for any two of them. Like before every one-sided curve of Σ is crossed by at least one of them. In particular, B has then non-empty intersection with every one-sided walk of G.

In the representation of Σ as a sphere with three identified holes these three curves go through one hole each.

The only fact that we will need about these examples is the trivial statement that the objects in the spheric representation with three holes are binary 1-blockers.

These were the binary 1-blockers. The following 1-blockers are not binary.

Toroidal 1-blockers

Let γ_1 and γ_2 be two simple curves of Σ which have one common point and cutting open along both of them we get the Möbius strip. B will then be called a *toroidal 1-blocker.*

In the same way as (3) we can prove that the one-sided curves of this Möbius strip are orienting curves of Σ. Toroidal 1-blockers intersect all one-sided non-orienting curves, but *they are not binary!*

Let γ_1 and γ_2 be two simple curves of a torus which intersect each other in a point, and so that cutting open along them we get a surface homeomorphic to the disc. (γ_1 and γ_2 constitute a "unimodular basis" of the torus, say one of them goes "around the thickness" the other "around the ring".) Cut a hole with a 0-homotopic curve disjoint from γ_1 and γ_2 on the torus, and identify its opposite points. We get Σ. γ_1 and γ_2 become the pair of curves in the definition of toroidal 1-blockers.

In the representation of Σ as a sphere with three identified holes both γ_1 and γ_2 go through two of the three holes, and the two pairs of holes are different.

Geometric 2-blockers

Single 2-blockers

Let a graph G be embedded on Σ, and let γ be a (separating) curve, for which (i) holds in (4) of Section 2, and which does not go through vertices of G. The edges crossed by γ (together with the number of times they are crossed) will be called a *single 2-blocker.* (It is easy to check directly (and it follows from Theorem 1), that every edge is crossed at most twice.) These are the only facets of $P(G, \Sigma)$ defined by curves which are not simple.

In a planar (spheric) drawing with 3 holes whose opposite points have been identified, a single 2-blocker corresponds to a two-sided curve which goes through each cross-cap exactly twice.

(2) *Every single 2-blocker intersects every non-orienting one-sided walk in a nonzero even number of edges.*

Proof. Since γ is a separating curve, it crosses every curve in an even number of edges. But one of the two regions into which γ, (more precisely the simple curve arising by a slight perturbation from γ) separates is orientable, and in the other, according to (3) of Section 2, all one-sided curves are orienting. □

Note that this immediately implies

(3) *If G is a graph Eulerian, then single* 2*-blockers have even size.*

Indeed, subdivide every edge of G into two edges with a new vertex. The cuts are still even. Single 2-blockers correspond to cuts (edges intersected by specific separating curves of Σ) of this graph.

We can feel already that single 2-blockers correspond indeed to the "$(2,3)$-cuts" in Karzanov's theorem: in the plane drawing with three holes a 2-blocker "goes through" all the three cross-caps twice: one of the two regions it bounds is divided to three parts by the holes, and the other into two parts.

Triple 2*-blockers*

Let γ_1, γ_2, γ_3 have the property that each two of them forms a double 1-blocker. Then B will be called a *triple* 2*-blocker*. (Every edge has multiplicity at most 2, (2) and (3) are also true for triple 2-blockers.)

4. Proof of Theorem 1

Let G be a connected graph embedded on Σ. Through arguments typical in polyhedral combinatorics, Theorem 1 is equivalent to the following claim:

> *For arbitrary $c : E(G) \longrightarrow \mathbb{Z}_+$ even on faces, the minimum weight τ of a non-orienting one-sided walk is equal to the maximum ν of $p + 2q$ where p is the number of* 1*-blockers, and q is the number of* 2*-blockers in a c-packing of* 1-2 *blockers.*

A *c-packing* is a family of edges-sets (with multiplicities), where for every $e \in E(G)$ the sum of the multiplicities of sets containing e is at most $c(e)$.

