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IMPERFECT AND NONIDEAL CLUTTERS: A COMMON APPROACH

GRIGOR GASPARYAN*, MYRIAM PREISSMANN, ANDRÁS SEBŐ

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We prove three theorems. First, Lovász's theorem about minimal imperfect clutters, including also Padberg's corollaries. Second, Lehman's result on minimal nonideal clutters. Third, a common generalization of these two. The endeavor of working out a 'common denominator' for Lovász's and Lehman's theorems leads, besides the common generalization, to a better understanding and simple polyhedral proofs of both.

Introduction

A *first goal* of this paper is to provide similar and simple proofs of two fundamental theorems in the theory of blocking and antiblocking polyhedra: Lovász's result on minimal imperfect clutters [8] (Section 1), and Lehman's result [6] on minimal nonideal clutters (Section 2). The two results themselves have already many similarities, pointed out by Shepherd [12]; deeper connections between the two results have been exhibited in [13]. The present approach arose from an effort of unifying both the statement and the proof: we provide one single proof to a common generalization. Moreover, for Lehman's theorem our proof is simpler than the previous ones; the proof of Lovász's theorem is also short once the prerequisites of Lehman's theorem are proved, but altogether it is longer and less elementary than Gasparyan's proof [4] concerning clique-clutters of graphs. However, the present approach also proves the polyhedral facts on the way, without any additional effort.

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The found 'common denominator' of the two classical theorems leads to simple proofs, and to a common generalization, – *the second goal* of this work (Section 3).

Our proof is probably closer to proofs of Lehman's theorem – with some exaggeration one could say that we provide a Lehman-type proof for minimal imperfect graphs as well. At the same time we warn the reader from deciding too early that the proof goes like some previous proof she knows. The solution we present seems to be an essentially uniquely determined mixture of new and existing ideas (with some modules from [10], [15], [4], and [14]) enabling a common treatment of minimal imperfect and minimal nonideal matrices.

The *third (but not least) goal* is to get rid of superfluous ingredients of the two proofs, to get to the bare essence of their common skeleton, and to communicate this in the cleanest and simplest way we can.

It is not among the purposes of this paper to reach the most general structure we can. The article [5] goes further in analysing the general *matrix-properties* that characterize minimal imperfect or minimal nonideal structures, and [14] studies the problem when systems of linear inequalities with both blocking and antiblocking types of constraints *are integer*.

0.1. Notation and terminology

Given a finite set V, a family $\mathcal{A}$ of subsets of V is called a *hypergraph* on V, $\mathcal{A}$ is the set of *hyperedges*. It is a *clutter*, if $A_1 \subseteq A_2$ ($A_1, A_2 \in \mathcal{A}$) implies $A_1 = A_2$.

Let us fix the notation $V := \{1, \dots, n\}$.

A subset $S \subseteq V$ and its $0-1$ incidence vector $\chi_S \in \{0,1\}^V$ will not be distinguished, in particular, we will mostly write S instead of χ_S. We will not distinguish between n-dimensional vectors, $1 \times n$ and $n \times 1$ matrices – in matrix multiplications the right shape is usually uniquely determined by another matrix. The only ambiguous case is the product of $x, y \in \mathbb{R}^n$: $x^T y$ will denote the scalar product.

Some more notation: $\mathbb{N}$ is the set of natural numbers $(1, 2, \dots)$; $\underline{1}$ is the all 1 vector of appropriate dimension (usually in $\mathbb{R}^n$), I and J are the identity matrix and the all 1 matrix of appropriate dimension (usually $n \times n$); $e_i (\in \mathbb{R}^n)$ denotes the incidence vector of $\{i\}$ $(i \in V)$.

If $v \in V$, then $\mathcal{A} - v$ is the hypergraph on $V \setminus v$ defined by $\{A \in \mathcal{A} : v \notin A\}$. If $\mathcal{A}$ is a clutter, then $\mathcal{A} - v$ is also a clutter. The degree of v in $\mathcal{A}$ is $d(v) := d_{\mathcal{A}}(v) := |\{A \in \mathcal{A} : v \in A\}|$.

