COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA **28** (2) (2008) 131–143
DOI: 10.1007/s00493-008-2197-0

CYCLIC ORDERS: EQUIVALENCE AND DUALITY

PIERRE CHARBIT, ANDRÁS SEBŐ

Received March 17, 2005

Cyclic orders of graphs and their equivalence have been promoted by Bessy and Thomassé's recent proof of Gallai's conjecture. We explore this notion further: we prove that two cyclic orders are equivalent if and only if the winding number of every circuit is the same in the two. The proof is short and provides a good characterization and a polynomial algorithm for deciding whether two orders are equivalent.

We then derive short proofs of Gallai's conjecture and a theorem "polar to" the main result of Bessy and Thomassé, using the duality theorem of linear programming, total unimodularity, and the new result on the equivalence of cyclic orders.

1. Introduction

In this paper we characterize – with a simple good-characterization and polynomial algorithm – the "equivalence" of (linear or cyclic) orders given on the vertices of a directed graph. This notion has been introduced by Bessy and Thomassé [1] in order to prove a forty years old conjecture of Gallai, and seems to be a basic concept that can be expected to have further applications.

Furthermore, we show the linear programming background of Bessy's and Thomassé's results. They prove two minmax theorems that are in 'antiblocking relation', and the related polyhedra have advantageous integrality properties that can be handled algorithmically with network flows [5]. We provide here proofs that are the simplest to our knowledge: these are based on total unimodularity and linear programming duality without any concern

Mathematics Subject Classification (2000): 05C38, 05C20, 05C70, 90C10, 90C27

of how the solutions of these are found. The proof of the second theorem uses the characterization of equivalent orders.

If D is a digraph then the *underlying graph* is the undirected graph $G=G(D)$ whose edges are the arcs of D without orientation. We will say that D is connected or 2-edge-connected, if the underlying graph has these properties. (Note the difference with strongly connected digraphs, which is a property of the digraph. For the standard definitions of graph theory or polyhedral combinatorics we refer to [4].) Strongly connected digraphs will shortly be said to be *strong*.

A *cycle* is a closed walk with distinct arcs (edges) and it is a *circuit* if all vertices are distinct, both in directed and undirected graphs. We will use the term *undirected circuits* of a digraph to design the circuits of the underlying graph. (The orientation of the arcs of such a circuit can be arbitrary.) A *multiset* is a set where each element has a nonnegative integer multiplicity. A *linear order* of a (di)graph is an order $O := (v_1, v_2, \dots, v_n)$ of its vertices. If in addition v_n is followed by v_1, we call it a *cyclic order*. A cyclic order of n elements, has n *openings* (v_i, v_{i+1}) $(i=1,\dots,n)$, that is, linear orders (starting with v_{i+1} and ending with v_i) which are cyclically equal to it. For a notation, a cyclic order can be represented by any of its openings. A *cyclic shift* of a linear order is another opening of the same cyclic order.

All digraphs considered here are without loops or parallel arcs, but may have directed 2-circuits, where a k-circuit $(k \in \mathbb{N})$ is a circuit of size k. A digraph is a pair $D=(V,A)$, where $V=V(D)$ is the vertex-set of D, and $A=A(D)$ is its arc-set. An arc $a=uv\in A$ has a *head* $h(a):=v$ and a *tail* $t(a):=u$. Sets $X\subseteq S$ will also denote their $0-1$ incidence (characteristic) vectors in $\{0,1\}^S$.

Given a digraph $D = (V, A)$, a *circulation* is a function $f : A \longrightarrow \mathbb{R}$ such that $f(\delta^{\text{in}}(x)) = f(\delta^{\text{out}}(x))$ for all $x \in V$, where $\delta^{\text{in}}(x)$ and $\delta^{\text{out}}(x)$ is the set of arcs entering, respectively leaving x. We do not require f to have nonnegative values. Clearly, any of the two ± 1 vectors associated to any undirected circuit of D (putting 1 on arcs in one direction, and -1 on those in the opposite direction) and any of their linear combinations are circulations; it is well-known and easy to see that conversely, any circulation is the linear combination of circuits signed in this way.