We proceed similarly to Schrijver (1989b): we cut open three times consecutively, always along a one-sided walk, in order to get a planar graph with three holes, and then apply Karzanov's theorem. There is one essential difference in the case of three cross-caps: *we have to be careful to choose the first of these walks to be non-orienting*, otherwise, after cutting open along this first curve all one-sided curves are already destroyed (see (1) in Section 2). On the other hand, after this first right choice, the second and third choices are automatic. Choose always the shortest curve among all possible choices.

We can suppose that every face of G is homeomorphic to the disc, otherwise we add new edges with big enough weight "through cross-caps", so that the minimum weight of a non-orienting one-sided walk does not change. (If the minimum is infinite, that is there is no non-orienting one-sided path, then choose the weight of the new edges to be very large, the sum of the other edge-weights say. It follows from the proof below that $\nu = \tau$ is then large, and in the original graph there is an empty geometric 1-2 blocker. It can also be easily seen directly that the non-existence of non-orienting one-sided walks is equivalent to the existence of geometric 1-2 blockers.)

We can also suppose that c is identically 1, and G is even on faces, otherwise we subdivide the edges. Then a $2,3$-metric can only take the values 1 or 0 on each edge.

Let Q_1 be a minimum length *non-orienting* one-sided walk. (Since every face of G is homeomorphic to the disc there exists such a walk: for every curve there exists a walk homotopic to it, because we can shift it to follow the boundary of the face it crosses.)

Let $t_1 := |Q_1| = \tau$. Cut open Σ along Q_1. What we get is a Klein bottle with one hole, with a graph G' embedded on it. (See (5) in Section 2 and the definition of cutting open in the Introduction.) The boundary of the hole is a circuit C_1 of G'_1 $|C_1| = 2t_1$. (Edges of G are twice as much in C_1 as their multiplicity in Q_1.)

Take a minimum size one-sided walk Q_2 on this new surface. (This will actually be a circuit of G', but the corresponding walk of G may not be simple.) This will not be orienting according to (2).

Cutting open along Q_2 we get the projective plane with two holes, and a graph G'' embedded in it. The boundary of the second hole is a circuit C_2 of G'', $|C_2| = 2t_2$. Similarly, take a third times a minimum weight one-sided walk Q_3 of G'', cut open along it, the boundary of the new hole will be a circuit C_3 of the arising graph G'''. $|C_3| = 2t_3$.

In the same way as before, Q_3 will not cross Q_1 and Q_2, and it will not be orienting.

Claim 1: *A subpath of C_i joining opposite points of C_i is a shortest path of G $(i = 1,2,3)$.*

Indeed, to see this we only have to note that a path of G''' between opposite vertices of the hole C_i corresponds to a walk of G homotopic to Q_i $(i = 1,2,3)$.

We apply now Karzanov's metric packing theorem (1990): let $\mathcal{K}$ and $\mathcal{M}$ be the edges-sets representing the cut-metrics and the $2,3$-metrics respectively in a packing provided by this theorem, where a metric is *represented* by its edges of weight 1. (If the number of cuts in the packing is maximum, then deleting the edges of weight 1 we must have one connected component if m is a cut metric, and 2 components on one side, 3 on the other for $2,3$-metrics. In this case the representation of a metric uniquely determines the metric.)

Because of Claim 1, according to any metric occurring in this packing, the distance between all pairs of opposite points on C_i must be the same: the only possibility for a metric to have this property is that the intersection of the representing edge-set with C_i consists exactly of two opposite edges of C_i.

With the help of this remark, we will prove now that the objects occurring in the metric packing can be "put together" to form the geometric objects described in Section 3, to provide altogether p 1-blockers and q 2-blockers, where all of these 1-2 blockers are pair wise disjoint, and $p + 2q = t_1$.

From $M \in \mathcal{M}$ there is nothing to put together: M intersects every subpath of C_i of length t_i in 2 edges, because it is easy to see that it must intersect C_i in 4 edges, and if the distribution of these 4 edges between the two parts of an arbitrary partition into two paths of length t_i were "unequal", then the distance between the end-points of these paths would decrease more than the metric value, in contradiction with the fact that K is in the packing. It follows that the intersection of K with each of the C_i, is two pairs of opposite edges of C_i. Moreover, the edges of weight 1 in K form a cut, that is a closed walk of the dual: K is the set of edges intersected by a circuit which goes through all the three cross-caps exactly twice, which means exactly that it is a single 2-blocker.