A matrix is called *uniform* if its column- and row-sums are all equal and nonzero. A uniform matrix is always a square matrix. If we want to emphasize that the row and column sums of M are all $r \in \mathbb{N}$, that is, $\underline{1}M = M\underline{1} = r\underline{1}$, we will write that M is r-uniform. Accordingly, a hypergraph will be called r-uniform if all cardinalities and also all degrees are equal to r.

A *polyhedron* is the set of all solutions of a system of linear inequalities. A *polytope* is a bounded polyhedron. For basic definitions and statements about polyhedra we refer to Schrijver [11], and we only repeat now shortly the definition of the terms we are using directly. A *face* of a polyhedron is a set we get if we replace certain defining inequalities with the equality so that the resulting polyhedron is nonempty. The *minimal face containing* w is the face defined by the equalities satisfied by w. A point $v \in \mathbb{R}^n$ is called *integer* if $v \in \mathbb{Z}^n$, otherwise it is called *fractional*. A polyhedron is *integer* if each of its faces contains an integer point, otherwise it is *noninteger*.

If $X \subseteq \mathbb{R}^n$, we will denote by $r(X)$ the (linear) *rank* of X, and by $\dim(X)$ the *dimension* of X, meaning the rank of the differences of pairs of vectors in X, that is, $\dim(X) := r(\{x - y : x, y \in X\})$.

If P is a polyhedron, then its faces of dimension $\dim(P) - 1$ are called *facets* and its faces of dimension 0 are called *vertices*. All (inclusionwise) minimal faces of P have the same dimension. We say that P *has vertices*, if this dimension is 0. *Neighboring vertices* share $n-1$ linearly independent facets. A vertex of a full dimensional polyhedron is *simplicial*, if it is contained in exactly n facets. A simplicial vertex is contained in n faces of dimension one of the polyhedron. If such a face is unbounded we will call it a *ray*, if it is bounded we will say it is an *edge*.

Given a clutter $\mathcal{A}$ on V, $P_{\leq}(\mathcal{A}) := \{x \in \mathbb{R}^n : x(A) \leq 1 \text{ for all } A \in \mathcal{A}, x \geq 0\}$ is called the *antiblocking polyhedron* (of $\mathcal{A}$), and $P_{\geq}(\mathcal{A}) := \{x \in \mathbb{R}^n : x(A) \geq 1 \text{ for all } A \in \mathcal{A}, x \geq 0\}$ is called its *blocking polyhedron*. The integer (that is, $0-1$) vertices of maximal, respectively minimal, support of these, constitute the *antiblocker*, respectively *blocker*, of $\mathcal{A}$. If $P_{\leq}(\mathcal{A})$ has only integer vertices, then we say that $\mathcal{A}$ is *perfect*, if $P_{\geq}(\mathcal{A})$ has only integer vertices, then $\mathcal{A}$ is said to be *ideal*.

If $x \in \mathbb{R}^n$, the *projection* of x parallel to the i-th coordinate is the vector $x^i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.

If $X \subseteq \mathbb{R}^n$, the projection parallel to the i-th coordinate of the set X is $X^i := \{x^i : x \in X\}$; if $I \subseteq V$, X^I is the result of successive projections parallel to $i \in I$ (the order does not matter).

If P is a polyhedron and $v \in V$, we define the *deletion* of v in P as $P \setminus v := \{x \in P : x_v = 0\}^v$, and the *contraction* as $P/v := P^v$. A *minor* of P is the polyhedron $P \setminus I/J$, $(I, J \subseteq V,\ I \cap J = \emptyset)$ one gets after successively deleting the elements of I and contracting the elements of J (to get $P\backslash I/J = (P\cap\{x:x_i=0 \text{ if } i\in I\})^{I\cup J}$).

We call a polyhedron $P\subseteq\mathbb{R}^V$ *critical* if it is not an integer polyhedron, but $P\setminus i$ and P/i are integer polyhedra for all $i\in V$.