Given a cyclic order $(v_1, v_2, \dots, v_n)$, the *length* of an arc (v_i, v_j) is $j-i$ if $j>i$ and $n+j-i$ if $i>j$. If C is a cycle of D, the sum of the lengths of its arcs is a multiple $\text{ind}(C)n$ of n. This integer $\text{ind}(C)$ is called the *index* (winding number) of C. The *index* $\text{ind}(\mathcal{C})$ of a family $\mathcal{C}$ of cycles is the sum of the indices of its constituent cycles. If every arc lies in a circuit of index 1, the cyclic order is said to be *coherent*. Bessy and Thomassé [1] showed that every strong digraph has a coherent cyclic order.

In the linear order $(v_1,v_2,\ldots,v_n)$ an arc v_iv_j is called a *forward* arc if $i<j$, and *backward* arc if $i>j$. It is an important observation in [1] that *the index of a circuit is equal to the number of its backward arcs in any opening; in particular, the number of backward arcs of a circuit does not change through cyclic shifts.*

If G is an undirected graph with a cyclic order, we can also define the index for its undirected circuits: the vertices of every undirected circuit can be ordered in two ways (so that consecutive vertices correspond to edges of C). For each circuit we fix one of these two as *reference orientation.* In undirected graphs we allow parallel edges, but we suppose throughout the paper that there are no loops. The index $\mathrm{ind}(C)$ of C is the index of the corresponding directed circuit; *equivalently it is equal to the number of backward arcs in any opening of the cycle endowed with the reference orientation.* Note that the sum of the indices of the two orientations of C (with respect to the fixed cyclic order of G) is $|C|$, and therefore one of these determines the other; in particular, it does not matter which of them we choose for reference orientation, either of them is good, or we can also keep both in mind.

2. Equivalence of Cyclic Orders

We first introduce the fundamental equivalence classes of cyclic orders promoted by Bessy and Thomassé's proofs [1]. Then we establish a basic invariance property of these equivalence classes, that will be used later on.

Suppose $G=(V,E)$ is an undirected graph. We will also use the following notions for directed graphs, but ignoring the orientation of the arcs. Two cyclic orders are *equivalent* if one can be obtained from the other by a sequence of elementary operations. An *elementary operation* is a permutation (interchange) of nonadjacent consecutive vertices, that is $(v_1,v_2,\ldots,v_n)$ maps to $(v_2,v_1,\ldots,v_n)$, where v_1 and v_2 are nonadjacent.

The motivation for this definition is that it preserves the index of every circuit of G. We show in this paper that two cyclic orders of a strongly connected digraph are equivalent if and only if all circuits have the same index in the two.

This does not hold for arbitrary digraphs: an acyclic orientation of a triangle has two non-equivalent cyclic orders, even though it does not even have directed cycles. We first prove a stronger condition involving all undirected circuits to be necessary and sufficient for arbitrary digraphs.

Theorem 2.1. *Let $G=(V,E)$ be an undirected graph. Let O_1 and O_2 be two cyclic orders of V. The following statements are equivalent:*

(i) $O_1 \sim O_2$.
(ii) *The index of every circuit is the same with respect to O_1 and O_2.*

It follows for instance that *for forests any two cyclic orders are equivalent* (which is easy to check directly). Note that this theorem provides a good characterization (a linear $NP \cap coNP$ certificate) for two orders to be equivalent. It is not surprising that this certificate depends only on the underlying undirected graph: the elementary operations depend only on this graph.

Note that this theorem has no condition on G. For strongly connected graphs the condition on directed cycles will turn out to imply the condition for undirected cycles.

Proof. Since an elementary operation does clearly not change the index of a circuit or of a closed walk, (i) implies (ii).

Let us prove now the essential statement "(ii) implies (i)", by induction on the number of edges. Let $e=xy \in E$ $(x,y \in V)$ be an arbitrary edge.

By the induction hypothesis the statement is true for $G-e$, that is, there exists a sequence $\pi_1,\dots,\pi_k$ of elementary operations that brings the order O_1 to the order O_2 in $G-e$. Every elementary operation on $G-e$ is also an elementary operation of G, except the permutation of x and y. If this operation does not occur, we are done: we have a sequence of elementary operations that brings O_1 to O_2.