The elements of $\mathcal{K}$ have to be "pasted" together now to form 1 blockers or triple 2-blockers. As opposed to the Klein bottle, this construction cannot be done here in a greedy way, but with some care, it works:

Let $\mathcal{K}_{123}$ denote the set of cuts in this packing which have a non-empty intersection with all of C_1, C_2 and C_3, let $\mathcal{K}_{ij}$ be the set of those which have non-empty intersection with exactly C_i and C_j and let $\mathcal{K}_i$ be the set of those which intersect $\mathcal{K}_i$ and not others $(i,j=1,2,3)$.

Let us denote the elements of $\mathcal{K}$ which have a non-empty intersection with C_i by $\mathcal{K}'_i$, $(i=1,2,3)$, for example $\mathcal{K}'_1 = \mathcal{K}_1 \cup \mathcal{K}_{12} \cup \mathcal{K}_{13} \cup \mathcal{K}_{123}$.

Claim 2: *There exists a list of triples, pairs or single elements of $\mathcal{K}$, where every element of $\mathcal{K}$ occurs at most once, every element of $\mathcal{K}'_1$ occurs, and each element of the list has one of the following forms:*

(i) $\{K_{123}\}$, *where* $K_{123} \in \mathcal{K}_{123}$
(ii) $\{K_1, K_2, K_3\}$, *or* $\{K_1, K_{23}\}$, *or* $\{K_{12}, K_3\}$, *or* $\{K_{13}, K_2\}$, *where* $K_i \in \mathcal{K}_i$, *and* $K_{ij} \in \mathcal{K}_{ij}$, $(i,j=1,2,3)$.
(iii) $\{K_{12}, K_{23}\}$ *or* $\{K_{13}, K_{23}\}$, *where* $K_{ij} \in \mathcal{K}_{ij}$.
(iv) $\{K_{12}, K_{23}, K_{13}\}$, *where* $K_{ij} \in \mathcal{K}_{ij}$.

The claim implies Theorem 1, because it is easy to see that the cuts in (i) and the union of the pairs or triples in (ii) form binary 1-blockers, the union of the pairs in (ii) form toroidal 1-blockers, and the union of a triple of the form (iv) forms a triple 2-blocker. Together with single 2-blockers all these form a packing of value t_1.

To prove the claim note that $t_1 \le t_2 \le t_3$, which implies $|\mathcal{K}'_1| \le |\mathcal{K}'_2|, |\mathcal{K}'_1| \le |\mathcal{K}'_3|$. We do not assume $t_2 \le t_3$ in order to have symmetry between the indices 2 and 3.

Although the following argument is straightforward, it takes some space:

These inequalities remain true if we delete triples, pairs, or single elements which have the form (i), (ii) or (iv). Let us delete such elements until we can, and suppose we remain with $\mathcal{K}''_1, \mathcal{K}''_2, \mathcal{K}''_3$. So, $|\mathcal{K}''_1| \le |\mathcal{K}''_2|$, $|\mathcal{K}''_1| \le |\mathcal{K}''_3|$.

If $\mathcal{K}''_1 = \emptyset$ we are done. If not, then $\mathcal{K}''_2 \ne \emptyset$ and $\mathcal{K}''_3 \ne \emptyset$.

It is not possible that $\mathcal{K}''_1$ contains an element of $\mathcal{K}_1$, because then all elements of $\mathcal{K}_{23}$ must already have been deleted, and also either all elements of $\mathcal{K}_2$, or those of $\mathcal{K}_3$, by symmetry, we can suppose that those of $\mathcal{K}_2$. But then $\mathcal{K}''_2 \subseteq \mathcal{K}_{12}$, and it follows that $|\mathcal{K}''_1| = |\mathcal{K}''_2| + 1$, contradicting our inequalities.

