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On the clique-rank and the coloration of perfect graphs

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On the clique-rank and the coloration of perfect graphs *

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Abstract: We are investigating the *clique-rank (linear rank of the ω-cliques)* of graphs and its relation to the optimal colorations of perfect graphs.

First, we observe a simple characterization of perfectness with an inequality about the clique rank. Then we notice that the extremal graphs with respect to this inequality are exactly the uniquely colorable perfect graphs (perfect graphs that have exactly one optimal coloration), and that a *good characterization for the unique colorability* can also be derived with the help of the clique-rank.

After this, we would like to understand the significance and the use of uniquely colorable perfect graphs (perfect graphs that have exactly one optimal coloration) for the perfect graph conjecture. In lack of a general *combinatorial* good characterization of unique colorability we state a conjecture for such a characterization, study its connection with other conjectures and theorems (including the Perfect Graph Conjecture). The validity of our conjecture, even for special cases, would imply new classes of perfect graphs. After settling it for $\omega=3$, we deduce the Perfect Graph Conjecture for diamond-free graphs, a result of Partasarathy, Ravindra and Tucker. This study relies again on the clique rank.

Finally, we show how a simply stated combinatorial algorithm for coloring perfect graphs can be designed with the help of the clique rank, an algorithm that unfortunately depends exponentially on ω. However, this algorithm is efficient for some classes perfect graphs, including "K_4-e"-free perfect graphs treated earlier by Tucker. Its performance in general is $O(n^{\omega+1})$, which, if ω is small, is better than the ellipsoid method, and than other algorithms we know .

1. Introduction

Let G = (V(G), E(G)) be a graph. A clique of G is a subset of pairwise adjacent vertices. Let ω(G) be the maximum cardinality of a clique. A *k-coloration* of G is a partition of V(G) *into k stable sets.* The chromatic number χ(G) is the minimum number of stable sets partitioning V(G). A graph is called *perfect* if ω(G')=χ(G') for every induced subgraph G'. G will be called *uniquely colorable*, if it has exactly one ω(G)-coloration.

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A chordless cycle is a cycle which is an induced subgraph. Berge's Perfect Graph Conjecture (PGC) states that *a graph is perfect if and only if it does not contain an odd chordless cycle or the complement of an odd chordless cycle as an induced subgraph .*

Let A(G) be the incidence matrix of the cliques of G and B(G) be the incidence matrix of the ω(G)-cliques of G. (B(G) is a submatrix of A(G)).The linear rank of the incidence (characteristic) vectors of the ω(G)-cliques of G will be called the *clique rank* of G, and will be denoted by r(G).Note that r(G) = r(B(G)).Sets and their incidence vectors will often not be distinguished. A maximum set of linearly independent ω(G)-cliques (that is a set that generates linearly all ω(G)-cliques) will be called a *clique-base*; if ω(G) =3, then it will be called a *triangle-base*; their span, that is the span of all ω(G)-cliques will be called the *clique-space* (for ω(G) =3 *triangle-space*).

Let us recall a basic result about perfect graphs :

(1.1) The following statements are equivalent :

(i) A graph is perfect.

(ii) The stable set polytope $P(G)=\{x : x \geq 0; A(G)\, x \leq 1\}$ is integral, that is the set of its extreme points is the set of incidence vectors of the independent sets of G.

(iii) (i) or (ii) hold for the complement of G.

The equivalence of (i) and (ii) is an immediate consequence of Lovàsz's main "blowing Lemma" in the proof of the Perfect Graph Theorem, a remark implicit in Lovàsz (1972), explicit in Chvàtal (1975) and conjectured by Fulkerson (1970). The equivalence of (iii) with the rest is just the Perfect Graph Theorem (Lovàsz 1972). (1.1) will be extensively used in the sequel.

Many results related to perfect graphs are based on this polyhedral characterization. Padberg's results (1974) on critically imperfect graphs are among them. We just mention two of the properties he proved, for later convenience:

(1.2) If G is a criticlally imperfect graph , then

a. r(G)=n

b. $G(V(G)\backslash\{v\})$ is uniquely colorable for every $v \in V(G)$.

(An imperfect graph is critically imperfect if any proper induced subgraph is perfect.) Another interesting result, that follows from Fulkerson (1971) is that there exists a polynomial algorithm for finding a minimum coloring of G provided that $\omega(G)$ is fixed. Again, this algorithm uses some powerful tools of linear algebra. Note however that the method proposed by Fulkerson is not combinatorial.