Claim. *If the permutation of x and y does occur among $\pi_1,\dots,\pi_k$, then there exist cyclic orders C_1,C_2, $C_1 \sim O_1$, $C_2 \sim O_2$ such that x is followed by y in both C_1 and C_2, or y is followed by x in both.*

This Claim finishes the proof of the theorem: since e joins neighboring vertices in both C_1 and C_2, and in the same order, these orders obviously define orders C'_1, C'_2 of G/e (the graph obtained after contraction of e, where the edges parallel to e are deleted before identifying the endpoints of e, in order to avoid loops); furthermore, since (i) implies (ii) (and this is already proven), the condition (ii) is still satisfied for C_1 and C_2, and therefore for C'_1 and C'_2 as well. Since G/e has less edges than G, by the induction hypothesis $C'_1 \sim C'_2$, and the elementary operations of G/e correspond obviously to one or two elementary operations of G.

In order to prove the Claim let i,j, $1 \le i \le j \le k$ be the first and the last index where the permutation of x and y occurs. Let O'_1 be the cyclic order we get from O_1 if we stop before executing π_i, and O'_2 the order we get by executing the permutations in reverse order from O_2 and stopping before executing π_j. Clearly, $O_1' \sim O_1$, $O_2' \sim O_2$, and therefore O_1', O_2' satisfy (ii).

In both O_1' and O_2' x and y are consecutive by definition. If they follow one another in the same order in O_1' and O_2', then we are done. If not, suppose without loss of generality (by possibly interchanging the notation x, y) that x precedes y in O_1', and y precedes x in O_2'.

Take a shift in O_2' so that x is the first, and y the last element. *There is no forward path now from x to y in $G-e$*, because if there was such a path $P=(x=x_0,x_1,\dots,x_p=y)$, then $p\geq 2$, and with the edge yx, P is in fact a cycle of index 1. On the other hand taking an opening of O_1' different from (x,y) we see that yx is a backward arc in P, and there must be another backward arc since otherwise $p=1$. Therefore the index of the cycle P in O_1' is at least 2, while it is 1 in O_2', contradicting (ii).

It follows that the set X of vertices that can be reached from x with a forward path (in O_2') have no forward neighbour outside X, and therefore X can be placed after y through a sequence of elementary changes. The vertices in $Y:=V\setminus(X\cup\{x,y\})$ have no backward neighbour in X, so similarly, they can be placed before x. Therefore $O_2'':=Y,x,y,X$ is an equivalent order, and y follows x as in O_1. So $C_1:=O_1'$ and $C_2:=O_2''$ are as claimed. ∎

We promised that for strongly connected digraphs condition (ii) is sufficient to hold for directed circuits in order to deduce (i). This sharpening follows by simple linear properties of circuits – roughly, the circuits of a strongly connected digraph "generate" all the undirected circuits of the underlying graph:

If $D=(V,A)$ is a directed graph, then each cycle of the underlying graph can be represented as a vector in $\{-1,0,1\}^A$ in the following usual way (see for instance *network matrices* in [4]):

Let C be an undirected circuit (with one of the two orientations fixed for reference), and define the vector $\overrightarrow{C}\in\{-1,0,1\}^A$ as follows: $\overrightarrow{C}(a)=1$ if $a\in C$ is oriented in the sense of the orientation of C, $\overrightarrow{C}(a)=-1$ if it is oriented in the opposite sense, and $\overrightarrow{C}(a)=0$ if $a\notin C$.

$$\mathcal{C}(D) := \mathrm{lin}\{\overrightarrow{C} : C \text{ is a circuit of } G(D)\}.$$

Note that the definition of $\mathcal{C}(D)$ does not depend on which of the two orientations of the circuits we chose, since the vector defined by the opposite orientation is just $-\overrightarrow{C}$. $\mathcal{C}(D)$ is the set of circulations.

Lemma 2.1. *A 2-edge-connected digraph $D=(V,A)$ is strongly connected if and only if $\mathcal{C}(D)$ is spanned (linearly) by the (directed) circuits of D (as vectors in $\{0,1\}^A$).*

Proof. Indeed, if D is not strongly connected, let e be an edge not contained in a directed circuit. Then since the underlying undirected graph is 2-edge connected, there exists an undirected circuit C in $G(D)$, $e\in E(C)$; since e is not contained in any directed circuit, C is not generated by circuits of D.

