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A generalized cut-condition for multiflows in matroids

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Abstract

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The class of binary matroids for which the so-called 'cut-condition' is not only necessary but also sufficient for the existence of a multiflow was characterized by P. Seymour. We formulate a natural generalization of the cut-condition and give a characterization of the corresponding larger class of matroids in terms of forbidden minors.

1. Definitions and notation

Let M be a binary matroid defined on the finite set $E(M)$ and p a function assigning integer values to the elements of $E(M)$. We think of the negative values of p as representing *demands* and of the nonnegative values as representing *capacities*. Define $F(p)=\{e\in E(M): p(e)<0\}$. A *flow problem* is a pair (M,p). It has a *solution* if there exists a *multiflow*, that is a function $\Phi:\mathscr{C}_p(M)\to\mathbb{R}_+$ defined on the set $\mathscr{C}_p(M)$ of all circuits C of M with $|C\cap F(p)|=1$ such that

$$\sum_{C\in\mathscr{C}_p, C\ni e}\Phi(C)=\begin{cases}\leqslant p(e) & \text{if } e\in E(M)-F(p),\\ =-p(e) & \text{if } e\in F(p).\end{cases}$$

If Φ can be chosen integer valued we say that (M,p) has an *integer solution*.

A function $m:E(M)\to\mathbb{R}$ is called a *metric* if $m\geqslant 0$ and $m(e)\leqslant m(C-\{e\})$ for all circuits C of M and for all elements e of C. (We use the notation $m(X)=\sum_{e\in X}m(e)$ for subsets X of $E(M)$.) Given $l:E(M)\to\mathbb{R}_+$ we call the

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function m defined by $m(e) = \min\{l(X): X = \{e\}$ or $X = C - e$ for some circuit C with $e \in C\}$ the metric *induced* by l. This is indeed a metric. Δ is a *family of metrics* if for every binary matroid M, $\Delta(M)$ is a set of metrics defined on $E(M)$. For example, we shall consider the family Δ_A of metrics having values in a subset A of $\mathbb{Z}_+$; thus $\Delta_A(M)$ is the set of metrics $m: E(M) \to A$.

The following proposition is easy to prove via linear programming (Farkas' Lemma). We shall actually use only the trivial 'only if' part of this statement.

Proposition 1.1. *A flow problem* (M, p) *has a solution if and only if*

$$m \cdot p \geqslant 0 \quad \text{for all } m \in \Delta_{\mathbb{Z}_+}.$$

For graphs this is the so-called *Japanese Theorem*, see Iri [1] and Onaga and Kakusho [5]. (In fact, it is easy to see that this statement holds for arbitrary, not necessarily binary matroids.)

Let Δ be a family of metrics, and let (M, p) be a flow problem. Consider the condition

$$m \cdot p \geqslant 0 \quad \text{for all } m \in \Delta(M). \tag{1}$$

A binary matroid M for which this condition (1) is sufficient for the existence of a solution of (M, p) for *arbitrary* functions p, will be called *flowing with respect to* Δ. If (1) is sufficient for the existence of an *integer* solution for all *Eulerian* problems (M, p), then M will be called *cycling with respect to* Δ. (A flow problem (M, p) is *Eulerian* if $p(D)$ is even for all cocircuits D of M.) It is easy to see that cyclingness with respect to Δ implies flowingness with respect to Δ. (Let (M, p) be a flow problem which satisfies (1); then $(M, 2 \cdot p)$ is Eulerian and satisfies (1) too; hence there exits an integer solution Φ of $(M, 2 \cdot p)$ and consequently $\frac{1}{2}\Phi$ forms a solution of (M, p).) Seymour's '∞-flowing' ('∞-cycling') corresponds to 'flowing (cycling) with respect to $\Delta_{\{0,1\}}$' or 'with respect to cut metrics' (see Section 2) in our terminology.

Note that the nontrivial direction of Proposition 1.1 asserts that every binary matroid is flowing with respect to $\Delta_{\mathbb{Z}_+}$.

We shall denote by $\mathscr{C}(M)$ the set of cycles (that is disjoint unions of circuits) of the matroid M and by $\mathscr{C}^*$ the set of cocycles. The symbols '\' and '/' will stand for deletion and contraction respectively. For a definition of these and others terms of matroid theory see for example Welsh [9].

The main problem we are interested in, is to characterize matroids cycling with respect to the family of all metrics; these matroids are also called 'routing'. Such a characterization would be an elegant extension of Seymour's basic theorems about integer flows in Eulerian matroids (that is, about matroids cycling with respect to cut metrics, see Section 2).

This problem seems to be difficult though. Seymour's method does not extend, because the sum operations fail to work in the usual way. However, a particular way of using them permits to extend Seymour's class of ∞-cycling matroids, allowing a characterization of routingness among matroids without certain minors, namely AG(2, 3), S_8 and $M(H_6)$ (see Section 2 and Sebő [7]). Unfortunately these three excluded minors are routing.

The main result of the present paper is that cyclingness with respect to a naturally arising special family of metrics can be completely characterized. It turns out that the above mentioned AG(2, 3), S_8 and $M(H_6)$ are not cycling with respect to these special metrics; the 'particular use' of sum operations remains possible, generating now all the matroids we want.

An additional technical difficulty here, which may require some patience from the reader too, is that checking the property for the 'bricks' of the decomposition becomes a nontrivial, sometimes complicated task.

The characterization of classes of binary matroids flowing or cycling with respect to certain families of metrics in terms of excluded minors is possible because of the following.