It is also impossible that both $\mathcal{K}''_1 \cap \mathcal{K}_{12} \ne \emptyset$, and $\mathcal{K}''_1 \cap \mathcal{K}_{13} \ne \emptyset$, because either one more pair of type (ii) or one more triple of type (iv) could be chosen then.

So, $\mathcal{K}''_1 \subseteq \mathcal{K}_{12}$ say. But then $\mathcal{K}''_3 \cap \mathcal{K}_3 = \emptyset$, implying $\mathcal{K}''_3 \subseteq \mathcal{K}_{23}$. But then $|\mathcal{K}''_1| \le |\mathcal{K}''_3|$ implies that pairing all elements of $\mathcal{K}''_1$ with elements of $\mathcal{K}''_3$ in an arbitrary way we get pairs of the type $\{K_{12}, K_{23}\}$, that is of type (iii).

The claim, and hence Theorem 1 are proved. □

Note that as a consequence we know: $P(G,\Sigma)$ has integer $0-1-2$ vertices, which are the minimal non-orienting one-sided walks; the vertices of $Q(G,\Sigma)$ are half-integer, and consist of the minimal 1-blockers and the halves of minimal 2-blockers.

We also remark that a free choice in the above proof can be exploited: it is possible to *fix in advance the homotopy classes of the walks Q_1, Q_2 and Q_3*, among an arbitrary choice of three disjoint simple non-orienting one-sided curves, and the theorem can be sharpened accordingly.

A different proof of Theorem 1, which does not rely on Karzanov's theorem—but uses a recent result of de Graaf and Schrijver (1992) —can be based on this sharper theorem, as well as an integer metric packing and fractional multiflow theorem for graphs embedded on the Klein bottle with all terminals on one hole in a special order (Sebő (1993)).

5. Connections with multiflows

Σ can be "cut open" in several ways. Using this liberty, Schrijver (1989b) has deduced a new multiflow theorem from his Klein bottle result.

We must be more careful when we cut open: the blocking polyhedron of our walks contains 2-blockers, which do not lead us to any result. *The new multiflow and metric packing theorems arise by cutting open along binary 1-blockers, or their dual on Σ respectively.* Karzanov's Theorem follows by cutting open along triple 1-blockers, and we will also study here the two other special cases: cutting open along double and single 1-blockers.

Karzanov's integer multiflow theorem for Eulerian graphs cannot be generalized to Σ: in the introduction we saw an example of an Eulerian graph where no optimal integer path packing exists. For this, the blame is the non-binary 1-blockers' and the 2-blockers': Theorem 3 below states that for some binary 1-blocker vertices of $Q(G,\Sigma)$ there exists an integer path packing, and we think (Conjecture 1 below) that it holds for every such vertex. Luckily, this set of vertices, (binary 1-blockers), coincide with those which yield multiflow theorems: if Conjecture 1 below is true, it implies integer multiflow theorems.

We will not have to worry of getting walks instead of circuits in the multiflow applications below: if for some weight-function c the minimum weight $1-2$ blocker is a binary 1-blocker, then in a maximum c-packing of paths we have only circuits.

5.1. *Triple 1-blockers*

The example of the introduction with no integer flow also shows that the multiflow problem for planar graphs with three holes is a proper special case of the surface problem, and we would like to understand in what consists this specialty: we will see that it corresponds to the special case when the optimal vertex of $Q(G,\Sigma)$ is a triple 1-blocker.

$Q(G,\Sigma)$ has various kinds of vertices, we classified them in Section 3, one of the classes is the set of characteristic vectors of *triple 1-blockers*.

Theorem 3. *For every triple 1-blocker vertex x_0 of $Q(G,\Sigma)$ and every Eulerian objective function c for which $c^Tx_0 = min\{c^Tx : x \in Q(G,\Sigma)\}$, this linear program (substituting in it the linear inequalities defining $Q(G,\Sigma)$), has an integer dual solution.*

The equivalence of Theorem 3 to Karzanov's integer flow theorem in planar graphs with three holes (Karzanov (1990 II)) is straightforward.

