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The Cographic Multiflow Problem: An Epilogue

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ABSTRACT. We start with "cographic multiflows," a basic special case of "matroid flow problems," exhaustively studied in Seymour's celebrated paper, "Matroids and multicommodity flows." This special case is closely related to matching theory and contains also the edge-disjoint paths problem in graphs still planar after adding the "demand couples" as edges. M. Middendorf and F. Pfeiffer have proved recently that this latter problem is NP-complete, closing the cographic multiflow problem in some sense.

We first give a quick survey of the polynomially solvable special cases of this problem, the axis of our setting being the parity of cuts: for Eulerian graphs a theorem of Seymour establishes the sufficiency of the Cut Condition for the existence of integer flows; for non-Eulerian graphs a conjecture of Frank and some related results use strengthenings of the Cut Condition taking into consideration the parities of the cuts. We present a compact unified formulation of these conditions in terms of binary matroids. Through the notions "Eulerian extension" and "routing matroids" we embed this approach in Seymour's theory of flows in Eulerian binary matroids, motivating new research about matroid flow problems.

These investigations crystallize into one main question interesting for its own sake: *when does the existence of a fractional flow imply the existence of an integer one*? The same question arises if we try to replace in Seymour's work the role of the Cut Condition by stronger constraints: the strongest among the possible (still necessary) conditions is the existence of a fractional flow. One can expect that stronger conditions provide good characterizations for the existence of integer flows in more general matroids.

The greatest part of the paper deals with the above-mentioned question. This epilogue, after Middendorf and Pfeiffer's surprisingly negative answer to the cographic problem, also leads to a number of new problems and conjectures on matroid flows.

1. Introduction

This paper can be considered an epilogue to the multicommodity flow problem in cographic matroids, a problem rich in interesting special cases,

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and in some sense closed recently by the negative results of M. Middendorf and F. Pfeiffer [**38**]. Before this problem is buried I would like to give a map of the results and pose some questions on matroid flow problems that seem to grow naturally out of the theory.

"Matroid" in this paper always means binary matroid; that is, representability over GF(2) is supposed throughout. Let M be a matroid. The ground-set of M will be denoted by $E(M)$ or simply E. "$\backslash$" will denote deletion and "$/$" contraction.

Let $R \subseteq E$, $r\colon R \to \mathbb{N}$, and $c\colon E\backslash R \to \mathbb{N}$ ($\mathbb{N}$ is the set of positive integers). We shall say that (M, R, r, c) is a *network*. Suppose $\mathscr{C}$ is a set of circuits, $f\colon \mathscr{C} \to \mathbb{Q}^+$ ($\mathbb{Q}^+$ is the set of nonnegative rationals) and they satisfy

(i) $$|C \cap R| = 1 \quad \text{for all } C \in \mathscr{C},$$

(ii) $$\sum_{e\in C\in\mathscr{C}} f(C) = r(e) \quad \text{for all } e \in R,$$

(iii) $$\sum_{e\in C\in\mathscr{C}} f(C) \leq c(e) \quad \text{for all } e \in E(G)\backslash R.$$

(Think of f as a function giving the *multiplicities* of elements of the *multiset* $\mathscr{C}$. The sum of the multiplicities of the elements of a multiset will shortly be said to be the *cardinality* of the multiset and will be denoted accordingly. We shall use this terminology even if the multiplicities are nonintegers.) $r(e)$ is called the *demand* of $e \in R$, and $c(e)$ is the *capacity* of $e \in E(G)\backslash R$. $(\mathscr{C}, f)$ or shortly the multiset $\mathscr{C}$ will be called a *flow*, if it satisfies (i), (ii), and (iii). (The notation in (ii) and (iii) can be simplified using the cardinality of multisets.) If f is integer-valued we shall say that the flow is *integer*. R is called the *demand-graph* and $E\backslash R$ the *supply-graph*. Determining whether a flow exists in a given network is the *matroid multicommodity flow problem*.

Note that the fact of choosing the demand-edges in the matroid is already a restriction of generality: in this way we can restrict the union of the supply and of the demand-graph. This is the approach of Seymour [**60**]. Some other models restrict only one of these, or both, but each in a different way; see Lomonosov [**32**], Karzanov [**24**], Okamura, Seymour [**41**]. The problems we shall be considering are of the first type but have analogues for these latter models that will only be touched in the sequel.

The *cographic multiflow problem* is the special matroid multicommodity flow problem where M is cographic (which of course contains the planar disjoint paths problem). This "cographic multiflow problem" has been investigated a great deal, due also to the underlying nice combinatorial structure and its appealing relation to matching theory. Seymour [**59**] discovered its relation to the Chinese Postman Problem and used it to solve the plane multicommodity flow problem for Eulerian networks and to settle the case of two demand-edges. We shall give a survey of some results about this problem in Section 2.

A set of edges will be called *even* (*odd*) if the sum of their demands and capacities is even (odd), and the same terminology will be used for an arbitrary integral weight function given on the edges of a graph. A weight function (or a matroid flow problem or its demand and capacity function) is called *Eulerian* if it has only integer values and every cut is even. (The word "cut" will also be used for matroids as a synonym of "cocircuit"; the cocircuits are the circuits of the dual matroid.) If every *circuit* is even we shall use the term *bipartite* instead. (Bipartite is the "dual" of Eulerian.)

After a survey of the known results, in which we shall put great emphasis to the relation between different matroid properties (Sections 2, 3), we study some questions growing naturally out of this theory. Our plans:

- We start with a compact common formulation of some theorems and a conjecture of András Frank about the necessity and sufficiency of a strengthening of the Cut Condition which takes into consideration the parities of the cuts. Our formulation involves binary matroids and is closely related to Seymour's matroid flow properties [**58**]. We are led to a general matroid property, which explains the interrelation between the different problems and raises a number of questions (Section 4).
- In Section 5 we concentrate on one of these questions that we find fundamental, namely: *when does the existence of a fractional flow imply the existence of an integer one*? Although this property does not behave well with respect to Seymour's "sum" operations, we work out tools of extending characterizations, from given matroids to their closure with respect to sums. As a result, good characterizations and polynomial algorithms for multiflow problems can be extended to larger classes of graphs or matroids.

These latter are the main new results of the present work. Readers who would like to concentrate only on these can read Sections 1, 3, 5 (the odd sections) independently of the others.

- In Section 6 we would like to make clearer the borders between NP-completeness and polynomial solvability of multiflow problems. We show some new polynomially solvable special cases that touch the border of NP-completeness. The endeavor to draw a clear map gives rise to a number of problems.

2. A survey

The cographic multicommodity flow problem is NP-complete in general (see Middendorf, Pfeiffer [**38**]), and even for planar graphs, and even under some more restrictions.

We are mainly interested in polynomiality or NP-completeness of the different problems and the type of good characterization they use: in the combinatorial content and not in small differences in complexity.

I. Restricting the underlying graph.

- Lovász [35] proved that in a cographic matroid a half-integer flow exists if and only if a trivially necessary condition for the existence of a flow, the *Cut Condition* (for every cut the sum of the demands is at most the sum of the capacities) is satisfied. Seymour [59] proved that in Eulerian networks, in addition, the Cut Condition is necessary and sufficient for the existence of an integer flow. These papers did not deal with algorithms. (For a quick proof of these results see Sebő [53] or [55]. This proof immediately implies a polynomial algorithm using weighted matching as subroutine; see [55]).

On the other hand a simple but, even from an algorithmic point of view, crucial observation was made by Seymour [59]. To explain this observation which connects multicommodity flows to matching theory and which will be used several times in the sequel, we need some definitions. Let G be a graph and let $t\colon V(G) \to \{0, 1\}$, with $t(V(G))$ even. (If f is a function defined on the elements of X, $f(X)$ means $\sum_{x\in X} f(x)$.) A *t-join* is a set of edges $F \subseteq E(G)$ such that $d_F(x) \equiv t(x) \bmod 2$ for all $x \in V(G)$. A *t-cut* is a cut $\delta(X)$ such that X is *t-odd*; that is, $t(X)$ is odd. If t is odd everywhere, we just say that $\delta(X)$ is an *odd cut*. ($\delta(X)$ denotes the set of nonloop edges with exactly one endpoint in X, and $d(X) := |\delta(X)|$. If we wish to emphasize that this set of edges is considered in the graph G, we write $\delta_G(X)$ and $d_G(X)$.)

Now let $w\colon E(G) \to \mathbb{Z}^+$. ($\mathbb{Z}^+$ is the set of nonnegative integers.) A *w-packing* of t-cuts is a set $\mathcal{T}$ of t-cuts with a function $g\colon \mathcal{T} \to \mathbb{Q}^+$, which has the property that for all $e \in E(G)$, $\sum_{e\in C\in\mathcal{T}} g(C) \le w(e)$. If g has only integer values it is an *integer w-packing.* (Think of g as a function giving the *multiplicities* of the elements of $\mathcal{T}$; that is, $g(C)$ is the number of *copies* of C, even if g is not integer.) $\sum_{T\in\mathcal{T}} g(T)$ will be called the *value* of the w-packing.

The minimum weight of a t-join will be denoted by $\tau(T, t, w)$, whereas the maximum value of a w-packing, and of an integer w-packing, of t-cuts will be denoted by $\nu^*(G, t, w)$ and $\nu(G, t, w)$, respectively. If F is a t-join and C is a t-cut, then obviously $|F \cap C|$ is odd. In particular, $|F \cap C| \ge 1$, and $\tau(G, t, w) \ge \nu^*(G, t, w) \ge \nu(G, t, w)$ follows. We shall say that (G, t, w) has the *Seymour property* if $\nu(G, t, w) = \tau(G, t, w)$. It is easy to prove that

> there is an integer flow in $(M^*(G), R, r, c)$ if and only if (G, t, w) has the Seymour property,

where $M^*(G)$ is the dual of the polygon matroid $M(G)$ of G; $t(x) :\equiv d_F(x) \bmod 2$ for all $x \in V(G)$, where F consists of the duals of edges of R, and $w(e) := c(e)$ if $e \in E(G)\backslash R$, $w(e) := r(e)$ if $e \in R$.