Similar remarks could be said about Grötschel Lovàsz and Schrijver's algorithm for coloring perfect graphs, the only known polynomial algorithm for this problem.

On the other hand many questions related to perfect graphs can be solved using purely combinatorial arguments; Berge's conjecture itself has a purely combinatorial flavor.

Our goal in this paper is to connect these two possible approaches; more specifically, we use linear algebra in order to get combinatorial methods and structures for the coloring problem of perfect graphs. In particular, we are interested in uniquely colorable perfect graphs which play a crucial role in perfect graph theory by the above mentioned result of Padberg. We shall see that from the point of view of linear algebra they are the "most saturated" perfect graphs.

In Section 2 we prove some general results based on the *clique rank* and the *clique space* of a graph. In particular, we show how perfectness can be characterized with the help of the clique rank. From the observation that uniquely colorable perfect graphs are those with maximum clique rank, we deduce a *good characterization of unique colorability* for perfect graphs.

In Section 3 we state a conjecture about a stronger, combinatorial characterization of unique colorability, and its relation to other conjectures and results. We would also like to explain why the special cases with fixed ω are interesting.

In Section 4 we prove the main conjecture for $\omega=3$, applying some algebraic considerations. This is a common generalization of the perfect graph conjecture for diamond-free and 3-chromatic perfect graphs, known earlier from the work of Parthasarathy and Ravindra (1979), and Tucker (1984), (1987a,b).

In the last Section 5 we show how the relation of the clique rank and colorations can be used for coloring algorithms.

2. The clique rank and the clique space

In this section we wish to make clear some simple connections between the clique rank and the colorations of perfect graphs.

Let us first define an upper bound for the clique rank of a perfect graph. We shall see that this bound is tight (equality is satisfied) exactly for uniquely colorable graphs (Theorem 2.3 below).

Proposition 2.1 If G is perfect, then $r(H) \le n - \omega(H) + 1$ for every induced subgraph H of G.

Proof: Let H be an induced subgraph of G. Since H is perfect there are $\omega(H)$ stable sets which partition V(H). These $\omega(H)$ vectors are clearly linearly independent, and are solutions of the linear system $B(H)x=1$. Thus $r(H) \le n - \omega(H) + 1$.

The following corollary will be important from an algorithmic point of view:

Corollary 2.2 : Let G be a perfect graph whose edge-set is non-empty.

a.) If $\mathcal{K}$ is a set of linearly independent $\omega(G)$ -cliques, then there exists $v \in V(G)$ such that v is contained in at most $\omega(G)-1$ elements of $\mathcal{K}$.

b.) There exists $v \in V(G)$ such that $r(G) \le r(G-v) + \omega(G) - 1$.

Proof: a)Let $\mathcal{K}(v) := \{K \in \mathcal{K} : v \in K\}$. If each vertex were contained in at least $\omega(G)$ elements of $\mathcal{K}$, then we would have $\omega(G)|\mathcal{K}| = \Sigma\{|\mathcal{K}(v)| : v \in V(G)\} \ge \omega(G)\ |V(G)|$, whence $|\mathcal{K}| \ge |V(G)|$. On the other hand $|\mathcal{K}| \le |V(H)| - \omega(H) + 1$, and since $\omega(H) \ge 2$, this is a contradiction.

b.) Take the vertex v given by a). $r(G-v) = r(G) - |\mathcal{K}(v)| \ge r(G) - (\omega(G) - 1)$, as we claimed

The following theorem summarizes the main remarks we made so far. It will be used to prove non-perfectness if the coloring algorithms fail.

Theorem 2.3: The following statements about the graph G are equivalent:

(i) G is perfect.

(ii) For every induced subgraph H of G: $r(H) \le n-1$, provided H contains an edge.

(iii) For every induced subgraph H of G: $r(H) \le n - \omega(H) + 1$.

Proof: "(i)$\Rightarrow$(iii)" is just Proposition 2.1 .

If H contains an edge, then $\omega(H) \ge 2$, whence "(iii)$\Rightarrow$(ii)" is obvious.

"(ii)$\Rightarrow$(i)" is just (1.2) a., and the Theorem is proved.

Note that the bounds in (ii) and (iii) are equal if and only if $\omega=2$, and the gap for $\omega>2$ is quite mysterious. (Has this particular role of $\omega=2$ something to do with the same role of this case in the perfect graph conjecture ?)