Minors of antiblocking (blocking) polyhedra are also antiblocking (blocking), and the minors of perfect (ideal) polyhedra are perfect (ideal). (It is easy to see that $P_{\leq}(\mathcal{A})\setminus I/J = P_{\leq}(\mathcal{A}'_{\leq})$, and $P_{\geq}(\mathcal{A})\setminus I/J = P_{\geq}(\mathcal{A}'_{\geq})$, where $\mathcal{A}'_{\leq}$, $\mathcal{A}'_{\geq}$ arise from $\mathcal{A}$ in a simple way: delete the columns indexed by I, and then delete those rows that are no more maximal, resp. minimal; for $P_{\leq}$ do the same with J as well; for $P_{\geq}$ the columns indexed by J have to be deleted together with all the rows having a 1 in at least one of these columns. The clutters $\mathcal{A}'_{\leq}$, $\mathcal{A}'_{\geq}$ are called minors of $\mathcal{A}$.) This fact is not necessary for understanding the results of this paper, we therefore omit the (easy) details.

Critical antiblocking polyhedra are called *minimal imperfect*, and critical blocking polyhedra *minimal nonideal*, and so is called the clutter $\mathcal{A}$ as well; by the previous remark, all proper minors of minimal imperfect and minimal nonideal polyhedra are perfect or ideal respectively, and therefore a certificate for the minor of a clutter to be imperfect (nonideal) certifies also that the clutter is imperfect (nonideal). We will usually speak about the minors of polyhedra instead of minors of the corresponding clutters, because the latter is different in the antiblocking and blocking case, and also because the way of using the critical property will be polyhedral. Note however for the sake of clarity, that according to the previous paragraph a clutter is minimal imperfect if and only if it is imperfect and the clutter consisting of the maximal members of $\{A\setminus\{v\}: A\in\mathcal{A}\}$ is perfect for all $v\in V$; it is minimal nonideal if and only if it is nonideal and $\mathcal{A}-v$, as well as the clutter consisting of the minimal members of $\{A\setminus\{v\}:A\in\mathcal{A}\}$ are ideal.

For $P = P_{\leq}(\mathcal{A}_{\leq})\cap P_{\geq}(\mathcal{A}_{\geq})$, with clutters $\mathcal{A}_{\leq}$, $\mathcal{A}_{\geq}$ in general it *is not true* that the minors of P are also intersections of antiblocking and blocking polyhedra:

Example 1. Let $V := \{1,2,3,4,5\}$, $\mathcal{A}_{\leq} := \{\{1,2\},\{3,4\},\{4,5\}\}$, $\mathcal{A}_{\geq} := \{\{2,3\},\{5,1\}\}$. The vector $\frac{1}{2}\underline{1}\in\mathbb{R}^5$ is a vertex of $P_{\leq}(\mathcal{A}_{\leq})\cap P_{\geq}(\mathcal{A}_{\geq})$. The following facts are straightforward: $\frac{1}{2}\underline{1}\in\mathbb{R}^4$ is still a vertex of (for instance) P^2; the inequality $x_1-x_3\leq 0$ is valid and facet inducing for P^2. This is a *noninteger polyhedron, that does not have a critical minor with only* $0-1$ *constraints*. (A refined definition of minors pulls this kind of polyhedron among minimal noninteger ones [14].)

We learnt: *a projection of the intersection of an antiblocking and a blocking polyhedron may easily contain non-*$0-1$*-constraints*. We do not care, since our goal here is not to apply the results to minors, but to capture the common essence of two theorems.

Let $\mathcal{A} := \mathcal{A}_{\leq} \cup \mathcal{A}_{\geq}$ and $w \in \mathbb{R}^V$. Define $\mathrm{core}(\mathcal{A}, w) := \{A \in \mathcal{A} : w(A) = 1\}$. The sets in the core maximize $w(A)$ among sets in $\mathcal{A}_{\leq}$, and minimize it in $\mathcal{A}_{\geq}$.