More generally, let M be a matroid represented by the set of vectors $E \subseteq \{0, 1\}^n$ $(n \in \mathbb{N})$ that will be imagined to be the columns of a matrix also denoted by M. It is well known that the linear space generated over GF(2) by the rows of M is the set of disjoint unions of cocircuits of M.

For given $t = (t_1, \dots, t_n) \in \{0, 1\}^n$, we shall call $F \subseteq E$ a *t-join* if $\sum_{v \in F} v \equiv t \mod 2$. (A t-join exists if and only if t is in the column space of M.) A cut C is a *t-cut* if writing it as a mod 2 sum of a set X of rows of M, $\sum_{i \in X} t_i$ is odd. (In other words, C is a t-cut if and only if $C \cup \{t\}$ is a cut in the matroid M' represented by t and the columns of M.) If the set of t-joins is nonempty, it is easy to see that the parity of $\sum_{i \in X} t_i$ depends only on C and not on X.

Now $\tau(M, t, w)$, $\nu^*(M, t, w)$, $\nu(M, t, w)$ can be defined in exactly the same way as for graphs and so can the Seymour property, and w and (r, c) define each other in the same way ($w(e) = r(e)$ if $e \in R$, $w(e) = c(e)$ if $e \notin R$). The same facts are true for them as well. The following trivial relation will be used several times in the sequel:

(2.1) *Suppose (M, R, r, c) satisfies the Cut Condition. There is an integer (fractional) flow in (M, R, r, c) if and only if (M^*, t, w) has the Seymour property (resp. $\nu^*(M^*, t, w) = \tau(M^*, t, w)$).*

The Chinese Postman Problem is the problem of finding a minimum t-join in a graph, and its dual is the fractional t-cut packing problem. Algorithms solving the dual of the Chinese Postman Problem also solve the multicommodity flow problem for cographic matroids through (2.1). Most of the following results are based on this observation. (We shall only mention the consequences to multicommodity flows.)

Barahona **[1]** and Korach **[26]** develop primal versions of Edmonds and Johnson's algorithm to the Chinese Postman Problem, and these can be improved to find integer solutions for the Eulerian cographic problem. Korach **[26]** finds an integer solution via a postoptimality method. Korach and Penn **[29]** prove that a flow "almost satisfying" all the demands can be found in an arbitrary graph. Barahona **[2]** finds an integer flow in a simple direct way. We recommend this paper for a quick understanding of this Edmonds-Johnson type approach, and for an efficient multiflow algorithm for planar graphs.

For the planar case Matsumoto, Nishizeki, and Saito **[37]** apply planar matching algorithms with low worst-case complexity to find integer flows in the Eulerian case (and hence half-integer flows in general). The best complexity is obtained by Barahona **[2]**, who builds into his primal algorithm the efficient data-structures used in planar matching algorithms.

All the above-mentioned good characterizations can be obtained in polynomial time via some "magic numbers" assigned to the vertices that seem to contain most of the structural information about the problem (see Sebő **[52]**, **[55]**). These make it possible to find integer solutions under various restrictions on the graph G, including all the previously known integrality results.

A different type of integrality result is proved by Seymour [**57**]: in series parallel graphs the Cut Condition is necessary and sufficient for the existence of integer flows (and actually much more is proved).

The most general theorem I know, in this direction of restricting the graph, is the main result of A. Gerards [**20**], generalizing both Seymour's Eulerian and series-parallel cases.

II. Restricting the number of demands. In the special case where all capacities are 1, the graph multicommodity flow problem specializes of course to the problem of finding edge-disjoint paths between a given set of pairs of vertices. Note that even this problem is NP-complete for graphs in general (Even, Itai, Shamir [**12**], and see Garey, Johnson [**19**]), but if the number of demand-edges is fixed and all demands are 1, it is polynomially solvable according to the celebrated papers of Robertson and Seymour [**43**]. The problems we are considering here are independent of this latter: we allow *arbitrary demands and capacities* but we have constraints for the matroid defined on the union of the demand and non-demand edges. Note that according to the above-mentioned papers of Even, Itai, and Shamir, or Garey and Johnson, even the special case of the graph multicommodity flow problem when the number of demand-edges is 2 is NP-complete if the demands and capacities are arbitrary.

It is implicit in Seymour [**59**] (see Sebő [**51**]) that for the cographic multiflow problem, with 2 demand-edges, the Cut Condition together with the following *Parity Condition* (P.C.) is necessary and sufficient for the existence of an integer flow:

(P.C.) *There is no odd cut contained in the union of tight cuts,*

where a *tight* cut is one for which the Cut Condition holds with equality, and a cut is *odd* if the sum of the capacities and demands in the cut is odd.

(The necessity of (P.C.) is easy to see.) For the case of 2 demand edges, Seymour's solution implies a polynomial algorithm as well.

For the case of 3 demand-edges, Korach [**26**] found a decomposition method that reduced the problem to some number of small graphs. Korach and Newmann [**28**] claim that the generalization of this approach works for 4 demand-edges, though the number of graphs to check is over 200.

Sebő [**54**] proves that the cographic multicommodity flow problem is polynomially solvable if the number of demands is fixed.

Frank [**17**] generalized the 2-demand case in the following way: *In a planar graph, if the demand-edges are in two faces of the non-demand graph,* then again the Cut Condition and the Parity Condition are necessary and sufficient for the existence of an integer multicommodity flow. (For more about the Parity Condition, see Section 3.) The same condition has been proved to be necessary and sufficient in networks with many tight cuts ("tense" and "half tense" networks in Sebő [**52**]).

Korach [27] proved a generalization of Frank's result [**18**] to cographic matroids in which the demand edges constitute a forest with two components in the underlying graph.

Middendorf and Pfeiffer [**38**] note that a polynomial solution for the edge-disjoint paths problem in a planar graph (that is, for the case when all demands and capacities are 1) follows from a homotopic routing theorem of Schrijver [**48**] provided all the demand-edges are on a fixed number of faces of the supply graph.

All of the three previous cases would be contained in one general theorem if the following question had a positive answer.

PROBLEM A. Is the multicommodity flow problem polynomially solvable for cographic matroids where the demand-edges constitute a forest with a bounded number of components?

Of course, the most interesting special case of this problem is the planar multiflow problem where the demand-edges are contained in a fixed number of faces of the supply graph.

3. Equivalence of some properties

In this section we wish to show the relations among various matroid properties. Our goal is first of all didactical: we collect and then paste into one theorem various well-known relations, and give the ideas of some simple proofs. We aim at giving a common compact description of phenomena known from several different papers (see for example (3.2)), first of all in order to provide the necessary preliminaries to the following sections.

Let M be a matroid, and $F \subseteq E$. M will be called *F-flowing* if for any subset $R \subseteq F$ of demand-edges, for arbitrary capacities and demands, the Cut Condition implies the existence of a flow. It is said to be F-flowing *in integers* if under the same conditions an *integer flow always exists*. M is *evenly F-flowing* (corresponds to "F-cycling" in Seymour [**60**] [1]) if for any $R \subseteq F$ as set of demand-edges, and an arbitrary *Eulerian* choice of capacities and demands, the Cut Condition implies the existence of an *integer* flow. We say that M is (*evenly*) *flowing* (*in integers*) if it is (evenly) F-flowing (in integers) for every F. (*Evenly*) *k-flowing* (*in integers*) means (evenly) F-flowing (in integers) for every $|F| \leq k$.

M is called *packing*, if $\nu^*(M^*, t, w) = \tau(M^*, t, w)$ for arbitrary $t \in \{0, 1\}^n$ and nonnegative w. If, moreover, $\nu(M^*, t, w) = \tau(M^*, t, w)$ (that is, the Seymour property for the dual matroid) holds for arbitrary or for arbitrary Eulerian w, then we say G is *packing in integers* or *evenly packing*, respectively. (Evenly packing is the property called "F-packing for every F" in Seymour [**60**], and (3.1) is a version of (12.2) there.)

[1] Note the slight difference that, for later convenience (see equivalence with the F-metric property through polarity), we allow R to be a proper subset of F. Evenly F-flowing in our sense means in Seymour's terminology R-cycling for every subset R of F. The same holds for the relation of F-flowing in the above sense and in Seymour's sense. Our properties might be somewhat weaker, and hence easier to characterize.

(3.1) *Let M be a matroid. M is (evenly) flowing (in integers) if and only if it is (evenly) packing (in integers).*

PROOF. (3.1) is an immediate consequence of (2.1). □

Clearly, flowing matroids contain evenly flowing matroids, which in turn contain matroids flowing in integers. K_4 shows that the latter containment is proper. One of the consequences of Seymour's theorem below is that *the former containment is in fact equality.*

Multicommodity flow problems can be considered to be circuit-packing problems with the restriction that the circuits we pack contain *exactly* one element of a given set. We can also be interested in packing arbitrary circuits. Though this is a completely different problem, it turns out that the multicommodity flow problem in a matroid is strongly related to the circuit-packing problem of the dual matroid, that is, to the cut-packing problem.