Proposition 1.2. *Let Δ be a family of metrics closed under minor taking, and let M be a matroid flowing (cycling) with respect to Δ. Then all the minors of M are also flowing (cycling) with respect to Δ.*

We omit the proof since it is easy and contains no new element compared to the analogous statements (3.4) and (3.5) of Seymour [8]. In this connexion 'closed under minor taking' means that the restriction of the metric to the elements of a minor defines a metric on that minor which also belongs to the family Δ.

2. A generalization of the cut-condition

Let $\Delta_{(CC)}(M)$ be the set of all *cut-metrics* of the binary matroid M, that is, $m \in \Delta_{(CC)}(M)$ if and only if m is the incidence vector χ^D of a cocycle D of M. Thus (M, p) satisfies the so-called *cut-condition* if and only if

$$m \cdot p \geqslant 0 \quad \text{for all } m \in \Delta_{(CC)}(M). \tag{CC}$$

The class of matroids flowing respectively cycling with respect to $\Delta_{(CC)}$ is Seymour's class of *∞-flowing* respectively *∞-cycling matroids* (see [8]). The following statement is obviously equivalent to [8, (4.5)].

Proposition 2.1. *A binary matroid M is flowing (respectively cycling) with respect to $\Delta_{(CC)}$ if and only if it is flowing (respectively cycling) with respect to $\Delta_{\{0,1\}}$.*

Thus, if we want to generalize the cut-condition, we have to go 'beyond' $\Delta_{\{0,1\}}$. Taking into account the previous proposition, Seymour's well-known characterization can be stated as follows.

Theorem 2.2. *For a binary matroid M the following are equivalent:*

(i) *M is cycling with respect to $\Delta_{(CC)}$;*

(ii) *M is flowing with respect to $\Delta_{\{0,1\}}$;*

(iii) *M has no F_7, R_{10} or $M(K_5)$ minor.*

$M(K_5)$ is the polygon matroid of the complete graph on 5 vertices; F_7 is the *Fano matroid* (the projective plane of dimension two over GF(2)) and R_{10}, a matroid with ten elements, is well known by the leading role it plays in Seymour's decomposition theorem of regular matroids. Binary representations of the latter two matroids can be found in Seymour [8].

A next natural question is to investigate the class of binary matroids flowing or cycling with respect to $\Delta_{\{0,1,2\}}$. Here too, it will turn out that we can actually restrict ourselves to a special subfamily of such metrics (although it is not in general true that $\Delta_{\{0,1,2\}}(M)$ is a subset of the cone of the special metrics, unlike Seymour's $\Delta_{\{0,1\}}(M) \subseteq \text{cone}(\Delta_{(CC)}(M))$). If we consider metrics of the form

$$m(e) = \begin{cases} \alpha & \text{if } e \in D_1 - D_2, \\ \beta & \text{if } e \in D_2 - D_1, \\ \gamma & \text{if } e \in D_1 \cap D_2, \\ 0 & \text{if } e \in E(M) - (D_1 \cup D_2), \end{cases}$$

where α, β, γ are nonnegative numbers and D_1, D_2 are arbitrary cocycles, then elementary calculations show that these metrics are nonnegative linear combinations of the vectors χ^{D_1}, χ^{D_2} and $\chi^{D_1 \Delta D_2}$ (Δ denotes the symmetric difference). Hence $m \in \text{cone}(\Delta_{(CC)}(M))$ and we do not get anything new. The situation changes if we proceed to the case of three cocycles: Let D_1, D_2 and D_3 be three cocycles of M and let $\Delta_{(CC3)}(M)$ be the set of all functions $m: E(M) \to \mathbb{Z}_+$ defined in the following way:

$$m(e) = \begin{cases} 1 & \text{if } e \in D_1, \\ 2 & \text{if } e \in (D_2 \cup D_3) - 1, \\ 0 & \text{if } e \in E(M) - (D_1 \cup D_2 \cup D_3). \end{cases} \tag{2}$$

It is easy to see that these functions are in fact metrics. The following 'generalized cut-condition' will turn out to be equivalent to the restriction of (1) to metrics in $\Delta_{\{0,1,2\}}$:

$$m \cdot p \geq 0 \quad \text{for all } m \in \Delta_{(CC3)}(M). \tag{CC3}$$

This generalizes some metrics introduced by Karzanov for graphs. Given an undirected graph $G = (V, E)$ and a partition of V in $r + s$ possibly empty classes $A_1, \ldots, A_r$, $B_1, \ldots, B_s$ such that $A_1 \cup \cdots \cup A_r$ and $B_1 \cup \cdots \cup B_s$ are nonempty, define a metric $m: E \to \mathbb{Z}_+$ as follows:

$$m(xy) = \begin{cases} 1 & \text{if } x \in A_i, y \in B_j, \\ 2 & \text{if } x \in A_i, y \in A_j \ (i \neq j) \text{ or } x \in B_i, y \in B_j \ (i \neq j), \\ 0 & \text{if } x, y \in A_i \text{ or } x, y \in B_i. \end{cases}$$

For fixed r and s let $\Delta_{\mathrm{bip}(r,s)}(G)$ be the set of metrics defined on G in this way. In particular $\Delta_{\mathrm{bip}(1,1)}(G) = \Delta_{(CC)}(G)$. Such metrics play a crucial role in the works Karzanov [2, 3] It can be shown that $\Delta_{(CC)}$ and $\Delta_{\mathrm{bip}(2,3)}$ do not generate all of $\Delta_{\mathrm{bip}(r,s)}$. (However, we shall see that for the graphs we are interested in this is true, see Corollary 2.6.)

The next theorem is proved in Karzanov [2].