Theorem 2.4: Let G be a perfect graph. G is uniquely colorable if and only if

$$r(G) = n-\omega(G)+1.$$

Proof: Let $P(G)=\{x\geq 0 : A(G)x\leq 1\}$ be the stable set polytope of G}.
For basic knowledge about polyhedra we refere to Schrijver (1986). Recall however that a d -dimensional face of a full dimensional polyhedron is the intersec-tion of n-d facets having a set of linearly independent coefficient vectors.

P(G) is obviously full dimensional. L et F be the face $F=\{x\in P(G) : B(G)x=1\}$ of P(G). The extreme points of F are exactly the stable sets which intersect every $\omega(G)$-clique. In other words *a stable set S belongs to F if and only if it occurs as a color class in some $\omega(G)$-coloration.* If S has this property it will be called a *color-class.* Since F is the intersection of the facets defined by the rows of B(G), dim(F)+ r(G)=n. (Recall that r(G)=r(B(G)).)

G is uniquely colorable if and only if the number of color classes is $\omega(G)$, which is true if and only if $\dim(F)=\omega(G)-1$. (A color class different from the classes of a given coloration, is also linearly independent of them.) . Thus the theorem is proved.

This is obviously a good characterization of unique colorability: an $\omega(G)$- coloration, and a set of $n-\omega(G)+1$ linearly independent $\omega(G)$-cliques of a graph on n vertices proves that the graph is uniquely $\omega(G)$-colorable. This good characterization can be presented in the following combinatorial way:

Suppose $\omega(G)\geq 3$. (The case $\omega(G)\leq 2$ is trivial.) $a,b\in V(G)$ will be said to be equivalent, in notation $a\sim b$ if there exist multisets (a multiset is a set whose elements have non-negative integer multiplicities showing the number of times they are contained in the multiset) $\mathcal{K}_1$ and $\mathcal{K}_2$ of $\omega(G)$-cliques such that $\mathcal{K}_1(x) = \mathcal{K}_2(x)$ holds if and only if $x\notin \{a,b\}$, where $\mathcal{K}_i(x) := |\{K\in\mathcal{K}_i: x\in K\}|$.

The following theorem is a combinatorial version of theorem 2.4, and we also included a characterization for a and b to have the same color in every coloration, even if G is not uniquely colorable:

Theorem 2.5: Let G be a perfect graph with $\omega(G)\geq 3$.

- The above defined relation is an equivalence relation;
- $a\neq b\in V(G)$ have the same color in all the ω-colorations of G if and only if they are equivalent.;
- G is uniquely colorable if and only if this equivalence relation has $\omega(G)$ classes.

Proof: One can remain within the framework of "combinatorial " arguments for the trivial parts. By simple counting arguments one can prove the following about $a, b\in V(G)$, $a\sim b$ and about the families of cliques in the definition of the equivalence:

1. $|\mathcal{K}_1|=|\mathcal{K}_2|$.
2. $\mathcal{K}_1(a)-\mathcal{K}_2(a)=-(\mathcal{K}_1(b)-(\mathcal{K}_2(b))$ (see the notations of Corollary 2.2)
3. " ~" is an equivalence relation

Let us now check the trivial if parts of the statements in a combinatorial way. Let $a\sim b$, and let $\mathcal{K}_i$ (i=1,2) be the $\omega(G)$-cliques which prove this equivalence. Let $S_1,\ldots,S_\omega$ be a coloration, and suppose indirectly $a\in S_1$, $b\notin S_1$. $|\mathcal{K}_i|=\sum_{x\in S_1}\mathcal{K}_i(x)$,

Since $\mathcal{K}_1(x)=\mathcal{K}_2(x)$ if $x\neq a$, but $\mathcal{K}_1(a)\neq\mathcal{K}_2(a)$, we can conclude $|\mathcal{K}_1|\neq|\mathcal{K}_2|$. Let now S_2 be a color class of the above coloration which contains neither a nor b. (There exists such a color class since $\omega(G)\geq 3$.) We have now, like above, $|\mathcal{K}_i|=\sum_{x\in S_2}\mathcal{K}_i(x)$, which implies now $|\mathcal{K}_1|=|\mathcal{K}_2|$, a contradiction. Thus a and b have the same color in every coloration, and the if part of both statements follows immediately.