In an arbitrary matroid, any nonnegative linear combination p of (characteristic vectors of) cocircuits is obviously a *metric*; that is, it satisfies

$$p(f) \le p(C \setminus \{f\}) \text{ for every circuit } C \text{ and every } f \in C.$$

Obviously, *weighting of the edges of a graph is a metric if and only if it can be extended to a function on all pairs of vertices so that the extended function satisfies the triangle inequality.*

The matroid M has the (even) *metric property* if an arbitrary nonnegative (bipartite) metric is a nonnegative (integer) linear combination of cuts. (Integer combinations of cuts are necessarily bipartite.) The metric property is the dual of the "sums of circuits property" of Seymour [**60**]. If for fixed $F \subseteq E$ an arbitrary (bipartite) metric is only required to be *greater than* or *equal* to a nonnegative linear (integer) combination of cuts, and it is only required to be equal to it on F, we say that M has the (even) F-metric property. The proof of the following relation shows that flowingness and the metric property are the same through polarity (compare with Seymour [**60**], Karzanov [**24**], Lomonosov [**34**], Schrijver [**49**].)

(3.2) *M is F-flowing if and only if it has the F-metric property.*

PROOF. (We suppose that the reader is familiar with the polarity relation of cones.) For every circuit C with $|C \cap F| \neq \varnothing$, and $f \in C \cap F$ define $v_{C,f}(x)$ to be -1 if $x = f$, 1 if $x \in C \setminus \{f\}$, and 0 otherwise. Let K be the cone generated by all the vectors $v_{C,e}$ and the "unit" vectors v_e $(e \in E \setminus F)$, which are defined to be 1 on e and 0 on the other edges. On the other hand, let L be the set of vectors that—if negative values are interpreted as demands and nonnegative values as capacities—consists exactly of the capacity and demand functions for which there exists a multicommodity flow with demand-edges in F.

CLAIM. $K = L$.

In fact, $L \subseteq K$ is obvious. $K \subseteq L$ is clearly equivalent to the following: for every nonnegative combination of the above given extreme rays of K

there exists another combination, in which all v_{C_1,e_1}, v_{C_2,e_2} with positive coefficients satisfy $e_1 \notin C_2$ and $e_2 \notin C_1$. For then, setting

$$R := \{e\colon \text{ there exists } v_{C,e} \text{ with positive coefficient}\} \subseteq F,$$

$C \cap R = \{f\}$ holds if $v_{C,f}$ has positive coefficient.

If v_{C_1,e_1} and v_{C_2,e_2} are not so, let us say they are a "bad pair." We shall define a new combination that decreases the number of "bad pairs." Consider a bad pair and let λ be the minimum of their two coefficients. If both $e_1 \in C_2$ and $e_2 \in C_1$, simply decrease both coefficients by λ and add the right number of unit vectors. Suppose now without loss of generality that $e_1 \in C_2$ and $e_2 \notin C_1$. Let $C := C_1 \triangle C_2$. Decrease both of the coefficients of v_{C_1,e_1} and v_{C_2,e_2} by λ and add v_{C,e_2} with coefficient λ and the right number of unit vectors. It is easy to see that the result of the linear combination remains the same and the number of bad pairs decreases. Thus the claim is proved.

Now denote the polar of a cone Q by Q^*. "M is F-flowing" means exactly that $L = N^*$ where N is the cone generated by the set of (characteristic vectors of) cuts and the vectors v_e, $e \in E \backslash F$. On the other hand, it is easy to see that "M has the F-metric property" means exactly that $N = K^*$. Since $K = L$ by the claim and $Q^{**} = Q$ for any cone, $L = N^*$ is equivalent to $N = K^*$. □

Lomonosov **[32]**, **[34]** and Karzanov **[24]** (see also Schrijver **[49]**) prove that for graphs with special R both properties can be strengthened.

Karzanov's and Lomonosov's Theorems. *Let G be a graph and $F \subseteq E(G)$. If F is the circuit on five vertices, or the complete graph on four vertices or it is the function of two stars, $M(G)$ is evenly F-flowing* (*Lomonosov*) *and has the even metric property* (*Karzanov*).

The authors show that the constraints on F cannot be weakened. Papernov (1976) proved earlier that F-flowingness holds for exactly the same F.

The well-known theorem of Okamura-Seymour **[41]** and Okamura's generalization **[40]** can also be formulated as stating "even F-flowingness" of some matroids for some F. Schrijver **[47]** proves the even F-metric property in Okamura's case. Okamura's and Schrijver's theorems constitute another pair related to each other in the same way as Karzanov's and Lomonosov's above theorems.

Finally we mention Seymour's main lemma **[60**, Section 7] to prove excluded minor characterizations of these properties. For the definition of k-sums ($k = 1, 2, 3$), and related notions and facts, see Appendix II.

(3.3) *If two matroids are* (*evenly*) *flowing* (*or flowing in integers*), *then so is their k-sum* ($k = 1, 2, 3$).

To prove the excluded minor characterizations Seymour **[60]** uses decomposition results based on Seymour **[58]** asserting that matroids without in which case we define the Eulerian extension to be equal to the original network; $\overline{R} := R \cup \{d\}$; $\overline{r}(d) := 1$, and otherwise (for $e \neq d$) $\overline{r}$ and $\overline{c}$ are defined to be the same as r and c. The Eulerian extension of the matroid we get from M by replacing every element $e \in E$ by $r(e)$ or $c(e)$ parallel copies is the matroid we get similarly from $\overline{M}$.

The following simple but important fact shows that we do not really restrict generality when we assume that our network is Eulerian.

(4.1) *In a network there exists an integer flow if and only if there exists one in its Eulerian extension.*

PROOF. The if part is obvious. To prove the only if part note that an integer sum of circuits is an Eulerian function on E. After subtracting such a function from the capacities and demands, we get a new function such that the columns with odd function values and d form a cycle. □

A necessary condition for the existence of an integer flow is *the existence of a fractional flow in the Eulerian extension.* This turns out to be much stronger than the Cut Condition, and even than the existence of a fractional flow, as the example of K_4 already shows. (With two matching edges as demand-set and all capacities and demands equal to 1, the Eulerian extension of K_4 is the Fano matroid F_7 with three demand-edges forming a cycle. It is easy to see that for any demand and capacity function on K_4, the existence of a fractional flow in the Eulerian extension is sufficient for the existence of an integer flow. This also follows from the "routingness" of F_7, see Appendix III.) The existence of a fractional flow in the Eulerian extension is strictly related to (F.C.).

(4.2) *If there exists a fractional flow in the Eulerian extension of a network, then* (F.C.) *also holds. For matroids that are both flowing and coflowing (with arbitrary set of demand-edges, capacities, and demands), the existence of a fractional flow in the Eulerian extension is equivalent to* (F.C.).

PROOF. Let (M, R, r, c) be a network, fix a binary representation of the matroid M, and let t be the sum of the vectors corresponding to odd r or c values. We get a representation of the Eulerian extension if we add t to the representation of M.

First observe that (F.C.) is equivalent to the following: *for every vector w which is the sum of cuts,*

$$(*) \qquad \nu^*(M, t, w) + \sum_{e\in R} w(e)r(e) \leq \sum_{e\in E\setminus R} w(e)c(e).$$

Let $\overline{w}(t) := \tau(M, t, w)$ and $\overline{w}(e) = w(e)$ if $e \in E(M)$. $\overline{w}$ is a metric on $\overline{M}$. For w is a metric on M, since it is a sum-of-cuts vector; and clearly

$$\tau(M, t, w) = \min\{w(C \setminus t) \colon C \text{ is a cycle of } \overline{M}, t \in C\}.$$

If there is a fractional flow in the Eulerian extension $(\overline{M}, \overline{R}, \overline{r}, c)$, then the Distance Criterion holds in particular for $\overline{w}$:

$$(**)\qquad \sum_{e\in\overline{R}} \overline{w}(e)\overline{r}(e) \leq \sum_{e\in E(\overline{M})\setminus\overline{R}} \overline{w}(e)\overline{c}(e).$$

This is the same as $(*)$ with the only exception that $\nu^*(M, t, w)$ is replaced by $\tau(M, t, w)$. Since $\nu^*(M, t, w) \leq \tau(M, t, w)$, $(**)$ implies $(*)$.

Now we prove the converse implication for the matroid M which is both flowing and coflowing. Since M is flowing, by (3.2) it has the metric property, which means that the metrics defined on it are exactly the sum-of-cuts vectors. Since M is coflowing, then by the same theorem, M is packing, that is, $\nu^*(M, t, w) = \tau(M, t, w)$. Thus, supposing that $(*)$ holds for an arbitrary sum-of-cuts vector w on M, we get that $(**)$ holds for an arbitrary metric $\overline{w}$ on $\overline{M}$. □

Hence, the following problem contains the characterization of flowing and coflowing matroids (e.g., planar graphs) for which Frank's conjecture is true.

Problem B. Characterize matroids in which the existence of an integer flow of an arbitrary set of demand edges, demands, and capacities is equivalent to the existence of a fractional flow in the Eulerian extension of the defined network.

We can neither prove nor disprove that "almost" Eulerian planar graphs satisfy this property:

Conjecture C. Let G be a planar graph. For an arbitrary set of demand edges, demands, and capacities such that there are at most two vertices with an odd sum of capacities and demands, the defined network has an integer flow if and only if its Eulerian extension has a flow.

Middendorf and Pfeiffer **[39]** have constructed counterexamples to various natural strengthenings of this conjecture. A trivially equivalent reformulation:

Let G be an Eulerian graph, $R \subseteq E(G)$, and suppose that $G - e$ is planar for some $e \in R$. Then, if there exists a fractional packing of paths in $G - R$ between the endpoints of the edges in R, then there also exists an integer packing of paths (that is, edge-disjoint paths).