Theorem 2.3. *Let (G, p) be an Eulerian flow problem where the demand edges $e \in F(p)$ are adjacent to at most five vertices. (G, p) has an integer solution if and only if $m \cdot p \geq 0$ for all $m \in \Delta_{\mathrm{bip}(2,3)}(M)$.*

$\Delta_{(CC3)}$ is a quite natural matroid theoretical analogon for $\Delta_{\mathrm{bip}(2,3)}$. Obviously $\Delta_{\mathrm{bip}(2,3)} \subseteq \Delta_{(CC3)}$ for a given graph G; choose the cocycles D_1, D_2 and D_3 of (2) as follows (δX denotes the set of edges of E with one end in $X \subseteq V$, the other one in $V - X$):

$$\begin{aligned} D_1 &= \delta(A_1 \cup A_2), \\ D_2 &= \delta(A_1 \cup B_1), \\ D_3 &= \delta(A_2 \cup B_3). \end{aligned} \tag{3}$$

Corollary 2.4. *K_5 is cycling with respect to $\Delta_{\mathrm{bip}(2,3)}$; $M(K_5)$ is cycling with respect to $\Delta_{(CC3)}$.*

Remember that $M(K_5)$ is not cyling with respect to $\Delta_{(CC)}$.

In Sebő [7] it is proved that all the six non-isomorphic 2-sums of the three matroids F_7, $M(K_5)$ and R_{10} listed in Theorem 2.2 are minimal noncycling with respect to $\Delta_{\mathcal{F}_+}$. (We define the 1-sum $M_1 \oplus M_2$ and the 2-sum $M_1 \oplus_2 M_2$ of binary matroids in the usual way, see Seymour [8]). These six matroids are called *bi-nonflowing* and denoted by $B_{i,j}$, where i and j are the indices of the two members of the 2-sum (for example $B_{5,7}$ is $M(K_5) \oplus_2 F_7$). Then it is shown in Sebő [7] that a matroid without AG(2, 3), S_8 and $M(H_6)$ minors is cycling with respect to the family of all metrics (shortly: routing) if and only if it does not contain any bi-nonflowing minors. While the class of routing matroids is much bigger than the class exhibited by this result, Theorem 2.5 below presents a complete characterization of cyclingness with respect to $\Delta_{(CC3)}$.

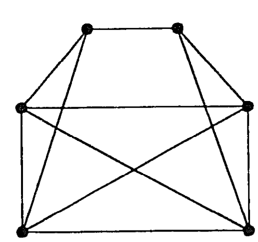

Fig. 1. H_6.

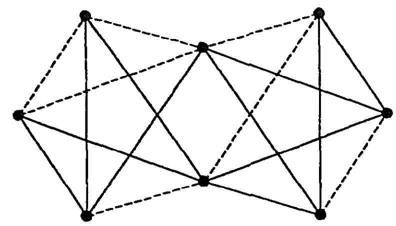

Fig. 2. $B_{5,5}$.

Fig. 2 depicts the graph corresponding to $B_{5,5}$, that was found and used as a basic example by Middendorf and Pfeiffer [4].

We are now ready to state our main result.

Theorem 2.5. *For a binary matroid M the following are equivalent.*

(i) *M is cycling with respect to* $\Delta_{(CC3)}$;

(ii) *M is flowing with respect to* $\Delta_{\{0,1,2\}}$;

(iii) *M has no* $AG(2,3)$, S_8, R_{10}, $M(H_6)$, $B_{5,5}$, $B_{5,7}$, *or* $B_{7,7}$ *minor.*

H_6 shown in Fig. 1 is Papernov's graph (Papernov [6]). Binary representations of the eight-element matroids $AG(2,3)$ (the affine geometry of dimension three over GF(2)) and S_8 are given in Seymour [8].

The proof gives a somewhat sharper statement for graphs.

Corollary 2.6. *For a graph G the following are equivalent*:

(i) *G is cycling with respect to* $\Delta_{\mathrm{bip}(2,3)}$;

(ii) *G is flowing with respect to* $\Delta_{\{0,1,2\}}$;

(iii) *G has no* H_6 *or* $B_{5,5}$ *minor.*

(Here of course $B_{5,5}$ denotes the graph rather than the graphic matroid.)

3. Proof of Theorem 2.5

The implication (i)⇒(ii) is trivial. (Remember that $\Delta_{(CC3)}$ is a subset of $\Delta_{\{0,1,2\}}$).

To prove that (ii) implies (iii), we have to show that all the seven matroids listed in (iii) are not flowing with respect to $\Delta_{\{0,1,2\}}$. Before doing this we formulate a well-known observation, which will be useful more than once in the sequel.

Lemma 3.1. *Let M be a binary matroid and* $A \subseteq \mathbb{Z}_+$. *M is flowing with respect to* Δ_A *if and only if* $\Delta_{\mathbb{Z}_+}(M) \subset \mathrm{cone}(\Delta_A(M))$

Proof. The 'if' part is a direct consequence of the *Japanese Theorem*. For the 'only if' part assume that the metric m is not expressible as a nonnegative linear combination of metrics in $\Delta_A(M)$; by Farkas' Lemma there exists a function $p: E(M) \to \mathbb{Z}$ with $m' \cdot p \geq 0$ for all $m' \in \Delta_A(M)$, but $m \cdot p < 0$. This is a contradiction to the flowingness with respect to Δ_A. □

Proposition 3.2. *AG(2, 3), S_8, R_{10} and $M(H_6)$ are not flowing with respect to $\Delta_{\{0,1,2\}}$.*