Conversely suppose that D is strongly connected. Then any circulation f is generated by directed circuits: indeed, for each of the negative coordinates $e_1,\dots,e_p$ of f choose a circuit C_i containing e_i ($i=1,\dots,p$); $f-\sum_{i=1}^{p} f(e_i)C_i$ is a nonnegative circulation, which is obviously a (nonnegative) combination of directed circuits, and then so is f. ∎

Theorem 2.2. *Let $D=(V,A)$ be a strongly connected digraph. Let O_1 and O_2 be two cyclic orders of V. Suppose that the index of each circuit is the same with respect to O_1 and O_2. Then $O_1\sim O_2$.*

Proof. We have to prove only that the condition is implied for every undirected circuit of the underlying graph, because then Theorem 2.1 implies the assertion. Denote by $\mathrm{ind}_i(C)$ the index of circuit C according to O_i ($i=1,2$).

Open both O_1 and O_2 to get the linear orders L_1 and L_2. Define the vectors $w_1,w_2\in\{1,-1\}^A$ to be -1 on backward arcs and 1 on forward arcs.

Note first that $w_i(C)=(|E(C)|-\mathrm{ind}_i(C))-\mathrm{ind}_i(C)=|E(C)|-2\,\mathrm{ind}_i(C)$ for every cycle ($i=1,2$). So *the assumption on the equality of indices according to the two orders is equivalent to $w_1(C)=w_2(C)$ for every circuit C.*

The equation

$$w_i^T\overrightarrow{C}=|E(C)|-2\,\mathrm{ind}_i(C)$$

holds for all the circuits of $G(D)$. Indeed, in the inner product $w_i^T\overrightarrow{C}$ we have four kinds of terms: $1\cdot 1$, $1\cdot(-1)$, $(-1)\cdot 1$, $(-1)\cdot(-1)$, and it is clear that the result is 1 if the corresponding edge goes forward in $\overrightarrow{C}$, and -1 if it goes backward, and the difference of the forward and backward edges is $|E(C)|-2\,\mathrm{ind}_i(C)$.

So for checking that the condition (ii) of Theorem 2.1 holds, it is sufficient to prove $w_1^T\overrightarrow{C}=w_2^T\overrightarrow{C}$ for every circuit C of $G(D)$. However, since we know that this holds for directed circuits, and by Lemma 2.1 the directed circuits generate $\mathcal{C}(D)$, it follows for every undirected circuit C of $G(D)$, as expected. ∎

3. Coherent Orders and Gallai's conjecture

3.1. Coherent cyclic orders

In this section we will investigate the notion of coherent cyclic orders of digraphs defined in [1]. Recall that the coherence of a cyclic order means that each arc lies in a circuit of index 1. We will prove, in fact, that as long as each arc lies in a circuit, the digraph admits such a cyclic order. That is why we only need to focus on strong digraphs.

We define the following reflexive and transitive relation on cyclic orders:

$$O_1 \leq O_2 \text{ if for each circuit } C \text{ of } D,\ \mathrm{ind}_{O_1}(C) \leq \mathrm{ind}_{O_2}(C).$$

Since indices of circuits are the same with respect to two equivalent cyclic orders, this relation extends to a reflexive and transitive relation on equivalence classes of cyclic orders. Theorem 2.2 states that in the case of a strong digraph, this relation is antisymmetric. Thus we can state the following result.

Theorem 3.1. *If D is a strong digraph, the relation "$\leq$" defines a partial order on equivalence classes of cyclic orders.*

The following statement is equivalent to Bessy and Thomassé's key-lemma about the existence of a coherent cyclic order in strongly connected graphs [1]. The following simple proof is an adaptation of the variant in [5], nevertheless the framework of any proof can be adapted.