II. Metrics. (4.2) shows that Frank's condition, although it is made typically to strengthen the Cut Condition for integer flows in *non-Eulerian networks*, is equivalent to the property "if there exists a flow then there exists an integer flow" of some *Eulerian networks*. Also, (4.1) permits us to reduce the existence of integer flows to Eulerian networks. *Thus the study of Eulerian networks is not really a restriction of generality.*

The property "if there is a flow for Eulerian data, then there is an integer flow" is a generalization of even flowingness. It is maybe the most general among matroid properties for which "nice" good characterizations hold. Indeed, the existence of fractional flows is well characterized (provided "minimum path problems" can be solved in the underlying matroid, see

Section 6):

(4.3) *There exists a flow in the network* (M, R, r, c) *if and only if the Distance Criterion holds.*

The Distance Criterion is a well-known necessary and sufficient condition for the existence of fractional multicommodity flows in graphs, claimed by the "Japanese Theorem" of Iri **[23]** and Onaga, Kakusho **[42]**. Here it is for matroids:

DISTANCE CRITERION. *For every metric* q,

$$\sum_{e\in R} q(e)r(e) \le \sum_{e\in E\setminus R} q(e)c(e).$$

The proof of (4.3) is easy: write the existence of a multiflow as the feasibility problem of a linear program, and apply Farkas' lemma. The result of this is (4.3) with the slight lack that q is not necessarily a metric. (What we got until now is Seymour **[60**, (4.4)].) Note now that replacing $q(e)$ for every edge e by $\min\{q(e), q(C\setminus\{e\})\}$, where C is the cycle for which $q(C\setminus\{e\})$ is minimum, we get a metric, and the difference between the right- and left-hand sides of the Distance Criterion has not increased.

(4.3) suggests the idea of studying the necessity and sufficiency of the Distance Criterion for the existence of a flow or for the existence of an integer flow in an Eulerian network, requiring the Distance Criterion for all or for a subset of metrics. (Even) flowingness (in integers) can be defined with respect to an arbitrary class of metrics: replace in the definitions the Cut Condition by the Distance Criterion restricted to the given class of metrics.

For example, in the previous section we studied these properties with respect to the "cut metrics." (The function which is 1 on a cut and 0 otherwise is clearly a metric.) Seymour **[60**, (4.5)] proves that the notion of (even) flowingness does not change if we replace the Cut Condition by the Distance Criterion restricted to arbitrary $(0, 1)$-metrics. Schwärzler and Sebő **[50]** characterize (even) flowingness with respect to $(0, 1, 2)$-metrics.

PROBLEM D. What is the class of matroids for which the Distance Criterion, maybe restricted to a subclass of metrics, is necessary and sufficient for the existence of an (integer) flow for every set of demand-edges and (Eulerian) demands and capacities?

Cut metrics are the most special, and the set of all metrics is the most general class of metrics we can be interested in. (The former fact is justified by noting that cuts are always extreme rays of the cone of all metrics; the latter by the Distance Criterion.) We shall see below that flowingness in integers cannot be really generalized, but the following section tries to show that evenly flowing matroids with respect to the set of all metrics (routing matroids) constitute a rich class.

From now on we are dealing only with the set of all metrics, that is, with the Distance Criterion as it is: this is equivalent to replacing the Cut Condition by the existence of a fractional flow. For a study of other metrics see Schwärzler and Sebő. **[50]**.

Flowingness in integers with respect to more general metrics than the cut condition does not give anything new:

(4.4) *A matroid has the property that for every set of demand-edges, integer capacity, and demand functions, the existence of a fractional flow implies the existence of an integer flow if and only if it has no* K_4 *minor.*

PROOF. It is easy to see that if a matroid has the above-mentioned property, then all its minors also have it (in the same way as in the proof of (5.1) below). Thus the only if part follows. (If R consists of a perfect matching of K_4 there exists a half-integer flow but there is no integer one.) On the other hand, if M has no K_4 minor, it is flowing in integers by Seymour [**58**, (8.1)]. Consequently, if there exists a fractional flow for an integer demand and capacity function, then the Cut Condition is satisfied, and thus there exists an integer flow as well. □

This also means that matroids *flowing in integers with respect to any class of metrics* (containing cut metrics) are the same.

III. F**-Routing matroids.** We think (4.4) also holds for F-flowingness in integers:

CONJECTURE E. Let M be a matroid and $F \subseteq E$. For a network in M with demand-set F and every integer capacity and demand function, the existence of a fractional flow implies the existence of an integer flow if and only if M is F-flowing in integers.

Some results and conjectures in Seymour [**60**] suggest that the characterization of the matroids that are F-flowing in integers is probably difficult. The above conjecture could be easier though. We actually think that the following somewhat stronger statement is also true:

If all the proper minors M' *of* M *are* $F \cap E(M')$*-flowing in integers, then* M *is* F*-flowing.* (To prove Conjecture E above from this, assume this is true and let M be a minimal counterexample to (the nontrivial part of) the conjecture: for an arbitrary integer choice of demands and capacities in M for which there exists a fractional flow there also exists an integer flow; M is not F-flowing in integers, but *all its proper minors are.* By the assumed claim M is F-flowing. Since it is not F-flowing in integers, there exist integer demands and capacities for which the Cut Condition holds but *there is no integer flow.* Since M is F-flowing, *there exists a fractional flow* in the same network, in contradiction to the choice of M.)

I do not actually know about any counterexample to the statement we get by replacing in Conjecture E "M is F-flowing" by "M is evenly F-flowing"; see Conjecture G (a) below. What about other analogous statements about other matroid flow properties? (Such statements would imply other equivalences and implications between various properties.)

In the case of $|R| \leq 2$ some obviously necessary conditions are also sufficient because of Seymour [**60**, (9.5) and (11.1)]:

If $|F| \leq 2$, M is F-flowing in integers if and only if it has neither an F_7^* minor containing at least one element of F nor a K_4 minor containing two elements of F forming a matching.

Using this, Conjecture E for $|F| \leq 2$ can be proved in the same way as (4.4).

If $|F| \geq 3$ the characterization of matroids F-flowing in integers is not known.

Let M be a matroid, and $F \subseteq E$. M will be called *F-routing* if for every subset $R \subseteq F$ of demand-edges, and for every Eulerian choice of demands and capacities, the existence of a fractional flow implies the existence of an integer flow. M is *routing* if it is *F-routing* for every $F \subseteq R$.

Evenly flowing matroids are obviously routing, because a fractional flow implies the Cut Condition. On the other hand, the minimal matroids that are not evenly flowing—that is, *the minimal nonflowing matroids K_5, F_7, R_{10}—are all routing* as well (see Appendix III).

Call a matroid k-routing ($k \geq 0$) if it is F-routing under the condition $|F| \leq k$. We saw that (4.4) remains true with the restriction $|F| \leq 2$. However, already the characterization of 1-routing matroids seems to be difficult: denote the Eulerian extension of the Petersen graph by P_{16}. P_{16} is not F-routing, where $F = \{d\}$: $r(d) = 3$ and $c(e) = 1$ if $e \neq d$ is an Eulerian weighting; putting $\frac{1}{2}$ as multiplicity on the circuits $F_i \cup \{d\}$ $(i = 1, \ldots, 6)$ of P_{16}, where the F_i's are the six different perfect matchings of the Petersen graph, we get a fractional flow; yet there is no integer flow, because it would give a factorization of the Petersen graph.

Werner Schwärzler noted that some "lifts" (see Seymour [**60**] of minimal nonrouting matroids are also not 1-routing, a consequence of (10.1) in Seymour's paper.

PROBLEM F. Find the minimal non-1-routing minors of P_{16}, and also of the non-1-routing "lifts" of the bi-nonflowing matroids, defined in the next section.

A wide class of F-routing matroids is provided by the following theorem:

KARZANOV'S THEOREM [**25**]. *Let G be an arbitrary graph and F the edge-set of a subgraph on at most five vertices. Then G is F-routing.*

We think (4.4) and the remarks we made afterwards can be refined in the following way. Part (a) of the following conjecture easily implies Conjecture E above (a fact proved in a remark in parentheses after Conjecture E).

CONJECTURE G. Let M be a matroid, and $F \subseteq E$.

(a) If all proper minors $M\backslash X/Y$ of M are $F\backslash X$-flowing in integers, where $X, Y \subseteq E$ and $Y \cap (F \cup X) = \emptyset$, then M is evenly F-flowing.

(b) If all proper minors $M\backslash X/Y$ of M are evenly $F\backslash X$-flowing, then M is F-routing.

If we delete "F-" in Conjecture G we get a statement that is easy to settle: K_4, the minimal matroid nonflowing in integers, is evenly flowing (see (4.4)); similarly, according to Appendix III, the minimal nonflowing matroids are routing.

5. Routing matroids

Let us recall that a matroid is *routing* if for any Eulerian capacities and demands for which there exists a fractional flow there also exists an integer one. We try to work out here some tools to study routing matroids and show some first application of these tools.

We have some examples of routing matroids: evenly flowing matroids are clearly routing, and so are the minimal nonflowing matroids K_5, F_7, and R_{10}, according to Appendix III. In this section we shall prove the routingness of a minor-closed class containing all these.

Note that according to Wagner's conjecture recently proved by Robertson and Seymour **[45]**, (5.1) implies that *routing graphs can be characterized by a finite number of excluded minors.* Since according to Robertson and Seymour **[44]** testing for any of these minors can be done in polynomial time. It follows that *routingness of graphs can be tested in polynomial time.* Since the same argument works for any minor closed class, in Section 6 we shall exclusively be interested in the complexity of multiflow problems in subclasses rather than testing for these subclasses.