Proof. As a consequence of the *Japanese Theorem* it is sufficient to exhibit for each matroid $M \in \{\mathrm{AG}(2,3), S_8, R_{10}, M(H_6)\}$ a vector $p_M: E(M) \to \mathbb{Z}$ such that $m \cdot p_M \geq 0$ for all $m \in \Delta_{\{0,1,2\}}$, but $m_p \cdot p_M < 0$, where m_p is the metric induced by the function $l_p: E(M) \to \mathbb{Z}_+$,

$$l_p(e) = \begin{cases} 1 & \text{if } p(e) \geq 0, \\ |E(M)| & \text{if } p(e) < 0. \end{cases}$$

Choose p_M for every matroid as in Seymour [8] in the proof of its nonflowingness. (For $p = p_{M(H_6)}$ we have $p(e) = -2$ on the upper horizontal edge of Fig. 1, $p(e) = -1$ on the two vertical edges and $p(e) = 1$ on the remaining eight edges; for $p = p_{\mathrm{AG}(2,3)}$, $p(f) = -3$ for an arbitrary element f and $p(e) = 1$ for $e \in E(\mathrm{AG}(2,3)) - f$; for $p = p_{S_8}$, $p(f) = -2$ for the element f contained in all circuits of cardinality 3, $p(g) = -1$ for the element g contained in no circuit of cardinality 3, and $p(e) = 1$ for $e \in E(S_8) - \{f, g\}$; finally for $p = p_{R_{10}}$, $p(e) = -1$ on a three element subset of a four element circuit and $p(e) = 1$ on the remaining seven elements.)

The strict inequality $m_p \cdot p_M < 0$ is easily checked, proving that there is no flow.

To show $m \cdot p_M \geq 0$ for all $m \in \Delta_{\{0,1,2\}}$, let $m \in \Delta_{\{0,1,2\}}$ be chosen arbitrarily and $p = p_M$; if moreover $m \in \Delta_{\{1,2\}}$, then we have immediately the result

$$m \cdot p \geq 1 \cdot p(E(M) - F(p)) + 2 \cdot p(F(p)) \geq 0$$

for all four matroids M. We thus may assume that there exists an element $e \in E(M)$ with $m(e) = 0$; denoting by f_e the restriction of the function f to $E(M) - e$, m_e is a metric on M/e, and the inequality $m \cdot p \geq 0$ to be proved is equivalent to $m_e \cdot p_e \geq 0$.

This is trivial if M/e is flowing with respect to $\Delta_{(CC)}$; p was chosen so that it satisfies the cut-condition (implying that the cut-condition is also satisfied in $(M/e, p_e)$), and clearly, in matroids flowing with respect to $\Delta_{(CC)}$ the cut-condition implies $m \cdot p \geq 0$ via Lemma 3.1. Similarly, if $m_e(f) = 0$ for some $f \in E(M) - e$, then we immediately have $m \cdot p \geq 0$, because contracting two different elements in any of the four considered matroids results in a matroid which is flowing with respect to $\Delta_{(CC)}$.

Thus the only thing remaining to be proved is $m_e \cdot p_e \geq 0$ for $m_e \in \Delta_{\{1,2\}}$. This follows easily for all needed cases in the same way as in the beginning of the proof of '$m \cdot p \geq 0$ for $m \in \Delta_{\{1,2\}}$'. (Since m is a metric, an element g which is parallel to f with $m_e(f) = 2$, must also have $m_e(g) = 2$, and this is the only place we use that m is a metric). □

The result stating the nonflowingness with respect to $\Delta_{\{0,1,2\}}$ of the bi-nonflowing matroids is postponed to Proposition 3.4.

Proposition 3.3. *F_7 is cycling with respect to $\Delta_{(CC3)}$.*

Proof. Let $p:E(F_7)\to\mathbb{Z}$ be Eulerian and suppose that (CC3) is satisfied. We have to show that (F_7,p) has an integer solution. Every proper minor of F_7 is cycling even with respect to $\Delta_{(CC)}$ (see Theorem 2.2). Hence we assume for the rest of the proof that $p(e)\neq 0$ for all $e\in E(F_7)$.

In accordance with the terminology in Seymour [8] we shall call a binary matroid M *F-cycling with respect to Δ* (where Δ is some family of metrics), if $F\subseteq E(M)$ and the validity of (1) implies the existence of an integer solution for (M,p) for all Eulerian p with $F(p)=F$.

If $|F|\leqslant 2$ then F_7 is F-cycling with respect to $\Delta_{(CC)}$ (see (13.4) of Seymour [8]).

If $|F|\geqslant 5$ then F contains a cocircuit, (CC) is always violated and F_7 is F-cycling with respect to $\Delta_{(CC)}$.

Let $|F|=4$ and $F=\{e_1,e_2,e_3,e_4\}$ be not a cocircuit. Then it is easy to check that there are circuits (and at the same time cocircuits) $C_1=\{e_1,e_2,e_3,e_5\}$, $C_2=\{e_1,e_3,e_4,e_6\}$ and $C_3=\{e_2,e_3,e_4,e_7\}$ such that (F_7,p) (with $F(p)=F$) has an integer solution if and only if $p(C_1)\geqslant 0$, $p(C_2)\geqslant 0$ and $p(C_3)\geqslant 0$, that is if and only if (CC) holds. Thus F_7 is again F-cycling with respect to $\Delta_{(CC)}$.