Graphs $G=(V,E)$ are always simple, undirected, $V=V(G)$ is the vertex-set, $E=E(G)$ the edge-set.

We define the set of *neighbors* of $v \in V$ as $N(v) = \{x \in V : vx \in E\}$, $d(v) := d_G(v) := |N(v)|$. A graph is called *regular* if all degrees are equal, r-regular, if all are equal to r.

We finish this subsection by introducing some particular structures, most of them are uniform hypergraphs.

The notation $\mathcal{H}_n^{n-1}$ will stay for the set of $n-1$-tuples of an n-set ($n \geq 3$). It is easy to see that $\mathcal{H}_n^{n-1}$ is minimal imperfect, and it is also an easy and well-known exercise to show that $0-1$-matrices not containing such a minor (or equivalently having the 'dual Helly property') can be represented as the (inclusionwise) maximal cliques of a graph.

The *degenerate projective plane* clutters

$$\mathcal{F}_n = \{\{1, \dots, n-1\}, \{1, n\}, \{2, n\}, \dots, \{n-1, n\}\}, \quad (n = 3, 4, \dots)$$

are minimal nonideal.

It is easy to show that the blocker of the blocker is always the original clutter. The antiblocker of the antiblocker of $\mathcal{H}_n^{n-1}$ is not itself but $\{V\}$, and this is the only exception: it is another well-known exercise to show that the antiblocker of the antiblocker of a clutter that has no $\mathcal{H}_n^{n-1}$ minor (dual Helly property), is itself.

The following definition of partitionability coincides with Bridges and Ryser's definition of 'binary $(r,s,1)$ systems' [1], and is close to Shepherd's definition [12]:

A pair (X,Y) of $n \times n$ matrices is called *partitionable*, if the following conditions are satisfied:

- $XY = YX$, and the diagonal elements of this product are all equal, and not to 1, whereas the non-diagonal elements are equal to 1.
- X and Y are uniform.

Then X is r-uniform, Y is s-uniform, and the diagonal elements are μ (for some integers $r, s > 0, \mu \geq 0$). We will speak about the r, s and μ of the partitionable matrices or clutters. There is a relation between these parameters: $n = rs - \mu + 1$. (Indeed, $(\mu + n - 1)\underline{1} = \underline{1}(XY) = (\underline{1}X)Y = (r\underline{1})Y = r(\underline{1}Y) = rs\underline{1}$.)

We call a clutter $\mathcal{A}$ on an n-element set $\leq$-*partitionable*, if there exists an $n\times n$ matrix X with rows from $\mathcal{A}$ and an $n\times n$ matrix Q with columns from the antiblocker $\mathcal{B}$ of $\mathcal{A}$, so that (X,Y) is partitionable with $\mu=0$.

A clutter $\mathcal{A}$ on an n-element set is $\geq$-*partitionable*, if there exists an $n\times n$ matrix X with rows from $\mathcal{A}$ and an $n\times n$ matrix Y with columns from the blocker $\mathcal{B}$ of $\mathcal{A}$ so that (X,Y) is partitionable with $\mu\geq 2$.

Note as a curiosity that $\mathcal{H}_n^{n-1}$ is $\leq$-partitionable for $n\geq 3$, but its antiblocker is not! If $\mathcal{A}$ is $\leq$-partitionable, and the antiblocker of the antiblocker $\mathcal{B}$ of $\mathcal{A}$ is $\mathcal{A}$, then $\mathcal{B}$ is also $\leq$-partitionable. (Indeed, $X':=Y^T$ and $Y':=X^T$ provide the definition for $\mathcal{B}$.) Similarly, the blocker of a $\geq$-partitionable clutter is also $\geq$-partitionable.