(5.1) *If a matroid is routing, then all its minors are also routing.*

PROOF. We have to prove that contracting or deleting an element e of a routing matroid M, the matroid M_e we get is also routing. Suppose (M_e, R, r, c) is an Eulerian network, and $(\mathscr{C}, f)$ is a flow in it.

- If e was deleted, then define $c(e) := 0$. (M, R, r, c) is Eulerian, $(\mathscr{C}, f)$ is a flow in it (an edge-set in M_e can also be considered to be an edge-set in M). Hence, since M is routing, there exists an integer flow in (M, R, r, c) which is also an integer flow in (M_e, R, r, c).
- If e was contracted, then for every cut C of M containing e, $r((C \cap F)\setminus\{e\}) + c((C\setminus F)\setminus\{e\})$ has the same parity. Define $c(e)$ to have this same parity, and to be "big enough" (for example the sum of all demands is enough). From a fractional flow in (M, R, r, c) we easily get one in (M_e, R, r, c), and then from an integer flow in (M_e, R, r, c), using that the capacity of e is big, we get one in (M, R, r, c). □

We were somewhat informal, because there is no essential difference between this proof and similar proofs in Seymour **[58]**. Nevertheless, the following "skew decomposition lemma" into 2- and 3-sums together with (5.3) seems to be a new feature of routing matroids:

(5.2) *If M_1 and M_2 are routing matroids, then their* 1*-sum is also routing. If M_2 is furthermore evenly flowing, then their* 2*- and* 3*-sums are also routing.*

Proof. The claim about 1-sums is clear.

Let M be the 2-sum of the routing matroid M_1 and the evenly flowing matroid M_2, let $E(M_1) \cap E(M_2) = \{d\}$, and let (M, R, r, c) be an Eulerian network for which there exists a flow, given by the multiset $\mathscr{C}$ (with maybe fractional multiplicities). We show that there exists an integer flow for (M, R, r, c).

Define R_i, r_i, c_i $(i = 1, 2)$ as follows. $r_i(e) := r(e)$ for $e \in (E_i \backslash \{d\}) \cap R$, and $c_i(e) := c(e)$ for $e \in (E_i \backslash \{d\}) \backslash R$. We still have to decide about d. Let

$$q := \max\{r((C\backslash\{d\}) \cap R) - c((C\backslash\{d\})\backslash R) : C \text{ is a cocircuit in } M_2, d \in C\}.$$

Furthermore, let $R_2 := (R \cap E_2) \cup \{d\}$ if $q < 0$, and otherwise $R_2 := R \cap E_2$; let $R_1 := (R \cap E_1) \cup \{d\}$ if $q \geq 0$, otherwise $R_1 := R \cap E_1$. Define $r_i(d)$ or $c_i(d)$ (depending on whether $d \in R_i$ or not) to be equal to $|q|$.

It can be seen immediately from the definition that (M_2, R_2, r_2, c_2) is Eulerian, and it satisfies the Cut Condition. Hence there exists an integer flow $\mathscr{C}_2$ in this network.

Our following step is to prove that there exists an integer flow $\mathscr{C}_1$ in (M_1, R_1, r_1, c_1) as well. Since this network is clearly also Eulerian, it is enough to prove that it has a fractional flow.

Let $\mathscr{D} := \{C \in \mathscr{C} : C \cap E_1 \neq \emptyset, C \cap E_2 \neq \emptyset\}$, and suppose $\mathscr{C}$ is such that $|\mathscr{D}|$ is minimum. Then the demand-edge of every $D \in \mathscr{D}$ is in E_1, or all of them are in E_2, because if $D_i \in \mathscr{D}$ had it in E_i $(i = 1, 2)$, then there exists a circuit in each of $(E_1 \cap D_1)\triangle(E_1 \cap D_2)$ and $(E_2 \cap D_2)\triangle(E_2 \cap D_1)$ which could replace D_1 and D_2 in the flow $\mathscr{C}$, thus decreasing $|\mathscr{D}|$. (It is easy to see from the definition of a 2-sum that these edge-sets are cycles of M_1 and M_2 respectively, and each of them contains exactly one demand-edge.) $\mathscr{C}_1' := \{C \in \mathscr{C} : C \subseteq E_1\} \cup \{(D\backslash E_2) \cup \{d\} : D \in \mathscr{D}\}$ is a flow in (M_1, R_1, r_1, c_1): the capacities and demands of the edges in $E_1\backslash\{d\}$ are obviously respected by $\mathscr{C}_1$ because they are respected by $\mathscr{C}$; in order to check that the capacity or demand of d is also respected, consider the cut Q of M_2 for which the maximum is reached in the definition of q. Clearly, if $q \geq 0$, then at least q circuits of $\mathscr{C}$ that have their demand-edge in Q must also intersect E_1, and these are exactly the circuits in $\mathscr{D}$; similarly, if $q < 0$, then at most $-q$ circuits of $\mathscr{C}$ intersect $(Q\backslash\{d\})\backslash R$ without intersecting $(Q\backslash\{d\}) \cap R$, and $\mathscr{D}$ is a subset of these.

It follows now that $\mathscr{C}_1'$ is a flow in (M_1, R_1, r_1, c_1), and hence there exists an integer flow $\mathscr{C}_1$ in this same network.

Finally note that the integer flows $\mathscr{C}_1$ and $\mathscr{C}_2$ can be combined to give an integer flow of (M, R, r, c): first, put in all circuits of both flows which do not contain d. Then observe that according to the definition of a flow, in $\mathscr{C}_1 \cup \mathscr{C}_2$ there are at least as many circuits which contain d as demand-edge as circuits containing it as supply-edge; all circuits of the former kind are in M_i, and all of the latter kind are in M_j $(i \neq j \in \{1, 2\})$. Delete some circuits of the former kind to have equality here. It is possible now to partition the multiset of circuits of $\mathscr{C}_1 \cup \mathscr{C}_2$ containing d into pairs of the form $\{C_1, C_2\}$ where $C_i \in \mathscr{C}_i$ $(i = 1, 2)$. Second, for each pair $\{C_1, C_2\}$, put into the flow $C_1 \triangle C_2$. It is easy to see that we have defined a flow in (M, R, r, c), and the statement concerning the 2-sum is proved.

We omit the proof concerning 3-sums, because it is quite complicated, and we will not need it in this paper. Let us note, however, that it can be easily worked out from the synthesis of the proof of Seymour [**60**, (7.3)] and the proof above. □

Clearly, the same proof works to prove that M is F-routing, if M_1 is $(F \cap E_1) \cup \{e\}$-routing and M_2 is evenly $(F \cap E_2) \cup \{e\}$-flowing, where e is the marker, and we can simply write $F \cap E_i$ here, if $F \cap E_j = \emptyset$ $(i \neq j \in \{1, 2\})$. This implies easily that *if both matroids are k-routing and there exist numbers $k_i \geq 0$ $(i = 1, 2)$ such that $k_1 + k_2 \geq k$ and M_i is evenly k_i-flowing $(i = 1, 2)$, then M is k-routing.*

(5.2) is sharp: for 2- and 3-sums (5.2) cannot be sharpened by writing "routing" instead of "evenly flowing." In a conversation, Paul Seymour suggested that even something like "*the 2-sum of two not evenly flowing matroids is not routing*" may hold. This is "almost" true; we only have to suppose that *the common element of the two matroids is contained in a minimal nonflowing matroid in both of them*, as the following statement shows:

(5.3) *If M_1 and M_2 are minimal nonflowing then their 2-sum is minimal nonrouting.*[2]

PROOF. Let R_i be a set of demand-edges in M_i for which the Cut Condition is satisfied with all demands and capacities equal to 1, but for which there exists no flow $(i = 1, 2)$. For all the three minimal not evenly flowing matroids there exists such a demand-set, and all of them are Eulerian. Take the 2-sum M of M_1 and M_2 with $d \in R_1 \cap R_2$ as marker, and define $R := (R_1 \cup R_2) \setminus \{d\}$. Let all demands and capacities be equal to 1. Since both M_1 and M_2 are Eulerian, so is M. *There exists a fractional flow in the defined network,* because $M_i \setminus d$ is flowing, and with $R := R_i \setminus d$ as demand-set, it satisfies the Cut Condition $(i = 1, 2)$.

We show now that *there is no integer flow* in the defined network. Such an integer flow would contain an integer flow either of $M_1 \setminus d$ with demands $R_1 \setminus \{d\}$ or of $M_2 \setminus \{d\}$ with $R_2 \setminus \{d\}$. But $M_i \setminus \{d\}$ with demand-set $R_i \setminus \{d\}$ *does not have an integer flow* either for $i = 1$ or for $i = 2$: if it had, then by (4.1), its Eulerian extension, which is just M_i with demands R_i, would also have an integer flow, in contradiction to the definition of M_i and R_i $(i = 1, 2)$. The statement about the proper minors is an immediate con-

[2] Right before the final submission of this paper we observed that the proof of this lemma works in a more general setting, providing more 2-separable minimal nonrouting matroids. For example, the 2-sum of $\overline{H}_6$ and K_5 is nonrouting, where $\overline{H}_6$ is the matroid we get from H_6 by adding a parallel copy of the edge e for which $H_6/e = K_5$, and defining the marker to be this parallel copy. Similar statements hold for the 2-sum of two copies of $\overline{H}_6$.

sequence of (5.2) using the routingness of the minimal nonflowing matroids; see Appendix III. □

Since a 2-sum is a special 3-sum (replacing the common edge of the two matroids by a triangle), the 3-sum of routing matroids is not necessarily routing. One could just conclude that "sums" are no more the right tool. It turns out, however, that the way of *using* the decomposition operations has to be changed instead: although routingness cannot be "summed," it can be "characterized among sums." The main purpose of this section is to show how. The answer lies in (5.2) and (5.3).