To prove the essential only if part we shall use some linear algebra in a similar way as in the proof of Theorem 2.4. Suppose $a, b\in V(G)$ have the same color in every $\omega(G)$-coloration of G, and define the vector w with $w(a)=1$, $w(b)=-1$ and $w(v)=0$ if $a\neq v\neq b\in V(G)$. By the assumption on a and b, $ws=0$ holds for the characteristic vector s of every color class. (A color class contains neither a nor b or both a and b.) Since G is perfect the color classes generates the set of all the solutions of the system $B(G)x=1$. Thus $wx=0$ is satisfied for every solution of the system $B(G)x=1$, and consequently w belongs tc the linear space generated by the rows of B(G): $w=\sum\lambda_i K_i$. Let d be their smallest common denominator of the

λ_i. The families $\{K_i : \lambda_i>0\}$ and $\{K_i : \lambda_i<0\}$ where the multiplicity of clique K_i is $|d\lambda_i|$ prove a~b, and the proof is finished.

Note however, that despite the tanslation into a combinatorial language, the above solution is actually algebraic. Unfortunately it does not say enough about the structure of uniquely colorable perfect graphs (see Section 3).

For $\omega=3$ we shall present in Section 4 a "more purely" combinatorial characterization.

3. Some conjectures

One of the most promising trials to approach the perfect graph conjecture has been Tucker's analysis of uniquely colorable perfect graphs, and their relation to the the perfect graph conjecture. Let us summarize Tucker's approach:

A.) Look for a good characterization of the unique colorability of perfect graphs: in a uniquely colorable perfect graph a *combinatorial* "forcing procedure" of the unique

$\omega(G)$-coloration should be constructed. (Theorem 2.5 gives already some forcing procedure based on linear algebra.)

B.) Suppose that a forcing procedure is applied to the (not necessarily perfect) graph G, and that a certain vertex is forced to have an $\omega(G)+1$-th color. Proving that in this case either an $\omega(G)+1$-clique or an odd chordless cycle or the complement of an odd chordless cyclecan be found, is equivalent to the PGC, as it is explained below. (It is obviously implied by the PGC.)

Let us explain how an appropriate answer to A.) and B.) would imply the Perfect Graph Conjecture. Let G be critically imperfect. By (1.2) G-v is uniquely $\omega(G)$colorable: apply now A.) for coloring G-v. Clearly, v must be colored with an $\omega(G)+1$-th color. Now we can apply B.) to find a hole or an antihole, and the Perfect Graph Conjecture is "proved".

Some results have been reached earlier in the direction of B.). Chvàtal's result (1984) can be interpreted as showing the existence of a hole or antihole under a strengthening of the unique colorability condition. See also Tucker (1983) for some generalizations. On the other hand there has not been anything done about A.), besides some conjectures of Tucker.

Let us state a conjecture for a combinatorial forcing of unique colorability:

Let v and w be two non-adjacent vertices of a graph G. We say that v and w are *forced* vertices if there exist two $\omega(G)$-cliques K_1 and K_2 such that $\{v\}= K_1 \setminus K_2$ and $\{w\}= K_2 \setminus K_1$.

Obviously, if G is perfect, in any optimal coloration the same color is assigned to v and w.

Starting from a graph G let H be the graph obtained by repeated identification of forced vertices. (Note that this operation does not necessarily preserve perfectness.)

Conjecture 1: A perfect graph is uniquely colorable if and only if H is a clique of size $\omega(G)$.

The sufficiency of this condition is obvious. In the following section we prove it for K_4-free graphs, a case whose significance is explained by Theorem 3.2 below. The validity of this Conjecture would be especially interesting for classes of graphs for which the Perfect Graph Conjecture is not proved yet.

Recently, Markossian and Gasparian (1988) proved a theorem in the direction of B.) using a similarly looking forcing procedure to the above defined. However, their forcing procedure is somewhat more particular. A proof of the Perfect Graph Conjecture using the above conjecture would in particular include their result and the main result of Tucker (1983). Such a proof may turn out to be less difficult than the conjecture itself.

Tucker (1983) had already a (more complicated) conjecture on unique colorability. We state without proof the equivalence of the two conjectures:

Theorem 3.1: The above conjecture is valid if and only if Tucker's conjecture is valid.

In fact we cannot even prove the existence of one forcing step:

Conjecture 2: If G is a uniquely colorable perfect graph, then it contains two $\omega(G)$-cliques whose intersection is an $\omega(G)$-1-clique.