Note that Karzanov's theorem (about metric packings), used in the proof of Theorem 1, is in fact equivalent to Theorem 1. The proof of the opposite implication is similar to the proof of Theorem 2 below.

5.2. *Double 1-blockers*

We state now the multiflow theorem which corresponds to dual solutions in case the optimal vertex of $Q(G,\Sigma)$ is a double 1-blocker.

Let H_1, H_2 and H_3 be the vertex sets of three distinct faces of a graph G embedded to the surface of the sphere, and suppose $\{s_1,\dots,s_r\} \subseteq H_1$, $\{t_1,\dots,t_r\} \subseteq H_2$, where the indices are given according to the clockwise order of the vertices on the corresponding faces, and some additional set of terminal pairs is given on H_3 in an arbitrary order. Define the edge-set of the fraph H be the set of all terminal pairs. $V(H) = H_1 \cup H_2 \cup H_3$. $G+H$ denotes the graph on

$V(G)$ whose edge-set is $E(G) \cup E(H)$. Let $c : E(G+H) \longrightarrow \mathbb{R}^+$. A family of paths where each path has a (maybe fractional) multiplicity and for every $e \in E(G)$ the sum of multiplicities of paths containing e is at most $c(e)$, moreover every $st \in H$ is contained in exactly $c(st)$ paths, is called a *multiflow*.

Theorem 4. *Under the above defined constraints there exists a multiflow if and only if for every cut-metric and* $2,3$*-metric* m $\sum_{st \in H} m(st)c(st) \leq \sum_{e \in E(G)} m(e)c(e)$ *holds.*

Proof. The necessity of this condition is trivial.

The sufficiency follows from Theorem 1 like for the multiflow theorems corresponding to the special cases (see Schrijver (1989b)). Let us sketch the details.

First, by a standard trick, we can reduce the theorem to the case when every pair of terminals on H_3 is crossing.

Second, for every $st \in E(H)$ add a "half-edge" to both s and t in a planar way, and draw a simple closed curve which contains all the "free" endpoints in their cyclical order, and otherwise does not intersect G; then identify opposite points of this curve so that the "free-end" of the half-edge at s is identified with the free-end of the one at t (for every pair of terminals).

What we get is Σ, with $G+H$ embedded on it.

It is easy to see that H is a double 1-blocker, and (by complementary slackness)

1. *The multiflow problem has a solution if and only if* $c(H) \leq cx$ *for every* $x \in Q(G, \Sigma)$.
   Substituting here that by Theorem 1 we know the vertices of $Q(G, \Sigma)$, we get that the condition of (1) is necessary and sufficient.
2. *For every* 1*-blocker* B_1, $c(B_1) \geq c(H)$, *and for every* 2*-blocker* B_2, $2c(B_2) \geq c(H)$.
   But it is straightforward to check, one by one, for each kind of $1-2$ blocker, (see the list of all of them in Section 3) that
3. *the edges crossed by one of the defining curves* γ *of a* 1*-blocker or triple* 2*-blocker constitute a cut in the planar graph* G'; *those crossed by the defining curve of a single* 2*-blocker (with multiplicity) constitute a* $2,3$*-cut. Moreover, if the condition of Theorem 4 is satisfied for these, then (2) holds.*

(1), (2) and (3) imply Theorem 4. □

Let us sketch the *proof of Theorem 2* now.

First, suppose that $\mathrm{dist}_G(u,v)$ is the same number d for all s_i, t_i pairs, and opposite vertices of H_3. This situation can be reached by adding paths with new vertices and edges, to complete the distance of the mentioned pairs to the maximum. Applying Theorem 2 to this graph, and dropping the cuts which contain new edges (these contain only new edges) we get Theorem 2 for the original graph.

Third, identify s_i and t_i, and the opposite vertices of H_3. We get a graph embedded to Σ. It is not completely evident, that

Claim: *The minimum cardinality of a non-orienting one-sided circuit of* Σ *is* d,

but it is true. The difficulty is that non-orienting one-sided paths may correspond to paths which go several (an odd number of) times through edges $st \in E(H)$. To prove this statement, either one has to solve an elementary problem in the planar graph G, or, and this is what we do, *use the fractional weakening of Theorem 2*: we get this fractional version from Theorem 4, by the Farkas Lemma. Now, like in the proof of Theorem 1, from a fractional packing of cuts, a fractional packing of $1-2$-blockers of size d can be constructed, proving the Claim.