According to (5.3) all the six nonisomorphic 2-sums of the three minimal nonflowing matroids are minimal nonrouting. These six matroids will be called *bi-nonflowing*. We shall refer to them with the notation $B_{i,j}$, where i and j are the indices of the two members of the 2-sum (for example, $B_{5,10}$ is the 2-sum of K_5 and R_{10}). We conclude that *all six bi-nonflowing matroids are minimal nonrouting.*

Note that all the minimal nonflowing matroids are evenly 2-flowing (Seymour **[60]**) and thus, by the remark after the proof of (5.2), the bi-nonflowing matroids are 3-routing. Lomonosov **[34]** constructed a (quite complicated) example for a not 3-routing *graph.*

PROBLEM H. Give other examples of minimal not 3-routing matroids.

Note also that the only graphic matroid among the bi-nonflowing ones is $B_{5,5}$, which is isomorphic to an important example in Middendorf and Pfeiffer **[38]**: it can serve as a basis for constructing other graphic counterexamples with little modification.

It might be useful to add that $B_{5,5}$ has three nonisomorphic choices of demand-edges for which there exists a flow but there is no integer flow, according to a choice between two nonisomorphic possibilities for d in the proof of (5.3).

Our goal now is to understand the relation between the unique "Cunningham–Edmonds decomposition" of a matroid into 2-sums and its routingness. Although Cunningham and Edmonds' uniqueness result [7] is not explicitly used in the presentation below, it is present behind most of the facts (see remarks also in Appendix II).

When taking the 2-sum of two routing matroids, we would like to use either (5.2) to prove that their 2-sum is also routing, or to find a bi-noncycling matroid, which is minimal nonrouting by (5.3). This will be possible due to the following fact:

(5.4) (a) *Suppose M_1, M_2 are connected matroids, let M be their 2-sum, and let N be an arbitrary matroid. If every element of $E(M_1)$ is contained in an N-minor, then so is every element of $E(M)$.*

(b) *Suppose that the connected matroid M can be decomposed into the matroids $M_1, \ldots, M_k$ by repeated 2-decompositions. Then the 2-sum of M_p and M_q $(p \neq q)$ is a minor of M.*

PROOF. We first prove (a). Let $e \in E(M)$ be arbitrary. If $e \in E(M_1)$ we have the statement by assumption. If $e \in E(M_2)$ let C be a circuit of M_2 containing both e and the marker d, and let $M_1/X\backslash Y$ be isomorphic to N and contain d. Clearly, $M/(X \cup (C\backslash\{e\})\backslash(Y \cup E(M_2)\backslash C)$ is isomorphic to N and contains e, proving (a).

To prove (b) note the following:

CLAIM. If the matroid N occurs as a member of a 2-sum in a repeated 2-decomposition of M, then every $e \in E(M)$ is contained in an N-minor.

Indeed, if $M_1 = N$, then the condition of (a) is trivially satisfied (every element of N is contained in an N-minor), and applying (a) to all the successive 2-sums we get the statement for M.

We proceed now by induction. By assumption, M can be decomposed into L_1 and L_2, which can be further decomposed into M_i $(i \in I_1)$ and M_i $(i \in I_2)$ respectively, where $I_1 \cap I_2 = \emptyset$, $I_1 \cup I_2 = \{1, \dots, k\}$. If p and q are both contained in I_1, or both are contained in I_2, we are done by induction. If not, suppose $p \in I_1$, $q \in I_2$, say. Let the marker of the 2-sum of L_1 and L_2 be d. According to the claim, d is contained in both an M_p-minor of L_1 and an M_q-minor of L_2. □

We now apply our lemmas to prove that the bi-nonflowing matroids are the only minimal nonrouting minors in the "closure" of the class of flowing and minimal nonflowing matroids:

(5.5) THEOREM. *Suppose that $\mathcal{M}$ is the minimal class of matroids that contains all the flowing matroids, and each of the minimal nonflowing matroids K_5, F_7, R_{10}, and for which $M_1, M_2 \in \mathcal{M}$ implies $M \in \mathcal{M}$, where M is a 1- or 2-sum of M_1 and M_2 with an arbitrary marker. Then $M \in \mathcal{M}$ is routing if and only if it does not contain any of the bi-nonflowing matroids as minors.*

PROOF. The "only if" part is an immediate consequence of (5.3) and (5.1).

Now let $M \in \mathcal{M}$. Decompose M first into the 1-sum of 2-connected matroids (connected components); then further decompose these until arriving at a list of 3-connected matroids. By assumption these 3-connected matroids are either flowing or isomorphic to one of the minimal nonflowing matroids K_5, K_7, R_{10}. If the list we get from a connected component contains at least two minimal nonflowing matroids, then by (5.4), M has a bi-nonflowing minor. Since M does not have bi-nonflowing minors by assumption, *among the 3-connected matroids arising from repeated 2-decompositions of a connected component of M, there is at most one which is not flowing.* Even this one is routing (since it is isomorphic to F_7 or R_{10} or K_5). Applying the second part of (5.2) we get first that each connected component of M is routing, and then, taking the 1-sums of these, by the first part of (5.2), M is routing. [3]

[3] (5.4) describes the trivial behavior of Seymour's "roundedness" property **[55]** with respect to 2-sums. Using Seymour's theorem about "rounded matroids" the way we applied (5.3) can be generalized: for graphs say, either (5.2) can be applied to M_1 and M_2, or both matroids contain a K_5 minor implying that the marker is contained in both in a K_5 or H_6 minor. But by the previous footnote, the 2-sum of a K_5 and $\overline{H}_6$ arising in this way, or of two copies of any of these, are minimal nonrouting matroids.

Corollary. *Suppose* M *is a matroid without* AG(2, 3), S_8, *and* H_6 *minors.* M *is routing if and only if it does not contain any of the bi-nonflowing matroids as minors.*

Proof. The class $\mathscr{M}$ of matroids defined in Theorem 5.5 is exactly the class of matroids without AG(2, 3), S_8, and H_6 minors, because of Seymour's Splitter Theorem (see Corollaries of the Splitter Theorem in Appendix II). □

Thus routingness is characterized above for a class of matroids closed under minor containment and containing both the evenly flowing matroids and the minimal nonflowing ones.

(5.2) and (5.3) explain the behavior of routingness with respect to 2-separation. For a characterization of routingness, though, we would first of all need more 3-connected minimal nonrouting matroids. The only one we know is actually P_{16} or some minor of it. M. Middendorf and F. Pfeiffer **[39]** have recently found a new example: they observed that K_7 is not routing; it is easy to see that K_7 does not have any 2-separable nonrouting minor (use (5.2) to prove this).

Problem I. Find 3-connected minimal nonrouting matroids.

Another application of (5.2): the generalization of Lomonosov's theorem for arbitrary regular matroids, or even to matroids without AG(2, 3) and S_8 minors, or even.... (It is possible to continue with the splitter theorem; use the version of (5.2) concerning F-routingness, mentioned after (5.2).) The proof of this generalization is similar to the proof of (5.5): luckily, in the 2-decomposition, at least one of M_1 and M_2 will be evenly flowing with the demand-edges it contains, because of the first (Lomonosov's) part of "Karzanov and Lomonosov's Theorem," Section 3.

The additional assumptions we made can be generalized by further applying Seymour's "Splitter Theorem", but Conjecture I seems to be difficult. It might be related both to the cycle cover conjectures of Seymour (for example, **[60]**) and Tutte's and Lovász's conjectures about partitioning the edges of a graph into perfect matchings (that imply the four-color conjecture for planar graphs). The following problem may lead to a common generalization of these two conjectures as well as to a special case easier to handle: we are interested in flows that leave us (after deleting the circuits of the flow) with weights that satisfy the sums-of-circuits property.

Problem J. Call a matroid M strongly routing if for arbitrary $F \subseteq E$ and Eulerian weighting w of E, if w is a nonnegative combination of circuits C with $|C \cap F| \leq 1$, then it is also the nonnegative integer combination of circuits with this property. Characterize strongly routing matroids in terms of excluded minors.

The existence of a fractional solution to this problem can be characterized with linear programming, and the result obtained in this way can be somewhat sharpened analogously to the Japanese theorem. Most questions asked for ordinary flow problems have their analogues for the set of circuits figuring in Problem J.

6. Complexity

(Evenly) flowing matroids were characterized by Seymour [**60**] (see Section 3), and in Section 2 we studied the more general concept of routing matroids. The most general question in this series is probably the following:

PROBLEM K. What are the classes of matroids closed under minor containment, for which the integer multicommodity flow problem is polynomially solvable? What are those classes for which it is polynomially solvable for arbitrary Eulerian weighting, or for an arbitrary choice of a fixed number of demand-edges?

Note that e.g. even solvability is more general than routingness, provided the Distance Criterion can be checked in polynomial time. This is the case if and only if shortest paths (minimum weight circuits containing a given element) can be found in polynomial time in the matroid. The complexity of a multiflow problem is thus strictly related to the complexity of shortest path algorithms:

PROBLEM L. Find a connection between "shortest path oracles" of the matroid, of its dual, matroid flow properties (for example, routingness) and the solvability of multicommodity flow problems. For example, are all routing matroids evenly solvable?

We shall say that a class of matroids closed under minor containment is *solvable* if the integer flow problem can be solved in polynomial time for an arbitrary set of demand-edges, demands, and capacities. We shall say it is *finitely solvable* if we require polynomial solvability only for instances with a prefixed number of demand-edges, and *evenly solvable* if we restrict the demand and capacity function to define an Eulerian network.