Finally let us mention a more modest goal than the PGC which also makes use of unique colorability:

Theorem 3.2 *If $\mathcal{G}$ is a class of graphs closed under taking induced subgraphs, and for some integer k every perfect graph in $\mathcal{G}$ with $\omega=k$ has at least two colorations and every graph in $\mathcal{G}$ with $\omega<k$ is perfect, then every graph in $\mathcal{G}$ is perfect.*

This statement is a simple consequence of the above mentioned result of Padberg:

Proof of Theorem 3.2: Suppose for $\mathcal{G}$ and k the conditions of the theorem are satisfied, but the statement is not true: let $G\in\mathcal{G}$ be not perfect and $|V(G)|$ minimum among such graphs. Since $\mathcal{G}$ is closed under taking induced subgraphs, G contains a critically imperfect induced subgraph which is in $\mathcal{G}$, and thus G itself is such a graph.

By our assumption on $\mathcal{G}$, $\omega(G)\geq k$. Take an arbitrary $v\in V(G)$. Since G is critically imperfect, $\omega(G-v)=\omega(G)\geq k$. Clearly, G-v is perfect, and by Padberg's result it is UC, whence the union of k color classes of its unique ω-coloration induces a graph H which is also UC. H is perfect, H belongs to $\mathcal{G}$, $\omega(H)=k$, and thus, by the assumption, it has two colorations, a contradiction.

Note that *k plays a role only in the condition of Theorem 3.2 , and it is not necessarily present in the definition of the class $\mathcal{G}$*. Thus, information about uniquely k-colorable perfect graphs can lead to new classes of perfect graphs. This gives some motivation to look at the case k=3 in more details.

4. The case $\omega=3$

The idea of Theorem 3.2 has already been exploited for k=2: clearly, *a bipartite graph is UC if and only if it is connected.* The proof of Giles, Trotter and Tucker (1984) of the PGC for $K_{1,3}$-free graphs repeatedly uses this fact.

For k=3 the following proof of the PGC for diamond-free graphs gives an example to the use of this idea. A *diamond* is a graph isomorphic to $K_4\backslash e$. *Diamond-free* graphs are graphs which do not contain a diamond as an induced subgraph. The following result can be checked through the decomposition procedure for diamond-free perfect graphs of Fonlupt and Zemirline (1988).

Theorem 4.1: A diamond-free non-bipartite perfect graph has at least two colorations.

This is Conjecture 2 for $\omega=3$. *Theorem 3.2 and Theorem 4.1 immediately imply the Perfect Graph Conjecture for diamond-free graphs,* a result of Parthasarathy and Ravindra (1979). The results of this section led us to a proof of Conjecture 1 as well, for $\omega=3$.

Despite the combinatorial nature of the results in this section, their proofs are based on an algebraic observation. In order to state this let us introduce some notations.

r_2 will denote the rank function over GF(2), and $r_2(G)$ will denote the rank over GF(2) of the set of characteristic vectors of all $\omega(G)$-cliques of a graph. Note that the incidence vectors of a family of subsets of V(G) are linearly independent over GF(2) if and only if for all subfamilies (including the original family) there exists $v \in V(G)$ which is contained in an odd number of sets belonging to the subfamily.

The essential part of Theorem 4.2 below is that there exists a basis for the triangle space over GF(2), which is also a basis over the rational numbers:

Theorem 4.2 : Let G be a 3-colorable perfect graph.

a.) $r_2(G) = r(G)$.

b.) If the integer vector z is in the triangle space and t is the 0-1vector such that $t \equiv z \pmod 2$, then t is in the triangle space over GF(2) as well.

Proof: Note first that $r_2(\mathcal{T}) \le r(\mathcal{T})$ is obvious. To prove the equality, let $k := n - r_2(\mathcal{T})$. It is well known from linear algebra that there are k vectors $t^1,\dots,t^k \in \{0,1\}^n$ linearly independent over GF(2), each of which is orthogonal to $\mathcal{T}$, that is: $t^i(T) \equiv 0 \bmod 2$ for $i=1,\dots k$ and for every $T \in \mathcal{T}$. We shall construct linearly independent vectors $z^1,\dots,z^k \in \mathbb{Z}^n$ such that $z^i(T)=0$ (for $i=1,\dots k$ and for all $T \in \mathcal{T}$). This will prove the statement, because then $r(\mathcal{T}) \le n-k = r_2(\mathcal{T})$.