Proposition 3.1. *Let D be a strong digraph. Any order in a class that is minimal (with respect to $\leq$) is coherent.*

Proof. Indeed, let O be an order and $e \in A$ an arc that is not contained in a circuit of index 1 of O. By replacing O with an equivalent order we can suppose that $e \in B$, where B is the set of backward arcs. The set $E(G) \backslash (B \backslash e)$ does not contain a cycle – since every cycle has a backward arc and not only e –, so it has an order where every arc in $E(G) \setminus (B \setminus e)$ is a forward arc. In this cyclic order O' the set of backward arcs B' satisfies $B' \subseteq B \setminus e$. Clearly, for every circuit C:

$$\mathrm{ind}'(C) = |C \cap B'| \leq |C \cap B| = \mathrm{ind}(C),$$

where ind' denotes the indices according to O'. Since G is strongly connected, e is contained in a circuit C, and for this circuit strong inequality holds, proving that O is not minimal with respect to $\leq$, and finishing the proof. ∎

3.2. Index-Bounded Weightings and a Min-Max theorem

The main result of Bessy and Thomassé in [1] is a Conjecture of Gallai, that we will refer to as *Gallai's conjecture.*

Theorem 3.2 (Bessy and Thomassé [1]). *Let D be a strong digraph and denote by α the stability of the graph, that is the maximum cardinality of a stable set. Then the vertices of D can be covered by at most than α circuits.*

This is a consequence of a minmax theorem proved by Bessy and Thomassé. In this section we provide a simple direct proof – differently from [1] and [5] – of a linear programming type corollary of Bessy and Thomassé's result which in turn easily implies Gallai's conjecture. We will use linear programming duality and total unimodularity in the proof, without any algorithmic aim.

A *weighting* of a digraph D is a function $w : V \to \mathbb{N}$. The *weight* of a vertex v of D is the value $w(v)$. By extension, the *weight* of a subgraph of D is the sum of the weights of its vertices. If D is endowed with a cyclic ordering O, and if $w(C) \leq \mathrm{ind}(C)$ for every circuit C of D, we say that the weighting w is *index-bounded* (with respect to O). We could also say "index-bounded multiset" of vertices. We prove the following:

Theorem 3.3. *Let D be a digraph and suppose each of its vertices lie in a circuit, and O is a cyclic order of D. Then*

$$\min\{\mathrm{ind}(\mathcal{C}) : \ \mathcal{C} \text{ is a circuit covering of } D\} \\ = \max\{w(D) : \ w \text{ is an index-bounded weighting}\}.$$

Gallai's conjecture can be easily deduced by applying this theorem to a coherent cyclic order:

- for every family $\mathcal{C}$ of circuits of D, $|\mathcal{C}| \leq \mathrm{ind}(\mathcal{C})$;
- since each vertex is the endpoint of an arc, it is also contained in a circuit of index 1, and therefore an index-bounded weighting of D is necessarily $(0,1)$-valued;
- there is no arc $a = v_i v_j \in A$ such that $w(i) = w(j) = 1$, because the circuit C of index 1 containing a satisfies $w(C) \leq 1$. So the support of w is a stable set, and $w(D) \leq \alpha(D)$ follows.

Our goal here is to gain in simplicity comparing to previous proofs by sacrificing the algorithm. We aim at the simplest possible proof of the same spirit as the other proofs of this paper.

Conversely, the theorem implies [1], Theorem 1 with the help of a simple combinatorial characterization of index-bounded weightings for coherent orders as "cyclic stable sets" [5].

Let us prepare the proof of Theorem 3.3 by recalling some basic well-known notations and facts:

– For a directed graph $D=(V,A)$, $M=M(D)$ denotes the $n\times m$ incidence matrix of D with entries $m_{v,a}$ ($v\in V$, $a\in A$) equal to -1 if v is the tail of arc a, 1 if it is the head, and 0 otherwise; if the arcs or (and) the vertices of D are indexed we will replace a or v by its index.
– The solutions $x\in\mathbb{R}^A$ of the equation $Mx=0$ are circulations.
– Given a function $d:A\longrightarrow\mathbb{R}$ the solutions of $\pi M\le d$, $\pi\in\mathbb{R}^V$ are called *potentials* (for d).
– Given d, there exists no negative cycle according to d if and only if $d^\top x\ge 0$ for every nonnegative circulation, and this holds if and only if (by Farkas's Lemma) there exists a potential for d. (Here we need only these facts, and can ignore the well-known combinatorial procedures that compute circulations or potentials [4].)

These are conform to notations and terminology in [4] which is a reference for more details or proofs, if necessary. Denote $M^+=M^+(D)$ the $n\times m$ matrix with entries $m^+_{v,a}=\max\{m_{v,a},0\}$ ($v\in V$, $a\in A$).