This is also true if $|F|=3$ and F is not a cocircuit. Say $\{e_1,e_2,e_4\}$, $\{e_1,e_3,e_5\}$, $\{e_1,e_6,e_7\}$, $\{e_2,e_3,e_6\}$, $\{e_2,e_5,e_7\}$, $\{e_3,e_4,e_7\}$ and $\{e_4,e_5,e_6\}$ are the circuits of cardinality three of F_7 and $F=\{e_1,e_2,e_3\}$. Suppose that the cut-condition (CC) holds for (F_7,p) $(F(p)=F)$ and define $p':E(F_7)\to\mathbb{Z}$ by

$$p'(e)=\begin{cases}p(e_i)+1 & \text{if } i=1,\\ p(e_i)-1 & \text{if } i\in\{6,7\},\\ p(e_i) & \text{if } i\in\{2,3,4,5\}.\end{cases}$$

We have to show that the cut-condition is satisfied for (F_7,p'). (Then the result follows by induction on $|p(e_1)|$, because $\{e_1,e_6,e_7\}$ is a circuit and a flow of value 1 through it together with an integer flow for (F_7,p') results in an integer flow for (F_7,p)). Assume not. Then necessarily $p'(D_1)<0$ or $p'(D_2)<0$, where $D_1=\{e_2,e_4,e_6,e_7\}$ and $D_2=\{e_3,e_5,e_6,e_7\}$. D_1 and D_2 are cocircuits, hence $p(D_1)\in\{0,1\}$ or $p(D_2)\in\{0,1\}$. This together with $p(\{e_1,e_2,e_3,e_7\})\geqslant 0$ implies that $p(e_i)=0$ for at least three elements e_i of one of D_1 or D_2, which is a contradiction.

Finally let F be a three element circuit, say $F=\{e_1,e_2,e_4\}$. Define p' as above and $p'':E(F_7)\to\mathbb{Z}$ by

$$p''(e)=\begin{cases}p(e_i)+1 & \text{if } i=1,\\ p(e_i)-1 & \text{if } i\in\{3,5\},\\ p(e_i) & \text{if } i\in\{2,4,6,7\}.\end{cases}$$

The functions p' and p'' are Eulerian. As in the preceding case we are done by induction on $|p(e_1)|$ if at least one of (F_7, p'), (F_7, p'') has an integer solution.

Assume not. Then by the induction hypothesis there are metrics $m', m'' \in \Delta_{(CC3)}(F_7)$ such that $m'p' < 0$ and $m''p'' < 0$.

Claim: $m', m'' \in \Delta_{(CC)}(F_7)$.

For an arbitrary cocircuit D let $m_D = \chi^D + 2 \cdot \chi^{E-D}$. It is easy to see that

$$\Delta_{(CC3)}(F_7) - \text{cone}(\Delta_{(CC)})(F_7) = \{m_D \colon D \text{ cocircuit of } F_7\}.$$

We conclude $0 \leq m_{D_1}p = m_{D_1}p' \leq m_D p'$ for an arbitrary cocircuit D and for $D_1 = \{e_3, e_5, e_6, e_7\}$. The same argument works with p'' instead of p'. This proves the Claim.

Now there is only one possible choice of m' and m''. Let $D' = \{e_2, e_4, e_6, e_7\}$, $D'' = \{e_2, e_3, e_4, e_5\}$, $m' = \chi^{D'}$ and $m'' = \chi^{D''}$. Then $m'p' = m''p'' = -2$ and $m'p = m''p = 0$. But

$$0 \leq m_{D_1}p - m'p - m''p = 2 \cdot p(e_1) < 0,$$

a contradiction. □

Proposition 3.4. *$B_{5,5}$, $B_{5,7}$ and $B_{7,7}$ are not flowing with respect to $\Delta_{\{0,1,2\}}$.*

Proof. Let $M_1, M_2 \in \{M(K_5), F_7\}$. M_1 and M_2 are—by Corollary 2.4 and Proposition 3.3—flowing with respect to $\Delta_{\{0,1,2\}}$, and—by Theorem 2.2—minimal not flowing with respect to $\Delta_{\{0,1\}}$. Therefore there exist functions $p_i \colon E(M_i) \to \mathbb{Z} - \{0\}$ $(i = 1, 2)$ with $m \cdot p \geq 0$ for all $m \in \Delta_{\{0,1\}}(M_i)$ and $m \cdot p < 0$ for some $m \in \Delta_{\{0,1,2\}}(M_i)$. Let $m \in \Delta_{\{0,1,2\}}(M_i)$ with $M(e) = 0$; M_i/e is flowing with respect to $\Delta_{\{0,1\}}$, by Lemma 3.1 the restriction of m to $E(M_i) - e$ is a nonnegative linear combination of metrics in $\Delta_{\{0,1\}}(M_i/e)$ and thus $m \in \text{cone}(\Delta_{\{0,1\}}(M_i))$. We conclude:

$$\begin{aligned} &m \cdot p_i \geq 0 \quad \text{for all } m \in \Delta_{\{0,1,2\}}(M_i) - \Delta_{\{1,2\}}(M_i), \\ &m \cdot p_i < 0 \quad \text{for some } m \in \Delta_{\{1,2\}}(M_i). \end{aligned} \tag{4}$$

To simplify matters let $E_i = E(M_i)$ and $F_i = F(p_i)$. Choose $f_i \in E_i$ such that $p_1(f_1) > 0$ and $p_2(f_2) < 0$.

Roughly speaking, we shall proceed as follows: First, by blowing up 'capacity' elements with an appropriate factor $\alpha_i > 1$ we guarantee the existence of a fractional flow; second, we again prevent M_1 from having a flow by multiplying $p_1(f_1)$ with a factor β less than but not too far from one; third we multiply 'capacities' and 'demands' in M_2 by some factor γ in order to have equal flows 'through' f_1 and f_2. More precisely, let

$$q_1(e) = \begin{cases} p_1(e) & \text{if } e \in F_1, \\ \alpha_1 \cdot p_1(e) & \text{if } e \in E_1 - (F_1 \cup f_1), \\ \beta \cdot \alpha_1 \cdot p_1(f) & \text{if } e = f_1, \end{cases}$$

and

$$q_2(e) = \begin{cases} \gamma \cdot p_2(e) & \text{if } e \in F_2, \\ \gamma \cdot \alpha_2 \cdot p_2(e) & \text{if } e \in E_2 - F_2, \end{cases}$$

where

$$\alpha_i = -2 \cdot p_i(F_i)/(p_i(E_i - F_i) \quad (i \in \{1, 2\}),$$

$$\beta = \max\{1/\alpha_1, 1/\alpha_2\},$$

$$\gamma = -\beta \cdot \alpha_1 \cdot p_1(f_1)/p_2(f_2).$$

It does not cause problems that the functions q_1 and q_2 are possibly fractional; they can be made integral at every stage of the proof by multiplying them with an adequate factor.