If $T \subseteq V(G)$ is a triangle of G, then it is a linear combination of elements of $\mathcal{T}$, whence $t^i(T) \equiv 0 \bmod 2$ $(i=1,\dots k)$. This means that the set $X_i := \{v \in V(G): t^i(v)=1\}$

intersects every clique in an even number of elements. In particular, it does not contain any triangle: in other words $\omega(G_i)\leq 2$ for the graph G_i induced by X_i. Since G is perfect, G_i is bipartite. Denote the classes of a bicoloration of G_i by A_i and B_i. Let $z^i(v):=1$ if $v\in A_i$ and -1 if $v\in B_i$. We know that every clique intersects $A_i \cup B_i$ in an even number of elements, so it is either disjoint from $A_i \cup B_i$, or intersects both A_i and B_i (because both A_i and B_i are stable sets). Consequently $z^i(T)=0$ ($i=1,\dots k$ and $\forall$ $T\in\mathcal{T}$). $z^1,\dots z^k$ are linearly independent, because $z^i\equiv t^i$ mod 2, and $t^1,\dots t^k$ are independent even over GF(2); Thus a.) is proved.

b.) is an easy consequence of a.): Let $\mathcal{T}$ be a triangle basis over GF(2) and z be an integer vector in the triangle space. Let t be the 0-1vector such that $t\equiv z$ (mod 2). Since $\mathcal{T}$ is also a triangle basis over the rationals, $\mathcal{T}\cup\{z\}$ is dependent, whence $\mathcal{T}\cup\{t\}$ is trivially dependent over GF(2). Since $\mathcal{T}$ is independent over GF(2), t must have nonzero coefficient in the linear dependence of $\mathcal{T}\cup\{t\}$ over GF(2), and the theorem is proved.

Note that we do not know about similar results for perfect graphs with arbitrary chromatic number.

By this theorem, we can restrict ourselves to (mod 2) linear relations for 3-colorable perfect graphs to get more combinatorial type theorems . Let us state for example the following more combinatorial version of Theorem 2.5 for this case:

Let G be a perfect graph with $\omega(G)=3$, and $a,b\in V(G)$. A set $\mathcal{K}$ of triangles such that a and b are contained in an odd number of triangles of $\mathcal{K}$, and all other vertices are contained in an even number of triangles of $\mathcal{K}$, will be called an *(a,b) path of triangles*. Let us define the relation a~b : a~b if and only if there exists an (a,b) path of triangles. If a~b let us say that a and b are *equivalent*.

Theorem 4.3: Let G be a perfect graph, $\omega(G)=3$. The above defined relation is an equivalence relation; $a\neq b\in V(G)$ have the same color in all the 3-colorations of G if and only if they are equivalent; G is uniquely colorable if and only if this equivalence relation has 3 classes.

Proof : The proof is similar to the proof of Theorem 2.5, in fact simpler. Let us just sketch it. First we prove that the defined relation is an equivalence relation:

Let a~b, and suppose a~b and b~c. Let $\mathcal{K}$ be an (a,b)-path of triangles and $\mathcal{L}$ a (b,c)-path of triangles. Clearly, $\mathcal{K} \Delta \mathcal{L}$ is an (a,c)-path of triangles, which proves transitivity.

The "trivial" if parts of the statements are even shorter here: Let a~b, and let $\mathcal{K}$ be the 3-cliques which prove this equivalence. Let S_1, S_2, S_3 be a coloration, and

suppose indirectly $a \in S_1$, $b \in S_2$. We obviously have $|\mathcal{K}| = \Sigma\{\mathcal{K}(x) : x \in S_i\}$ (i=1,2,3). For i=1,2 this sum is odd, because all terms of it but one ($\mathcal{K}$(a) and $\mathcal{K}$(b) respectively) are even; on the other hand for i=3 the sum is even, because all terms of it are even, a contradiction.

The essential only if part can be proved similarly to the corresponding part of Theorem 2.5, arguing over GF(2) instead of the rational numbers.

Actually the following strengthening of Theorem 4.3 can be proved:

THEOREM 4.4 : Conjecture 1 is valid for ω=3.

Here, we omit the proof which is difficult and will be published in a forthcoming paper. Note only, that one of the main difficulty is that contraction by forcing rule does not preserve perfectness.

We saw that Theorem 4.1, which is an easy consequence of Theorem 4.4 immediatly implies the Perfect Graph Conjecture for diamond-free graphs. Another consequence of Theorem 4.4: Tucker's (1984) result about the perfect graph conjecture for K_4-free graphs. (See also Tucker (1987a,b).)

We think that Theorem 4.4 might have further corollaries, that should be exploited in the futur.