It is easy to see that the $2n\times m$ matrix $\widetilde{M}$ whose first n rows constitute a matrix identical to M and the second n rows a matrix identical to M^+ is totally unimodular: indeed, in any square submatrix M', subtract the i-th row of M^+ from the i-th row of M for all i for which both rows are present in M'. We get a matrix with at most two nonzeros per column, and if there are two nonzeros, then one of them is 1, the other -1; such a matrix is the submatrix of the incidence matrix of a graph, and as such, has determinant 0 or ± 1, and the determinant of M' is the same.

3.3. Proof of Theorem 3.3

Let D be a digraph, with vertex set V and an opening $(v_1,\dots,v_n)$ of the cyclic order O. We note its arc set $A=\{a_1,\dots,a_m\}$, and define the objective function $c\in\mathbb{R}^m$ with $c_i:=1$ if a_i is a backward arc and 0 otherwise. Consider the linear program $x\in\mathbb{R}^m$,

$$(P)\qquad \text{minimize } c^\top x \text{ subject to } Mx\ge 0,\ M^+x\ge 1, x\ge 0.$$

Since $\widetilde{M}$ is totally unimodular, the linear program (P) has integer primal and dual optima and by the duality theorem of linear programming [4] the two optima are equal.

Claim 1. *The primal optimum of (P) is equal to the left hand side of the minmax equality.*

Let x be an integer primal solution (with objective value $c^\top x$). We first show that x is an integer circulation: indeed, the sum of the rows of M is 0 and therefore

$$0 = 0^\top x = (1^\top M)x = 1^\top (Mx),$$

where $1^\top Mx$ is the sum of the coordinates of Mx, all nonnegative by the condition, and therefore there is equality throughout. Thus $Mx = 0, x \geq 0$, that is, x is a circulation and it is an integer vector: there exists a multiset $\mathcal{C}$ of circuits with $x = \sum_{C \in \mathcal{C}} C$. Moreover, because of $M^+ x \geq 1$, every element is covered by at least one of the circuits.

Conversely, for any multiset $\mathcal{C}$ of circuits that cover the vertex-set $x := \sum_{C \in \mathcal{C}} C$ is an integer vector that satisfies (P). Moreover,

$$c^\top x = \sum_{C \in \mathcal{C}} c(C) = \sum_{C \in \mathcal{C}} \operatorname{ind}(C),$$

establishing the claim.

Claim 2. *The dual optimum of (P) is equal to the right hand side of the minmax equality of the theorem.*

An integer dual solution is of the form (π, y), $\pi \in \mathbb{Z}^n$, $y \in \mathbb{Z}^n$, where π is a potential for $c(a) - y_{h(a)}$ (see the definition of potentials). Such an integer potential exists for an integer y if and only if there is no negative circuit for the edge-weights $c(a) - y_{h(a)}$ $(a \in A)$, that is, if and only if for every circuit C,

$$\sum_{v \in V(C)} y_v = \sum_{a \in C} y_{h(a)} \leq \sum_{a \in C} c(a) = \operatorname{ind}(C),$$

that is, if and only if y is an index-bounded weighting. The dual objective value is $\sum_{v \in V} y_v$. ∎

4. Cyclic Colourings

The authors of [1] prove a theorem that can be considered to be the "antiblocker" of Theorem 3.3, and actually a sharpening of such a theorem. We provide a simple proof of this theorem as well, in the spirit of the proof of Theorem 3.3, and using Theorem 2.2. (Our definitions and terminology are slightly different from [1]: the definitions do not depend on orientation, so they concern only undirected graphs; the terminology and notation are simplified and unified.)

Let $G = (V, E)$ be an undirected graph. According to Zhu [6] a *circular r-coloration* of G is a function $f : V \longrightarrow [0, r) = \{x \in \mathbb{R} : 0 \leq x < r\}$, such that for all $(x,y)\in A$: $\mathrm{dist}(x,y)\geq 1$, where $\mathrm{dist}(x,y):=\mathrm{dist}_{f,r}(x,y):=\min\{|f(x)-f(y)|, r-|f(x)-f(y)|\}$ – the *distance* of (x,y) on the circle of perimeter r.