For $k \in \{0, 1, 2\}$ and $i \in \{1, 2\}$ define

$$S_k(M_i) = \min\{m \cdot q_i : m \in \Delta_{\{0,1,2\}}(M_i), m(f_i) = k\}.$$

Claim 1. $1 < \alpha_i < 2$ $(i = 1, 2)$, $\beta < 1$;

Let p_i' be the function obtained from p_i by multiplying 'capacities' with α_i. The values α_i are chosen such that $\min\{m \cdot p_i' : m \in \Delta_{\{1,2\}}(M_i)\} = m_i^* \cdot p_i' = 0$, where m_i^* is the metric with value 2 on F_i and value 1 on $E_i - F_i$. Thus, by (4), $\alpha_i > 1$ and $\beta < 1$.

Assume $\alpha_i \geqslant 2$ and let $m \in \Delta_{\{0,1\}}(M_i)$ be the everywhere one metric; then $m_i^* \cdot p_i' \geqslant 2 \cdot m \cdot p_i \geqslant 0$, a contradiction.

Claim 2. $S_0(M_1) = 0$; $S_1(M_1) = \alpha_1 \cdot (\beta - 1) \cdot p_1(f_1)$; $S_2(M_1) \geqslant 0$.

Claim 1 yields $q_1(e) \geqslant p_1(e)$ for all $e \in E_1 - F_1$. Hence $m \cdot q_1 \geqslant 0$ for all $m \in \Delta_{\{0,1,2\}}(M_1) - \Delta_{\{1,2\}}(M_1)$. Now to get the values of S_0, S_1, S_2 respectively, consider the everywhere zero metric, the metric m_1^* and the metric having value 1 on $E_1 - (F_1 \cup f_1)$ and value 2 on $F_1 \cup f_1$.

Claim 3. $S_0(M_2) = 0$; $S_1(M_2) = \gamma \cdot (1 - \alpha_2) \cdot p_2(f_2)$; $S_2(M_2) = 0$.

$S_0 = S_2 = 0$ again is an immediate consequence of the choice of α_2. To find the value of S_1, we first consider metrics without zeroes:

$$\min\{m \cdot q_2 : m \in \Delta_{\{1,2\}}(M_2), m(f_2) = 1\}$$
$$= m_2^* \cdot q_2 - q_2(f_2) = -q_2(f_2) > (1 - \alpha_2) \cdot q_2(f_2) = \gamma \cdot (1 - \alpha_2) \cdot p_2(f_2),$$

because $\alpha_1 < 2$ by Claim 1.

Now we turn to metrics with at least one zero value; they are nonnegative linear combinations of metrics in $\Delta_{\{0,1\}}(M_2)$ and it is easy to see that

$$\min\{m \cdot q_2 : m \in \Delta_{\{0,1,2\}} - \Delta_{\{1,2\}}, m(f_2) = 1\}$$
$$= \min\{m \cdot q_2 : m \in \Delta_{\{0,1\}}, m(f_2) = 1\}.$$

Thus if $m \in \Delta_{\{0,1\}}(M_2)$ and $m(f_2) = 1$, then

$$m \cdot q_2 = \gamma \cdot m \cdot p_2 + \gamma \cdot (\alpha_2 - 1) \cdot \sum_{e \in E_2 - F_2} m(e) \cdot p_2(e)$$
$$\geqslant \gamma \cdot m \cdot p_2 + \gamma \cdot (1 - \alpha_2) \cdot \sum_{e \in F_2} m(e) \cdot p_2(e)$$
$$\geqslant \gamma \cdot (1 - \alpha_2) \cdot p_2(f_2).$$

Claim 4. $S_k(M_1) + S_k(M_2) \geqslant 0$ $(k \in \{0, 1, 2\})$, $2 \cdot S_1(M_1) + S_2(M_2) < 0$.

This can be seen by simple calculations using the values of $S_k(M_i)$ found in Claim 2 and Claim 3.

Let now $M = M_1 \oplus_2 M_2$ with $E(M) = (E_1 - f_1) \cup (E_2 - f_2)$ and

$$q(e) = \begin{cases} q_1(e) & \text{if } e \in E_1 - f_1, \\ q_2(e) & \text{if } e \in E_2 - f_2. \end{cases}$$

Claim 5. *If $m \in \Delta_{\{0,1,2\}}(M)$, then there exists a number $k \in \{0, 1, 2\}$ such that m_1 and m_2 defined by*

$$m_i(e) = \begin{cases} m(e) & \text{if } e \in E_i - f_i, \\ k & \text{if } e = f_i \end{cases}$$

are metrics on M_1, M_2 respectively.