5. Coloring algorithms

In this section we shall deduce some algorithmic consequences of the results developed in the previous sections. The general coloration algorithm we suggest below has $O(n^{\omega+1})$ running time, which might be interesting if ω is small. Unfortunately we are unable to find a polynomial time combinatorial algorithm for arbitrary perfect graphs, even if the clique basis is given.

Recall that it is also possible to use Gaussian elimination in order to get a polynomial algorithm for minimum coloring, if the chromatic number is fixed, see Fulkerson(1971). However, the complexity of such an algorithm is much bigger.

The initial idea of the coloring algorithm presented below is the following statement:

Theorem 5.1: Let G be a perfect graph with $\omega = \omega(G)$. There exists an ordering of the vertices of G: $x_1, \ldots, x_n$, such that $\{x_{n-\omega+1}, x_{n-\omega}, \ldots, x_n\}$ is an ω-clique, and for $1 \le i \le n-\omega+1$: $0 \le r(G_i) - r(G_{i+1}) \le \omega-1$, where G_i is the graph induced by $\{x_i, x_{i+1}, \ldots, x_n\}$, $(i=1,\ldots,n-\omega)$.

Proof: We proceed by induction on n. If G contains only one ω-clique, the proposi- tion is clear. Otherwise the statement we have to prove is clearly the following: there exists $x_1 \in V(G)$ such that $0 \le r(G) - r(G-x_1) \le \omega-1$. But this is just Corollary 2.2 b.) .

Let us call a clique-base $\mathfrak{B}$ *normal* with the order $x_1, \ldots, x_n$ of the vertices, if:
a.) $\{x_{n-\omega+1}, x_{n-\omega+2} \ldots, x_n\} \in \mathfrak{B}$
b.) for $1 \le i \le n-\omega+1$: $d_i = r(G_i) - r(G_{i+1}) \le \omega-1$, where d_i is the number of cliques $B \in \mathfrak{B}$, $B \subseteq \{x_i, \ldots, x_n\}$containing x_i $(i=1,\ldots,n-\omega)$.
(Equivalently: $\mathfrak{B}_i := \{B \in \mathfrak{B} : B \subseteq \{x_i, \ldots, x_n\}\}$ is a triangle basis of G_i).

Theorem 5.2: There exists a normal clique basis.

Proof: Take an order ensured by Theorem 5.1, and define $\mathfrak{B}$ recursively:
$\mathfrak{B}_{n-\omega+1} := \{ \{x_{n-\omega+1}, x_{n-\omega}, \ldots, x_n\} \}$;
$\mathfrak{B}_i$:=clique basis of the graph induced by $\{x_i, x_{i+1}, \ldots, x_n\}$ containing
$\mathfrak{B}_{i+1}$ $(i=n-\omega,\ldots,1)$
(Clearly, the clique basis $\mathfrak{B}_{i+1}$ of G_{i+1} can be completed to a clique basis $\mathfrak{B}_i$ of G_i.)
If $B \in \mathfrak{B}_i \setminus \mathfrak{B}_{i+1}$, then $x_i \in B$, because if not, then $B \subseteq \{x_{i+1}, x_{i+2}, \ldots, x_n\}$, in contradiction with the maximality of $\mathfrak{B}_{i+1}$. Thus $d_i = |\mathfrak{B}_i \setminus \mathfrak{B}_{i+1}| = r(G_i) - r(G_{i+1})$. Q.E.D.

For $\omega=3$: There exists a triangle basis $\mathfrak{B}$ such that $\{x_{n-2}, x_{n-1}, x_n\} \in \mathfrak{B}$, and $\{x_i, x_{i+1}, \ldots, x_n\}$ contains 0, 1 or 2 more elements of $\mathfrak{B}$ than $\{x_{i+1}, x_{i+2}, \ldots, x_n\}$ depending on whether $r(G_i) - r(G_{i+1})$ is 0, 1 or 2; these new elements contain x_i.

We shall now see a new relation between linear algebra and colorations, a relation that can be used algorithmically. Here we shall restrict ourselves to the case $\omega=3$ in order to show the ideas on a simple special case.