If a cyclic order O is given, then a circular r-coloration is called a *cyclic r-coloration* (with respect to the cyclic order O) if in addition the order $(v_1,v_2,\dots,v_n)$, $0\leq f(v_1)\leq f(v_2)\leq\ldots\leq f(v_n)<r$ is equivalent to O. We will use these terms for directed graphs as well, whenever the properties hold for the underlying undirected graph.

The infimum (which is clearly a minimum) of all reals $r>0$ such that O has a circular r-colouring is the *circular chromatic number*, denoted by χ_{circ}. The minimum of $r>0$ for which there exists a cyclic r-coloring (with respect to O) will be denoted by ξ_O. (It is easy to check that $\chi_{\mathrm{circ}}\leq\chi=\lceil\chi_{\mathrm{circ}}\rceil$ and $\chi_{\mathrm{circ}}\leq\xi_O$, where $\chi=\chi(D)$ is the (usual) chromatic number.)

We define for any circuit C, the *cyclic length* of C as $l_O(C):=|C|/\mathrm{ind}(C)$ (with respect to the fixed cyclic order O).

Theorem 4.1. *Let D be a nontrivial strongly connected digraph and O a coherent cyclic order on its vertices. Then*

$$\xi_O(D)=\max\{l_O(C):\ C \text{ a circuit of } D\}.$$

Proof. Let $D=(V,A)$ be a digraph, with vertex set V, and let $(v_1,\dots,v_n)$ be a linear order of V, which is an opening of the coherent cyclic order O. We denote the arc set of D by $A=\{a_1,\dots,a_m\}$.

Consider the linear program $x\in\mathbb{R}^m$,

$$(P*)\qquad \text{maximize} \sum_{i=1}^{m} x_i \text{ subject to } Mx\leq 0,\ x(B)\leq 1, x\geq 0,$$

where $B\subseteq A$ is the set of backward arcs. Clearly, this linear program is feasible and bounded.

Claim 1. *The primal optimum of $(P*)$ is equal to the right hand side of the theorem.*

Indeed, again, $Mx\leq 0$ implies $Mx=0$, so primal solutions are circulations x with $x(B)\leq 1$. Since D contains at least one circuit, and for any circuit C, $C/\mathrm{ind}(C)$ is a primal solution (vertex) of $(P*)$ with objective value $|C|/\mathrm{ind}(C)=l_O(C)$, the primal optimum is positive, and *greater than or equal to the right hand side of the theorem.*

Conversely, any primal solution x is a nonnegative circulation, that is, a nonnegative linear combination of circuits. We write $x=\sum_{C\in\mathcal{C}}\lambda_C C$, $(\lambda_C\geq 0)$, for some set $\mathcal{C}$ of circuits.

The constraint $x(B)\le 1$ is equivalent to $\sum_{C\in\mathcal{C}}\lambda_C \operatorname{ind}(C)\le 1$ and therefore:

$$1^\top x = \sum_{C\in\mathcal{C}}\lambda(C)|C| = \sum_{C\in\mathcal{C}}\lambda(C)\operatorname{ind}(C)l_O(C)$$
$$\le \max\{l_O(C):\ C \text{ a circuit of } D\}$$

finishing the proof of the claim.

Note that Claim 1 does not use that the given order is coherent. This will be exploited for Claim 2.

Fix $(\pi_1,\dots,\pi_n,r)$ to be the dual optimum. Starting with this vector we construct a cyclic colouring.

According to Claim 1, $r=\max\{l_O(C)\colon C \text{ a circuit of } D\}>1$.

Claim 2. *For every forward arc uv, $1\le\pi_v-\pi_u\le r-1$. For every backward arc uv, $1\le\pi_u-\pi_v\le r-1$.*

First, $(\pi_1,\dots,\pi_n,r)$ satisfies the dual constraints for each $a=uv\in A$, that is:

$$\pi_v-\pi_u\ge\begin{cases}1 & \text{if } uv \text{ is a foward arc}\\ 1-r & \text{if } uv \text{ is a backward arc.}\end{cases}\tag{1}$$

Furthermore, if uv is a backward arc, by coherence, there exists a forward path P between v and u, and adding up the inequalities concerning the arcs of this path: $\pi_u-\pi_v\ge|P|-1\ge 1$.