Let $\mathscr{D} = \{C - f_1 : f_1 \in C \in \mathscr{C}(M_1)\} \cup \{C - f_2 : f_2 \in C \in \mathscr{C}(M_2)\}$: m_1 and m_2 are metrics on M_1, M_2 respectively, if and only if

$$2 \cdot m(e) - m(D) \leqslant k \leqslant m(D) \quad \text{for all } D \in \mathscr{D}.$$

Assume there does not exist such a number k. Then there exist $D, D' \in \mathscr{D}$ with $2 \cdot m(e) - m(D) > m(D')$, that is $2 \cdot m(e) > m(D \triangle D') + 2 \cdot m(D \cap D')$. If $e \in D \triangle D'$, then $m(e) > m((D \triangle D') - e)$, a contradiction, because $D \triangle D'$ is a cycle of M, and if $e \in D \cap D'$ we conclude $M(D \triangle D') < 0$, again a contradiction.

Now choose $k = \max\{0, \max\{2 \cdot m(e) - m(D) : e \in D \in \mathscr{D}\}\}$; this implies $k \leqslant 2$, because if $2 \cdot m(e) - m(D) > 2$, then $2 \cdot m(e) > m(D) + 2 \geqslant m(e) + 2$ and hence $m(e) > 2$, a contradiction.

Claim 6. *M is not flowing with respect to $\Delta_{\{0,1,2\}}$.*

Giv. $m \in \Delta_{\{0,1,2\}}(M)$, choose k as in Claim 5. Then

$$m \cdot q = m_1 \cdot q_1 + m_2 \cdot q_2 \geq S_k(M_1) + S_k(M_2) \geq 0$$

by Claim 4.

By Claims 2 and 3 there exist metrics $m_i \in \Delta_{\{0,1,2\}}(M_i)$ $(i = 1, 2)$ with $m_1(f_1) = 1$, $m_2(f_2) = 2$, $m_1 \cdot q_1 < 0$ and $m_2 \cdot q_2 = 0$. Obviously m defined by

$$m(e) = \begin{cases} 2 \cdot m_1(e) & \text{if } e \in E_1 - f_1, \\ m_2(e) & \text{if } e \in E_2 - f_2 \end{cases}$$

is a metric in $\Delta_{\{0,\ldots,4\}}(M)$, but $m \cdot q = 2 \cdot m_1 \cdot q_1 + m_2 \cdot q_2 < 0$ by Claim 4. Thus (M, q) has no solution. □

Fig. 2 showing $B_{5,5}$ illustrates Proposition 3.4. If $q(e) = -3$ for all dotted edges and $q(e) = 4$ for the remaining ones, then $m \cdot q \geq 0$ for all $m \in \Delta_{\{0,1,2\}}(B_{5,5})$, but there is no flow.

It remains to show that (iii) implies (i). By combining Theorem 2.2 with a twofold application of Seymour's 'Splitter Theorem' ((6.3) in Seymour [8]) we obtain the following.

Proposition 3.5. *Every binary matroid with no* AG(2, 3), S_8, R_{10} *or* $M(H_6)$ *minor may be obtained by* 1- *and* 2-*sums from matroids cycling with respect to* $\Delta_{(CC)}$ *and copies of* F_7 *and* $M(K_5)$.

Restricting Proposition 3.5 to graphic matroids—that is to those binary matroids without F_7, F_7^*, $M^*(K_5)$ and $M^*(K_{3,3})$ minors—one gets the following.

Corollary 3.6. *Every graphic matroid with no* $M(H_6)$ *minor may be obtained by* 1- *and* 2-*sums from graphic matroids with no* $M(K_5)$ *minor and copies of* $M(K_5)$.

Proof. F_7 is a minor of AG(2, 3) and S_8, $M^*(K_{3,3})$ is a minor of R_{10} and $M(K_5)$ is minor of $M(H_6)$. □

It was shown in Seymour [8] that taking the 1-sum or 2-sum of matroids cycling with respect to $\Delta_{(CC)}$ results in a matroid cycling with respect to $\Delta_{(CC)}$; it is also trivial to verify that $\Delta_{(CC3)}$-cyclingness is preserved under taking the 1-sum of binary matroids. The example of $B_{5,5}$ shows that the same is not true for 2-sums. However, for our purposes the following 'skew' decomposition lemma, which seems to be a characteristic feature of metrics more general than cut-metrics (see Sebő [7]), turns out to be sufficient.

Proposition 3.7. *Any* 2-*sum* $M_1 \oplus_2 M_2$ *of a matroid* M_1 *cycling with respect to* $\Delta_{(CC3)}$ *and a matroid* M_2 *cycling with respect to* $\Delta_{(CC)}$ *is cycling with respect to* $\Delta_{(CC3)}$.

Proof. Let $E(M_1) \cap E(M_2) = \{f\}$ and $M = M_1 \oplus_2 M_2$. Choose $p : E(M) \to \mathbb{Z}$ such that (M, p) is Eulerian and (CC3) is satisfied. We define functions $p_i : E(M_i) \to \mathbb{Z}$

($i \in \{1, 2\}$) in the following way:

$$p_i(e) = \begin{cases} p(e) & \text{if } e \in E(M_i) - f, \\ (-1)^{i-1}q & \text{if } e = f, \end{cases}$$

where $q = \min\{p(D-f): f \in D \in \mathscr{C}^*(M_2)\}$. Let D_0 be a cocycle of M_2 with $p(D_0 - f) = q$.

Claim 1. p_i ($i \in \{1, 2\}$) *is an Eulerian function.*

Let D_i be a cocycle of M_i. If $f \notin D_i$, then $p_i(D_i) = p(D_i) \equiv 0 \bmod 2$, because D_i is also a cocycle of M. If $f \in D_i$, then

$$\begin{aligned} p_i(D_i) &= p_i(D_i - f) + p_i(f) \\ &\equiv p(D_i - f) + p(D_0 - f) \equiv p(D_i \triangle D_0) \equiv 0 \bmod 2, \end{aligned}$$

because $D_i \triangle D_0$ is a cocycle of m.