Finally, apply Theorem 1: there exists an integer packing of p 1-blockers and q 2-blockers so that $p + 2q = d$. But in the proof of Theorem 4 we have checked already that the parts of 1-blockers and triple 2-blockers correspond to cuts, and those of single 2-blockers to $2,3$-metrics. □

5.3. *Single 1-blockers*

We deduce now the multiflow special case corresponding to those vertices of $Q(G, \Sigma)$ which are single 1-blockers. We do not state the (integer) metric packing polar of this theorem, it is very close to Theorem 1.

Let G be a graph embedded on the torus, let $V(H)$ be the vertex set of one particular face of G, and $E(H) := \{(s_i, t_i) : i = 1, \ldots, k\}$, where $s_1, \ldots, s_k, t_1, \ldots, t_k$ is the clockwise order of these vertices on the particular face, and $c : E(G+H) \longrightarrow \mathbb{R}^+$. An (s_i, t_i) path $(i = 1, \ldots, k)$ is *0-homotopic*, if contracting H to one point on the torus (in the face it bounds), it becomes a 0-homotopic circuit. 1- *and* 2*-cuts* are defined as edge-sets which become 1- or 2-blockers after the identification of opposite vertices of the particular face so that s_i is identified with t_i $(i = 1, \ldots, k)$. For example, if a curve intersects only edges and it is 0-homotopic on the torus, moreover the disc it bounds contains the particular face, the edges crossed by this curve form a 2-cut. We can see directly that every non-0-homotopic path has at least two common edges with such a 2-cut, and similarly, with every 2-cut, and at least one common edge with every 1-cut.

Theorem 5. *Under the above defined constraints there exists a multiflow consisting only of non-0-homotopic paths, if and only if*

$$\sum_{st \in H} m(st)c(st) \leq \sum_{e \in E(G)} m(e)c(e)$$

if m *is a* 1*-cut, and*

$$2\sum_{st \in H} m(st)c(st) \leq \sum_{e \in E(G)} m(e)c(e)$$

if m *is a* 2*-cut.*

The necessity of this condition is trivial, and the sufficiency follows from Theorem 1, in a similar way as Theorem 4. For Eulerian c there may always be an integer solution:

Conjecture 1. $Q(G, \Sigma)$ *has an integer optimal dual solution for the (fixed) Eulerian objective function* c *if and only if it has a binary-1-blocker-vertex* x_0 *for which* $c^T x_0 \leq c^T x$ *for all* $x \in Q(G, \Sigma)$.

This conjecture is the special case of a general conjecture I stated on the relation of integer metric packings in bipartite graphs, integer path packings in Eulerian graphs, and the bipartiteness of the "essential" metrics. (See the problems of the "Graph Minors" meeting (Seattle), to appear in the *Journal of Graph Theory*.) We state now a special case of that conjecture, which is closer to Conjecture 1, and also contains all integer multiflow theorems for Eulerian graphs I know about:

Conjecture 2. *Let G and H be two graphs on the same vertex set, the edges of H are terminal pairs to be joined by paths. If the validity of*

$$\sum_{st \in H} m(st)c(st) \leq \sum_{e \in E(G)} m(e)c(e)$$

for every cut-metric and 2, 3*-metric* m *is sufficient for the existence of a multiflow for all* $c : E(G+H) \longrightarrow \mathbb{R}^+$, *then for every Eulerian* $c : E(G+H) \longrightarrow \mathbb{R}^+$ *it also implies the existence of an integer flow.*

References

A. Frank, "Packing paths, circuits and cuts—A survey," Technical Report 88532-OR, Bonn, 1988.

P.J. Giblin, *Graphs, Surfaces and Homology* (Chapman and Hall, London and New York, 1981).