Likewise, if uv is a forward arc, uv lies in a circuit C of index 1. Let $u'v'$ be the unique backward arc of C. Then $\pi_{v'}\le\pi_u\le\pi_v\le\pi_{u'}$, and therefore $|\pi_v-\pi_u|\le|\pi_{v'}-\pi_{u'}|\le r-1$. This finishes the proof of Claim 2.

For any dual solution $(\pi_1,\dots,\pi_n,r)$ of $(P*)$ define $q: V(D)\longrightarrow[0,r)$ with $\pi_i=:p(v_i)r+q(v_i)$, that is, $q(v_i)$ is the remainder of π_i modulo r. It is straightforward to check that Claim 2 implies that q is a circular r-coloration. Moreover, the linear order O_π of the vertices defined by the increasing order of π_v, $(v\in V)$ has the same set of backward arcs as O, so these two orders are equivalent.

Claim 3. *The function q is a cyclic r-coloration with respect to O.*

In addition to Claim 2 we have to check that q defines a cyclic order *equivalent to O*. According to Theorem 2.1, it is sufficient to check that in the linear order O_q where the vertices are in increasing order of q, *every circuit has exactly the same number of backward arcs as in O, that is, as in O_π*, since the latter two have been proved to be equivalent. Let C be an arbitrary circuit. Thanks to Claim 2, we know that for an arc uv, $p(u)-p(v)$

equals either -1, 0, or 1. Clearly, arcs uv with $p(u)=p(v)$ are forward arcs or backward arcs in both O_π and O_q; we also see from Claim 2 that in case $p(u)-p(v)=1$, uv is a backward arc in O_π, and it is a forward arc in O_q; similarly, if $p(u)-p(v)=-1$, then uv is a forward arc in O_π, and a backward arc in O_q; since $\sum_{uv\in C} p(u)-p(v)=0$, we have:

$$|\{uv \in C : p(u) - p(v) = 1\}| = |\{uv \in C : p(u) - p(v) = -1\}|,$$

that is, the number of backward arcs remains the same in O_q and O_π in every circuit.

Claims 1 and 3 assert that $\max\{l_O(C)\colon C \text{ a circuit of } D\}=r\geq\xi_O(D)$. To finish the proof of the theorem note first that for any cyclic $\xi_O(D)$-coloration, and any circuit C, with the distances defined by the cyclic coloration, $|C|\leq \sum_{xy\in A(C)} \operatorname{dist}(x,y)=\xi_O(D)\operatorname{ind}(C)$. ∎

With the starting idea of defining a coloring from a potential, the proof can be finished in two ways, see [2] and [5]. We have chosen here a third, simpler way.

Acknowledgment. The authors wish to thank Adrian Bondy for suggesting a purely linear programming approach to Theorem 3.3 – a suggestion that was independent from the network flow approach in [5], led to a different proof, and was a crucial motivation for the present work.

References

[1] S. Bessy and S. Thomassé: Spanning a strong digraph by α circuits: A proof of Gallai's conjecture; *Combinatorica* **27(6)** (2007), 659–667.

[2] P. Charbit: Circuits in Graphs and Digraphs via Embeddings, PhD thesis, Université Claude Bernard, Lyon, December 2005.

[3] T. Gallai: Problem 15, in: *Theory of Graphs and its Applications (M. Fiedler, ed.)*, Czech. Acad. Sci. Publ., 1964, p. 161.

[4] A. Schrijver: *Combinatorial Optimization*, Springer, 2003.

[5] A. Sebő: Minmax Relations for Cyclically Ordered Digraphs, *Journal of Combinatorial Theory, Ser. B* **97(4)** (2007), 518–552.

[6] X. Zhu: Circular chromatic number: a survey; *Discrete Mathematics* **229(1–3)** (2001), 371–410.

Pierre Charbit

LIAFA
Universite Paris 7 Denis Diderot
Case 7014
75205 Paris, Cedex 13
France
charbit@liafa.jussieu.fr

András Sebő

CNRS
Laboratoire G-SCOP – INPG, CNRS, UJF
46, Avenue Felix Viallet
38000 Grenoble
France
andras.sebo@g-scop.inpg.fr