It is not difficult to give examples of such classes: compositions of solvable classes are solvable, because the decomposition itself can be carried out in polynomial time according to Bixby, Cunningham [**4**] or Cunningham, Edmonds [**7**]. For example, using Seymour's [**60**] decomposition results (6.5), (6.7), (6.10), it is easy to see that evenly flowing matroids are also evenly solvable, and finite solvability of cographic matroids (see below) implies in the same way that flowing matroids are also finitely solvable. Similarly, matroids without $AG(2, 3)$, S_8, and H_6 minors are evenly (and also finitely) solvable (use the decomposition given by Appendix II, second corollary).

Furthermore, given a routing matroid composed with the method of Section 5 with Eulerian capacities and demands, the proofs provide us an algorithm to find an integer flow, or a metric proving the nonexistence of a fractional flow. More precisely:

(6.1) THEOREM. *There exists a polynomial algorithm which starts with the Eulerian network* (M, R, r, c) *as input, where* M *is a matroid without* AG(2, 3), S_8, *and* H_6 *minors, and stops with one of the following outputs:*

(i) *an integer flow in this network;*
(ii) *a metric proving that no fractional flow exists;*
(iii) *a bi-nonflowing minor of* M *proving that the matroid is not routing.*

This theorem is a clear consequence of the proof of (5.5) and remarks made above in this section.

We saw how to produce a growing sequence of solvable classes of matroids. But is it possible to find the exact border of NP-hardness and polynomiality? Can the solvable classes be characterized?

PROBLEM M. What is the complexity of the problem of deciding whether for given matroids (as input) the class of matroids not containing these as minors is (simply or finitely or evenly) solvable?

Note that Problem M may also be undecidable (a warning of Osamu Wattanabe, Tokyo Institute of Technology). However, a first result related to this problem is an observation of Middendorf and Pfeiffer [**38**]: their NP-completeness results and Robertson and Seymour's algorithm [**43**] imply a complete characterization of minor closed classes of matroids for which the "disjoint paths problem" (the integer multiflow problem with all capacities and demands equal to 1) is solvable in polynomial time.

In the following we shall first sketch a proof of the fact that cographic flow problems are finitely solvable. (For the details see Sebő [**54**]. Our goal is just to say enough for the problems we would like to state.) If $P \neq NP$ they are not solvable, because of the result of Middendorf and Pfeiffer [**38**]. Then we would like to ask some related questions and make some general remarks on the complexity of multicommodity flow problems.

Let us formulate now the finite solvability problem more precisely:

Suppose k is a positive integer and define the *integral planar* k*-commodity flow problem* to be a matroid flow where the matroid is restricted to be both graphic and cographic and the number of demand-edges to be at most k.

Recall now the connection of multicommodity flow problems and the Chinese Postman Problem (see (2.1)). Let G be a planar graph. Note that $|R| \leq k$ implies $t(V(G)) \leq 2k$ in the corresponding t-join–t-cut optimization problem, and thus

(6.2) *If the* t*-cut packing problem with bounded* $t(V(G))$ *can be solved in polynomial time, then graphic matroids are finitely solvable.*

Let us recall a well-known fact about packings of t-cuts in graphs: for any w-packing of t-cuts there exists another w-packing of t-cuts that is laminar, and whose value is equal to the value of the original packing. (A family of sets is called *laminar* if for any two X_1, X_2 of its elements, $X_1 \subset X_2$ or $X_2 \subseteq X_1$ or $X_1 \cap X_2 = \emptyset$. A packing of cuts is called laminar if it is the set of coboundaries of a laminar family.) We shall not need to know the complexity of finding a laminar system of cuts, since the size of our graph is bounded.

Note that the fractional relaxation of a t-cut packing problem is just the dual of the Chinese Postman Problem (see Edmonds, Johnson **[11]**). Edmonds and Johnson observed that the Chinese Postman Problem can be reduced to a matching problem in a related weighted auxiliary graph. The main observation that leads to the finite solvability of cographic matroids is that (integer) *optimal dual* solutions for the Chinese Postman Problem also correspond to (integer) optimal odd cut packings in this weighted auxiliary graph, whose vertices are $T := \{x \in V(G) : t(x) = 1\}$, a set whose cardinality is bounded by $2k$ if the number of demands in a corresponding flow problem is bounded by k. This also implies that the well-known matching algorithms can be converted into algorithms for the Chinese Postman Problem, with the same complexity.

Suppose we are given the graph G, the function t where the set $T := \{x \in V(G) : t(x) = 1\}$ has even cardinality, and the function $w : E(G) \to \mathbb{Z}^+$. Define the *map* of (G, t, w) to be the pair (K, d) consisting of the complete graph K on T and of the *distance function* $d(x, y) := d_{G,w}(x, y) := \min\{w(P) : P \text{ is an } (x, y) \text{ path in } G\}$.

Let us state now the observation that is the key to the complexity of flow problems with a fixed number of demands:

(6.3) *Let (G, t, w) be arbitrary, and (K, d) be its map. If $(\mathscr{C}, h)$ is an (integer) d-packing of odd cuts in K, then there exists an (integer) w-packing $(\mathscr{T}, g)$ of t-cuts in G such that $\sum_{C\in\mathscr{T}} g(C) = \sum_{C\in\mathscr{C}} h(C)$, and it can be determined in strongly polynomial time.*

Note that the converse statement is obvious: if $x, y \in T$ and P is an (x, y) path, then clearly, the number of cuts in $\mathscr{T}$ separating x and y is at most $w(P)$. This is true, in particular, for a minimum weight path, and consequently x and y are separated by at most $d(x, y)$ elements of $\mathscr{T}$. Hence $(\mathscr{T}_K, g_K$ is a d-packing in K. The theorem states the surprising statement that conversely, from a d-packing in the "rougher" structure (K, d), a w-packing of the "finer" (G, w) can always be "reconstructed." The reconstructed packing will always have the nice property that its restriction to the map is the originally given packing.

(6.4) *The problem of finding maximum integer packing of odd cuts in a graph K can be formulated as an integer program with a $\binom{k}{2} \times 2^{k-2}$ constraint matrix, where $k := V(K)$.*

In fact, let (K, d) be the pair to be tested. Define one variable x_I for each odd subset I of vertices of K, which makes 2^{n-1} variables, and put one constraint $\sum\{x_I : |I| \text{ odd}, I \text{ separates } v_1 \text{ and } v_2\} \le d(v_1, v_2)$ for each pair $v_1, v_2 \in V(G)$. Add the nonnegativity constraints $x_I \ge 0$ for all the variables.

Combining (6.2), (6.3), and (6.4), and applying Lenstra's result ([**31**]; see also Schrijver [**46**]) on integer programming in fixed dimension, we easily get:

(6.5) THEOREM. *Cographic matroids are finitely solvable.*

Note that the worst case performance of the algorithm provided by (6.4) is doubly exponential in k. However, this comes only from the application of Lenstra's algorithm. For $k = 5$, say, we just have a linear integer program with 45 constraints and 256 variables, and a very particular structure. It is very likely that by looking more carefully into this integer programming problem, we get a better bound, and the algorithm becomes applicable in practice. However, we cannot expect much more than the following conjectures, because the problem is NP-complete in general.

PROBLEM N. Is it true that Chvátal rank of the above polyhedron is bounded by a function of $|T|$? Find a polynomial algorithm for integral multicommodity flows in planar graphs with a bounded number of demands without using Lenstra's algorithm.

Let us finally mention some immediate consequences of the k-sum operations on the complexity of matroid flow problems:

- Matroids without AG(2, 3), S_8, and H_6 minors are finitely and evenly solvable. Note that graphic matroids are even not "2-solvable." (According to Even, Itai, Shamir [**12**], even the 2-commodity flow problem is NP-complete.)
- If the demand graph is restricted to be a K_5, then multiflow problems on matroids without AG(2, 3), S_8 minors can be solved in polynomial time.

The reader can deduce more general and other similar results by composition. Since any finite class is, for example, finitely solvable, we can always take all possible 2-sums, say, of a finite class and classes we know to be solvable: there is an infinite sequence of more and more general classes of finitely solvable matroids. To make a real step forward though, one should be able to describe the difference between "...-solvable" classes of matroids and those which are not. In other words the real question about the complexity is Problem M.

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Appendix

I. Matroid examples. First we give the definition of some matroids used throughout the paper, following Seymour [**60**].

K_i is the polygon matroid of the complete graph on i vertices, and $K_{i,j}$ that of the complete bipartite graph with i and j element classes; H_6 is the only graphic matroid for which $H_6/e = K_5$. R_{10} is the binary matroid represented by the characteristic vectors of the three-element subsets of a set of cardinality 5;

$$F_7 := \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}, \qquad S_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix};$$

AG(2, 3) (3-dimensional Affine Geometry over GF(2)) is the binary matroid defined on the points of the 3-dimensional affine space over GF(2), where independence is affine independence. Equivalently, it is the binary matroid represented over GF(2) by all 4-dimensional (0, 1)-vectors whose last coordinate is 1. (Similarly, F_7 is the 2-dimensional projective geometry, or equivalently, the 3-dimensional linear space over GF(2) without 0.)

In order to understand these matroids better, note that F_7 is the Eulerian extension (for the definition, see Section 4) of K_4, and R_{10} is the Eulerian extension of $K_{3,3}$; the automorphism group of all the matroids defined here except that of H_6 and S_8 is transitive; R_{10}, AG(2, 3), S_8 are isomorphic to their dual; AG(2, 3), K_5, F_7, R_{10}, are Eulerian; AG(2, 3) and S_8 are the only matroids in which deleting or contracting some nonseries and nonparallel element we get F_7 or its dual (important for Section 5); if $M\backslash e = K_5$ or $M/e = K_5$ holds for some matroid M, and M is not H_6, then M has an F_7 minor.