This definition of the partitionability of a pair of matrices is highly redundant (this can be seen from Bridges and Ryser's theorem [1] see also the Commutativity Lemma of Subsection 0.2). These properties can (will) be easily seen to provide good certificates for a matrix to be imperfect (if $\mu=0$), or not ideal (if $\mu\geq 2$), (see the final remarks of the paper). The reader may find it useful to discover the combinatorial statements reflected in a compact way in this definition. They all rely on the following interpretation: Denote the rows of X by X_i, and the columns of Y by Y_i $(i=1,\dots,n)$. Then the entry in the intersection of the i-th row and j-th column of XY is $|X_i\cap Y_j|$, and that of YX is $d_{ij}=|\{k\in\{1,\dots,n\}: i\in X_k, j\in Y_k\}|$. In particular we get from this 'combinatorial meaning' of the elements of the matrix YX that the following statements are equivalent:

– The non-diagonal elements of YX are equal to 1.
– For all $v\in V$, $\{X_i\setminus\{v\}: i\in\{1,\dots,n\}, v\in Y_i\}$ is a partition of $V\setminus\{v\}$.
– For all $u\in V$, $\{Y_i\setminus\{u\}: i\in\{1,\dots,n\}, u\in X_i\}$ is a partition of $V\setminus\{u\}$.

We will not use these statements but they may help the reading of some proofs. (In alternate variants of proofs it can play a more important role: with its help one can avoid using the inverse of X in the proof of Lemma 0.3.)

The ultimate goal of each of sections 1, 2, 3 is to prove that minimal noninteger systems of inequalities are partitionable. The lemmas proved in the following subsection will be the technical tools that allow to finish these proofs when all the essential combinatorial properties have been collected. The reader can postpone the reading of Section 0.2 until it is used.

0.2. Prerequisites

The main parts of the proofs of the following sections consist in converting polyhedral constraints into combinatorial structure. For then deducing the main structural results we are going to use some simple facts about matrices. We state and prove these in the introduction, because all the three following sections will use them directly, in the same way.

The next (or some trivially equivalent) statements occur in all previous work proving Lehman's theorem on minimal nonideal matrices; on the other hand the properties of minimal imperfect matrices are usually treated in other terms. The first in the series appeared more than fifty years ago as the key-lemma of Erdős and de Bruijn's theorem [2] (see Conways's proof in [7]). It can be considered as a refinement of the following trivial fact: if G is bipartite with bipartition $\{X,Y\}$, $|X|\geq|Y|$ and any degree in X is greater than or equal to any degree in Y, then G is regular.

Lemma 0.1 (Erdős-de-Bruijn-Lemma). *Let $G=(X,Y;E)$ be a bipartite graph, $|X|\geq|Y|$, and $d(x)\geq d(y)$ for all $xy\in E$, $(x\in X,\ y\in Y)$. Then the equalities hold, that is, $|X|=|Y|$ and each connected component of G is regular.*

Proof. $|X|=\sum_{x\in X}\left(\sum_{y\in N(x)}\frac{1}{d(x)}\right)=\sum_{xy\in E,x\in X}\frac{1}{d(x)}$ $\leq\sum_{xy\in E,y\in Y}\frac{1}{d(y)}=\sum_{y\in Y}\left(\sum_{x\in N(y)}\frac{1}{d(y)}\right)=|Y|$, and now the constraint $|X|\geq|Y|$ implies the equality throughout, that is, $|X|=|Y|$, and $d(x)=d(y)$ for every edge xy of G. ∎

Lemma 0.2 (Unicity Lemma). *Let $G=(X,Y,E)$ and $G'=(X',Y,E')$ be bipartite graphs whose connected components are regular, with vertices $v\in X$, $v'\in X'$ such that $G-v=G'-v'$ (so $X'\setminus\{v'\}=X\setminus\{v\}$). If G and G' have no isolated vertex then $N_G(v)=N_{G'}(v')$.*

Proof. Assume $N_G(v)\neq N_{G'}(v')$, say $u\in N_G(v)\setminus N_{G'}(v')$.