M. de Graaf and A. Schrijver, "Decompositions of graphs on surfaces," preprint, 1992.

C.A.J. Hurkens, A. Schrijver, and É. Tardos, "On fractional multiflows and distance functions," *Discrete Mathematics* 73 (1989) 99–109.

A. Karzanov, "Paths and metrics in planar graphs with three or more holes I, II," manuscript, 1990.

S. Lefschetz, *Introduction to Topology* (Princeton University Press, 1949).

L. Lovász, *Combinatorial Problems and Exercises*, (Akadémiai Kiadó and North Holland, 1979).

S. Lins, "A minimax theorem on circuits in projective graphs," *Journal of Combinatorial Theory B* 30 (1981) 253–262.

H. Okamura, "Multicommodity flows in graphs," *Discrete Applied Mathematics* 6 (1983) 55–62.

H. Okamura and P.D. Seymour, "Multicommodity flows in planar graphs," *Journal of Combinatorial Theory B* 31 (1981) 75–81.

A. Schrijver, "Distances and cuts in planar graphs," *Journal of Combinatorial Theory B* 46 (1989a) 46–57.

A. Schrijver, "The Klein-bottle and multiflows," *Combinatorica* 9 (1989b) 375–384.

A. Schrijver, "Decomposition of Graphs on Surfaces and a Homotopic Circulation Theorem," *Journal of Combinatorial Theory B* 51 (1991) 161–210.

A. Sebő, "On surfaces, multiflows, and metric packings," in preparation, 1993.

J. Stillwell, *Classical Topology and Combinatorial Group Theory* (Springer Verlag, New York, Heidelberg, Berlin, 1980).

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Routing in grid graphs by cutting planes

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In this paper we study the following problem, which we call the weighted routing problem. Let be given a graph $G = (V, E)$ with non-negative edge weights $w_e \in \mathbb{R}_+$ and integer edge capacities $c_e \in \mathbb{N}$ and let $\mathcal{N} = \{T_1, \ldots, T_N\}$, $N \geq 1$, be a list of node sets. The weighted routing problem consists in finding edge sets $S_1, \ldots, S_N$ such that, for each $k \in \{1, \ldots, N\}$, the subgraph $(V(S_k), S_k)$ contains an $[s,t]$-path for all $s, t \in T_k$, at most c_e of these edge sets use edge e for each $e \in E$, and such that the sum of the weights of the edge sets is minimal. Our motivation for studying this problem arises from the routing problem in VLSI-design, where given sets of points have to be connected by wires. We consider the weighted routing problem from a polyhedral point of view. We define an appropriate polyhedron and try to (partially) describe this polyhedron by means of inequalities. We briefly sketch our separation algorithms for some of the presented classes of inequalities. Based on these separation routines we have implemented a branch and cut algorithm. Our algorithm is applicable to an important subclass of routing problems arising in VLSI-design, namely to problems where the underlying graph is a grid graph and the list of node sets is located on the outer face of the grid. We report on our computational experience with this class of problem instances.

Key words: Routing in VLSI-design, Steiner tree, Steiner tree packing, cutting plane algorithm.

1. Introduction

One of the main topics in VLSI-design is the routing problem. Roughly described, the task is to connect so-called terminal sets via wires on a predefined area. In addition, certain design rules are to be taken into account and an objective function like the wiring length must be minimized. The routing problem in general is too complex to be solved in one step. Depending on the user's choice of decomposing the chip design problem into a hierarchy of stages, on the underlying technology, and on the given design rules, various subproblems arise. Many of the routing problems that come up this way can be formulated in graphtheoretical terms as follows:

Problem 1.1. (The Weighted Routing Problem)

Instance:

A graph $G = (V, E)$ *with positive, integer edge capacities* $c_e \in \mathbb{N}$ *and non-negative edge weights* $w_e \in \mathbb{R}_+$, $e \in E$.
A list of node sets $\mathcal{N} = \{T_1, \ldots, T_N\}$, $N \geq 1$, *with* $T_k \subseteq V$ *for all* $k = 1, \ldots, N$.

Problem:

Find edge sets $S_1, \ldots, S_N \subseteq E$ *such that*