II. Sums and decompositions. We give here some basic definitions and facts from Seymour [**58**], [**60**].

Let M_1, M_2 be matroids on the sets E_1, E_2 and with cycle-set $\mathscr{C}_1$, $\mathscr{C}_2$ respectively, where $E_1 \cap E_2$ satisfies one of the following:

(i) $= \emptyset$;
(ii) $= \{d\}$ where d is neither a loop nor a coloop of either M_1 or M_2;
(iii) $= \{e_1, e_2, e_3\}$, where $\{e_1, e_2, e_3\}$ is a cycle in both M_1 and M_2.
The *sum* of these matroids is the matroid M on the set $E := E_1 \triangle E_2$ with cycle-set $e_1 \cup e_2$ in case (i), $\{C_1 \Delta C_2 \colon d \notin C_1, d \notin C_2 \text{ or } d \in C \cap C_2\}$ in case (ii), and $\{C_1 \Delta C_2 \colon C_1 \in \mathscr{C}_1, C_2 \in \mathscr{C}_2, C_1 \cap \{e_1, e_2, e_3\} = C_2 \cap \{e_1, e_2, e_3\}\}$ in case (iii).

It is called the 1-, 2-, or 3-sum respectively, depending on whether (i), (ii), or (iii) holds. M_1 and M_2 are the *members* of the sum. d, or the cycle $= \{e_1, e_2, e_3\}$, is called the *marker* of the 2- or 3-sum respectively.

For 1- and 2-sums it is easy to see that both M_1 and M_2 are minors of M. (For more about sums see Seymour [**58**, Sections 2, 3, 4].) A binary matroid is called *connected* or 2-*connected* if it is not the 1-sum of two of its proper minors. It is called 3-*connected* if it is neither the 1- nor the 2-sum of two of its proper minors. Otherwise it is called 1-, 2-, or 3-*separable*, and the sum operation giving it as result is called a 1-, 2- *or* 3-*separation* or *decomposition.*

It is well known that every matroid has a unique 1-decomposition (the components of the matroid). By Cunningham, Edmonds [7] the list of 3-connected matroids resulting from repeated 2-decompositions of a matroid is also unique (and not only up to isomorphism). This result is not directly applied in the paper, but explains why we can deduce from properties of a given decomposition (for example, from the number of minimal nonflowing 3-connected matroids in the list) other properties depending on the matroid only (for example, routingness; see proof of Theorem 5.5).

Cunningham and Edmonds [7] also design polynomial algorithms to do the decompositions; for simpler algorithms see Bixby, Cunningham ([**4**] or [**5**]). These algorithms make it possible to turn our proofs algorithmic (see Section 5 and Theorem 6.1 and the paragraphs above it).

Let $\mathscr{M}$ be a class of matroids closed under minors. $S \in \mathscr{M}$ is said to be a *splitter* for this class if every 3-connected matroid in the class is either isomorphic to S or does not have a minor isomorphic to S. In other words, S is a splitter for $\mathscr{M}$ if every matroid in $\mathscr{M}$ can be obtained from matroids without S-minors and matroids isomorphic to S with the repeated application of the 1- and 2-sum operation. *To see whether a given matroid is a splitter of a given class one only has to see whether the one-element extensions of the given matroid contradict this fact or not:*

SPLITTER THEOREM. (Seymour [**58**], (7.3)) *Let $\mathscr{M}$ be a minor-closed class of matroids, and $N \in \mathscr{M}$, $N \neq \emptyset$, a minor that has no loops, coloops, parallel or series elements, and is different from the polygon matroid of a wheel. If $M \in \mathscr{M}$, $M \backslash e = N$ implies that e is a loop, coloop, or parallel element, and $M/e = N$ implies that e is a loop, coloop, or series element of M, then N is a splitter for $\mathscr{M}$.*

Seymour [**60**] gives many examples of splitters. In the paper we only need the following statements. ((a) and (c) are immediate consequences of the Splitter Theorem using the last two remarks in Appendix I.)

COROLLARY.

(a) F_7 *and its dual are splitters in the class of matroids without* AG(2, 3) *and* S_8 *minor (Seymour* [**60**], (6.6)).

(b) R_{10} *is a splitter in the class of matroids without* F_7 *and its dual as minors (Seymour* [**60**], (6.7)).

(c) K_5 *is a splitter in the class of matroids without* H_6 *and* F_7 *minors (Seymour* [**60**], (6.8)).

As an immediate consequence we have:

COROLLARY. *K_5, F_7, and R_{10} are splitters in the class of matroids without* AG(2, 3) *and S_8 minor.*

III. Routing matroids. We are finally proving the following theorem:

THEOREM. *The minimal nonflowing matroids K_5, F_7, and R_{10} are routing.*

PROOF. Suppose (M, R, r, c) is an Eulerian network, where M is a minimal nonflowing matroid. For K_5 the statement follows immediately from Karzanov's theorem (Section 4.III), so we shall suppose M is either F_7 or R_{10}.

Let us represent F_7 and R_{10} with the columns of the vertex-edge incidence matrix of K_4 and $K_{3,3}$ respectively, completed by a last column 1 everywhere. We shall assume that $E(M)$ is the set of columns in this representation, and the graph we get by deleting the last column will be denoted by G. G is isomorphic to K_4 or $K_{3,3}$.

Let $t \in R$ be arbitrary, and $\mathscr{C}$ with the multiplicities f be a flow in this network. Suppose indirectly that there exists no integer flow, and that $\sum_{e \in R} r(e) + \sum_{e \in E(M) \setminus R} c(e)$ is minimum among the counterexamples.

By symmetry (since the automorphism group of all the minimal nonflowing matroids is transitive), we can assume that t is the last (all 1) column of the representation. Since M is Eulerian, the circuits of M containing t are exactly the t-joins of G.

CLAIM 1. $f(C) < 1$ for all $C \in \mathscr{C}$.

Indeed, if $f(C) \geq 1$, then replace $f(C)$ by $f(C) - 1$, and decrease $r(e)$ and $c(e)$ $(e \in C)$ by 1. We shall call this operation the *subtraction* of C from the flow and the network respectively; the inverse operation is the *addition* of C.

The new network (after having subtracted C) is clearly Eulerian and has a fractional flow, so by the minimality of our counterexample it also has an integer flow. Adding C to this flow we get an integer flow in the original network, a contradiction.

CLAIM 2. t is contained in at most two circuits of $\mathscr{C}$.

Indeed, if there are three different circuits $t \in C_1, C_2, C_3 \in \mathscr{C}$, then $C_i \setminus t$ $(i = 1, 2, 3)$ are different t-joins of G, none of which contains a demand-edge. But the union of 3 different t-joins of K_4 or $K_{3,3}$ leaves out at most one edge of these graphs, whence M *has at most one demand-edge besides* t. But it is not difficult to see (and follows immediately from Seymour **[58]**, 13.4, 14.7) that F_7 and R_{10} are evenly 2-flowing, a contradiction with the nonexistence of an integer flow in (M, R, r, c).

CLAIM 3. $r(t) = 1$. $r(t) > 0$ because $M \setminus t$ is evenly cycling; $r(t) < 2$ because of Claims 1 and 2.

CLAIM 4. If $t \in C \in \mathscr{C}$ $(f(C) > 0)$, then $|C| \geq 5$.

Suppose indirectly $|C| \leq 4$. We show that subtracting C from the network the Cut Condition is still satisfied. This is a contradiction with the nonexistence of an integer flow, because by Claim 3, $r(t) = 0$ holds after the subtraction, and $M\backslash t = G$ is evenly flowing.

Since there exists a flow in (M, R, r, c), the Cut Condition is satisfied. By the cardinality of C, for every cut Q: $0 \leq c(C \cap Q\backslash R) - r(C \cap Q \cap R) \leq 3$, and since our network is Eulerian, in fact $q := c(C \cap Q\backslash R) - r(C \cap Q \cap R)$ is either 0 or 2. Note that if we subtract εC, $c(Q\backslash R) - r(Q \cap R)$ decreases by εq. Thus, if $q = 0$, Q clearly satisfies the Cut Condition after subtracting C. If $q = 2$, then $f(C)q > 0$, and consequently $c(Q\backslash R) - r(Q \cap R) > 0$. (Because, after subtracting $f(C)C$, the Cut Condition is still satisfied.) Since our network is Eulerian, $c(C\backslash R) - r(C \cap R) \geq 2$, and after the subtraction it decreases by $1q \leq 2$.

Claim 5. Suppose $t \in C_1, C_2 \in \mathscr{C}$, $|C_1|, |C_2| \geq 5$. Then there exist circuits $t \in C_1'$, C_2' of M such that the sum of C_1' and C_2' is at most as much as the sum of C_1 and C_2, and $|C_1'|, |C_2'| \leq 4$.

If $M = F_7$ there is no circuit of cardinality 5 (so Claim 4 is already a contradiction). If $M = R_{10}$ then $G = K_{3,3}$, and it is easy to see that in $K_{3,3}$ the union of two t-joins bigger than a matching contains two disjoint matchings.

Claim 6. $\mathscr{C}$ has only one circuit containing t.

For if there were two, by Claim 4 both would be of cardinality at least 5. Let them be C_1 and C_2, and apply Claim 5: replacing $\min\{f(C_1), f(C_2)\} > 0$ "copies" of C_1 and C_2 by the same number of copies of C_1' and C_2', we get a feasible flow, and $|C_1'| \leq 4$. But this contradicts Claim 4.

Now the proof of the theorem is straightforward: by Claim 6 there is a unique circuit $C_t \in \mathscr{C}$ containing t, and thus $f(C_t) = r(t)$. But then Claim 1 and 3 contradict each other. □

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