If $N_G(u)=\{v\}$, then because $G-v=G'-v'$, u is an isolated vertex in G', a contradiction to our hypothesis. Otherwise there exists $x\in N_G(u)\setminus\{v\}$, which is impossible since $d_{G'}(x)=d_G(x)$, and $d_{G'}(u)=d_G(u)-1=d_G(x)-1$, contradicting $ux\in E(G')$ and the regularity of the connected components of G'. ∎

These two lemmas will be applied to graphs $G:=G_{\mathcal{H}}$, where $\mathcal{H}$ is a hypergraph on V with $|\mathcal{H}|=|V|$: $G_{\mathcal{H}}=(X,Y;E)$ where $X:=\mathcal{H}$ and $Y:=V$, $xy\in E$ $(x=H\in\mathcal{H},y\in V)$ if and only if $y\notin H$. Note that $G_{\mathcal{H}}$ is regular if and only if $\mathcal{H}$ is uniform (using $|X|=|Y|$).

Necessary remarks. Since frequent switching between $\mathcal{H}$ and of $G_{\mathcal{H}}$ can be tiring, it is good to notice once for ever how some properties of $G_{\mathcal{H}}$ have to be read in terms of the hypergraph $\mathcal{H}$. We assume that $\mathcal{H}$ does not have two equal hyperedges, that V is not a hyperedge of $\mathcal{H}$ and that all connected components of $G_{\mathcal{H}}$ are regular – these will be satisfied in the applications. Notice that from these assumptions it is easy to deduce that $G_{\mathcal{H}}$ has no isolated vertex: if it has one then it has one on each side of the bipartition (by the regularity of the connected components and the fact that $|X|=|Y|$) and then V would be a hyperedge of $\mathcal{H}$, a contradiction. A connected component of $G_{\mathcal{H}}$ will be called *big*, if the (equal) degrees of its vertices are at least 2, that is, if and only if it contains at least two vertices of V and at least two hyperedges from $\mathcal{H}$.

Let $u \in V$. We first observe:

(0.1) *If $H \in \mathcal{H}$ and u are not in the same connected component of $G_{\mathcal{H}}$, then $u \in H$. If u belongs to a big connected component of $G_{\mathcal{H}}$, then in this component there also exists $H' \in \mathcal{H}$ so that $u \in H'$.*

Indeed, the first part is just repeating the definition of $G_{\mathcal{H}}$. The second part follows from the fact that a big component cannot be a complete bipartite graph, because then $\mathcal{H}$ would contain the same set more than once, contradicting one of the assumptions above.

From this observation we get another property that will be very useful:

(0.2) *Suppose $\mathcal{H}$ is not uniform. Then for every $u,v \in V$ there exists $H \in \mathcal{H}$, $H \supseteq \{u,v\}$. If furthermore $\mathcal{H}$ has no element of cardinality $n-1$ then there exist two different sets $H_1, H_2 \in \mathcal{H}$, $H_1, H_2 \supseteq \{u,v\}$.*

Indeed, if $\mathcal{H}$ is not uniform then $G_{\mathcal{H}}$ contains connected components of different degrees and so, since there is no isolated vertex, it contains at least one big component. Any $H \in \mathcal{H}$ in a component not containing u nor v would have the required property, so we can suppose: there are exactly two components, and u, v lie in different ones. Say the component of u is big. Then take in the component of u: $H \in \mathcal{H}$, $u \in H$ (see the second part of (0.1)). Since v is in a different component, $v \in H$ also holds.

Now the additional statement follows in the same way: if $\mathcal{H}$ has no element of cardinality $n-1$ then all its connected components are big.