Theorem 5.3: Let G be perfect and $\omega=3$. Let $\mathfrak{B}$ be a normal triangle basis with the order $\{x_1,\dots, x_n\}$, and use the notation above. Suppose that the graph G_{i+1} has already been colored, and denote by $H_{x,y}$ the graph induced by the vertices of color x and y.

a.)If $r(G_i) - r(G_{i+1})=0$, then for arbitrary two colors a and b, in every component of $H_{a,b}$, all the neighbors of x_i have the same color.

b.)If $r(G_i) - r(G_{i+1})=1$, then for all triangles T of G_i containing x_i, $T\backslash x_i$ is colored with the same two colors, say a and b. Moreover in every component of $H_{c,b}$ and $H_{c,a}$ all the neighbors of x_i have the same color.

c.)If $r(G_i) - r(G_{i+1})=2$, then there exists a color c, such that all triangles of G_i containing x_i have a vertex of color c. Moreover, in every component of $H_{a,b}$, all the neighbors of x_i have the same color.

Proof: First note the following:

(*) Suppose that x_i has two neighbors y and z of different colors in the same component of a graph H induced by their two colors, and let P be a minimal path in H with end-nodes y and z. Then *there exists a triangle consisting of x_i* and *two neighboring vertices of P*.

(Indeed, $P \cup x_i$ induces an odd cycle and cannot be bipartite since G is perfect.)

Now, if a.) were not true, then by (*) we would have a triangle through x_i, a contradiction.

To prove b.), suppose that there exist triangles T_1,T_2 of G_i, where $T_1\backslash x_i$ is colored with colors "a" and "b", and $T_2\backslash x_i$ is colored with colors "b" and "c". We show that $\mathfrak{B}_{i+1} \cup \{T_1, T_2\}$ is linearly independent, in contradiction with $r(G_i) - r(G_{i+1})=1$. (Recall that $\mathfrak{B}_j:=\{B\in\mathfrak{B} : B\subseteq\{x_j, \dots , x_n\}\}$ for j =1,...,n.)

$\mathfrak{B}_{i+1} \cup \{ T_1 \}$ is linearly independent because of $x_i \in T_1$. *Define now $v(x):=1$ if x has color "a",or "b", and $v(x):=0$ for all other vertices of $V(G)$. Clearly, $v(T)$ is even for all triangles in $\mathfrak{B}_{i+1} \cup \{T_1\}$, and $v(T_2)$=-is odd, proving the linear independence of T_2 of all the other triangles of $\mathfrak{B}_{i+1} \cup \{T_1\}$* over GF(2) as well. This contradiction proves that all triangles of G_i containing x_i are colored with the same two colors, a and b say.

The second sentence of b.) follows now immediately from (*): if this second sentence were not true, by (*) there would exist a triangle of G_i containing x_i and having a vertex of color c, in contradiction with the already proven first sentence. Thus b.) is proved.

Let us prove c.) now. Again, the second sentence follows from the first one: using (*) the indirect assumption would imply the existence of a triangle T in G_i containing x_i, where $T \setminus x_i$ is colored with the colors "a" and "b". But this is in contradiction with the first sentence.

To prove the first sentence, suppose indirectly that $T_{a,b}$, $T_{b,c}$, $T_{a,c}$ are triangles of G_i containing x_i, whose two other vertices are colored with the colors shown in the indices. According to the proof of b.), $\mathfrak{B}_{i+1} \cup \{T_{a,b}, T_{b,c}\}$ is linearly independent over GF(2).

Let $v(x):=1$ if x is of color a or c or if $x=x_i$. Clearly, $v(T) \equiv 0 \mod 2$ if $T \in \mathfrak{B}_{i+1}$, and also if $T= T_{a,b}$ or $T= T_{b,c}$. On the other hand, $v(T_{a,c}) \equiv 1 \mod 2$. Hence $T_{a,c}$ does not depend linearly from $\mathfrak{B}_{i+1} \cup \{T_{a,b}, T_{b,c}\}$ over GF(2). Equivalently, $\mathfrak{B}_{i+1} \cup \{T_{a,b}, T_{b,c}, T_{a,c}\}$ is linearly independent, in contradiction with $r(G_i) - r(G_{i+1})=2$.

Note how this Theorem implies an algorithm: in a.), by interchanging the two colors in some components of the graphs induced by 2 color classes, all neighbors of x_i will have the same color. In b.) and c.) similarly, after maybe interchanging colors in some components of $H_{x,y}$, x_i will have neighbors of two different colors only. The precise details of the algorithm are omitted here.

Note also that in the proof of c.) the equivalence of the independence over the rationals and over GF(2) played again an important role (Theorem 4.2). Since we do not know this for $\omega \geq 4$, the algorithm for general ω will be less efficient. However, Theorem 5.3 can be straightforwardly generalized to arbitrary ω and the complexity of the corresponding algorithm is $O(n^{\omega+1})$.

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