Claim 2. (M_2, p_2) *satisfies* (CC).

Let $D \in \mathscr{C}^*(M_2)$. If $f \notin D$, then again D is a cocycle of M and $p_2(D) = p(D) \geqslant 0$, because we assumed that (CC3) and so in particular (CC) is satisfied for (M, p). If $f \in D$, then the definition of q implies the following inequality: $p_2(D) = p_2(D-f) + p_2(f) = p(D-f) - p(D_0 - f) \geqslant 0$.

Claim 3. (M_1, p_1) *satisfies* (CC3).

To each subset A of $E(M_1)$ we assign a subset A^0 of $E(M)$ in the following way:

$$A^0 = \begin{cases} A & \text{if } f \notin A, \\ A \triangle D_0 & \text{if } f \in A. \end{cases}$$

It follows from the definition of p_1 and D_0 that

$$p_1(A) = p(A^0) \tag{5}$$

and that if A is a cocycle of M_1 then A^0 is a cocycle of M. The following two properties are easily checked: $(A \cup B)^0 = A^0 \cup B^0$ and $(A - B)^0 = A^0 - B^0$.

We have to show that $p_1(D_1) + 2 \cdot p_1((D_2 \cup D_3) - D_1)$ is nonnegative for every choice of cocycles $D_1, D_2, D_3 \in \mathscr{C}^*(M_1)$. As (M, p) satisfies (CC3) it is sufficient to verify the following equality:

$$p_1(D_1) + 2 \cdot p_1((D_2 \cup D_3) - D_1) = p(D_1^0) + 2 \cdot p((D_2^0 \cup D_3^0) - D_1^0). \tag{6}$$

This can now be shown by an easy calculation applying the above rules. In particular we get

$$p((D_2^0 \cup D_3^0) - D_1^0) = p(((D_2 \cup D_3) - D_1)^0),$$

from where the result follows by an application of rule (5). Thus Claim 3 is proved.

As M_1 (respectively M_2) was assumed to be cycling with respect to $\Delta_{(CC3)}$ (respectively $\Delta_{(CC)}$), the above claims guarantee the existence of integer flows Φ_i in (M_i, p_i) $(i \in \{1, 2\})$. Φ_i consists of a list of cycles of $\mathscr{C}_{p_i}(M_i)$. Suppose without loss of generality that $q \geqslant 0$ (to treat the case $q < 0$, simply interchange the roles of M_1 and M_2) and that precisely the first k_i cycles of each list contain the element f. It follows from the definition of a flow that $k_1 \leqslant q = k_2$. After deleting the first $k_2 - k_1$ cycles from the second list Φ_2, the union of the two lists contains exactly k_1 cycles of $\mathscr{C}(M_1)$ and k_1 cycles of $\mathscr{C}(M_2)$ passing through the element f. Build k_1 pairs (C_1, C_2) $(C_i \in \mathscr{C}(M_i))$ of the cycles passing through f and replace each pair by $C_1 \triangle C_2$. It is easy to see that the list of cycles obtained in this way represents an integer flow of (M, p). □

Corollary 3.8. *Any 2-sum $G_1 \oplus_2 G_2$ of a graph G_1 cycling with respect to $\Delta_{\text{bip}(2,3)}$ and a graph G_2 cycling with respect to $\Delta_{(CC)}$ is cycling with respect to $\Delta_{\text{bip}(2,3)}$.*

Proof. The proof of Proposition 3.7 can be copied step by step, replacing $\Delta_{(CC3)}$ by $\Delta_{\text{bip}(2,3)}$. In particular equation (6) holds. To convience ourselves that the right-hand side of (6) is nonnegative, we observe that if a metric $m_1 \in \Delta_{\text{bip}(2,3)}(G_1)$ is defined by three cocycles $D_1, D_2, D_3 \in \mathscr{C}^*(G_1)$ (just as in (3)), then the cocycles D_1^0, D_2^0, D_3^0 of $\mathscr{C}^*(G)$ $(G = G_1 \oplus_2 G_2)$ represent a metric $m \in \Delta_{\text{bip}(2,3)}(G)$. □

The following result is proved in Sebő [7].

Proposition 3.9. *Let the connected binary matroid M be built up by 2-sums from $M_1, M_2, \ldots, M_k$ $(k \geqslant 2)$, and suppose that there are indices i and j, $1 \leqslant i < j \leqslant k$, such that $M_i, M_j \in \{M(K_5), F_7\}$. Then M contains a minor $M_i \oplus_2 M_j$.*

To continue with the proof of Theorem 2.5 we assume that M does not have any of the minors listened in (iii). M is isomorphic to the 1-sum of its connected components, and by Proposition 3.5 every connected component N of M may be obtained by 2-sums from matroids $N_1, N_2, \ldots, N_{k(N)}$, which are either cycling with respect to $\Delta_{(CC)}$ or copies of F_7 and $M(K_5)$. N does not have a minor $B_{i,j}$ $(i, j \in \{5, 7\})$, and hence, by Proposition 3.9, at most one of the terms N_i is isomorphic to F_7 or $M(K_5)$. Thus by Corollary 2.4 and Proposition 3.3 and 3.7, N is cycling with respect to $\Delta_{(CC3)}$, and so is M. This completes the proof of Theorem 2.5. □

Proof of Corollary 2.6. (i)$\Rightarrow$(ii) is trivial. (ii)$\Rightarrow$(iii) follows from Propositions 3.2 and 3.4. (iii)$\Rightarrow$(i) follows from Corollaries 2.4, 3.6 and 3.8 and from the (graph-theoretical version of) Proposition 3.9. □

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