We show now another statement used in all three sections, stating connections between different combinatorial properties of $0-1$-matrices. These have a similar flavor to (and some implications have a big correlation with) Bridges and Ryser's theorem [1]. However, the following result is not a straightforward corollary of Bridges and Ryser's result:

Lemma 0.3 (Commutativity Lemma). *Let X and Y be $n \times n$ $0-1$ matrices such that the non-diagonal elements of XY are equal to 1, and*

the diagonal elements are either all smaller or all bigger than 1*. Then the following statements are equivalent:*
(i) X *is uniform.*
(ii) $XY = YX = J + (\mu - 1)I$, $(\mu \in (\mathbb{N} \setminus \{1\}) \cup \{0\})$.
(iii) Y *is uniform.*
(iv) (X, Y) *is partitionable.*

Proof. Suppose (i) holds, that is, $\underline{1}X = X\underline{1} = r\underline{1}$ $(r \in \mathbb{N})$. Denote by $\mu_1, \ldots, \mu_n$ the diagonal elements of XY. Clearly,

$$(\mu_1 + n - 1, \ldots, \mu_n + n - 1) = \underline{1}(XY) = (\underline{1}X)Y = r\underline{1}Y.$$

We show now $\mu_1 = \ldots = \mu_n$. If $\mu_i = 0$ for some $i \in \{1, \ldots, n\}$, then by the condition, $\mu := \mu_1 = \ldots = \mu_n = 0$, and we are done. Similarly, if $\mu_i \geq 2$ $(i = 1, \ldots, n)$, since there is at most one μ, $2 \leq \mu \leq r$, such that $\mu + n - 1$ is a multiple of r, all the μ_i $(i = 1, \ldots, n)$ must be equal to μ, as claimed. (The inequality $\mu_i \leq r$ $(i = 1, \ldots, n)$ is true, since μ_i is the scalar product of two $0-1$-vectors, that is the cardinality of the intersection of two sets, one of which is of cardinality r.)

Since X is uniform it commutes with both I and J and so it also commutes with $XY = J + (\mu - 1)I$, that is $XXY = XYX$. On the other hand, as $J + (\mu - 1)I$ is nonsingular (because $\mu \neq 1$), and is equal to XY it follows that X is nonsingular as well. Hence $X^{-1}XXY = X^{-1}XYX$ and (ii) is proved.

Now suppose (ii): $YJ = Y(XY - (\mu - 1)I) = (YX - (\mu - 1)I)Y = JY$. But YJ is a matrix all of whose columns are equal, and JY is a matrix all of whose rows are equal: any two elements m_{ij} of the matrix $M := YJ = JY$ are equal, since for all $i, j, k, l \in \{1, \ldots, n\}$, $m_{ij} = m_{kj} = m_{kl}$. So Y is uniform, that is, (iii) holds.

Suppose now that (iii) holds. By symmetry (applying to $Y^T X^T$ that (i) implies both (ii) and (iii)) we get that (ii) holds and X is also uniform, so (X, Y) is partitionable.

Last, (iv) implies (i) by definition. ∎

Note that it is not sufficient to write $XY = YX$ instead of (ii), as the (unique) example of $\mathcal{F}_n$ shows.

Example 2. We show that the condition of the lemma is essential: if some of the diagonal elements of X are smaller than 1 and some others bigger than 1, (i) can hold without any of the others to hold.

Let the rows of X be 11000, 01100, 00110, 00011, 10001, and the columns of Y be 00101, 01101, 01001, 10100, 10101. We see that the non-diagonal elements of XY are 1, X is uniform, but Y is not uniform! (Note that X is the constraint matrix of Example 1, and the columns of Y satisfy the constraints. In general, 'mixed odd circuits' (see Example 1) provide counterexamples, and the only relevant counterexamples [14]. Conversely and more sharply, if X is uniform, Y is not uniform, the non-diagonal elements of XY are equal to 1, and the diagonal elements not equal to 1, then it follows that X is the incidence matrix of an odd hole [5].)

0.3. The common part of the proofs

This section contains starting observations for the proof of the general case that cannot be simplified in the minimal imperfect case nor in the minimal non ideal case.

Let $\mathcal{A}_\leq$ and $\mathcal{A}_\geq$ be clutters on V. Recall that a polyhedron P is *critical*, if P has a noninteger vertex, but $P\setminus v$ and $P/v=P^v$ have only integer vertices for all $v\in V$.

Let $P:=P_\leq(\mathcal{A}_\leq)\cap P_\geq(\mathcal{A}_\geq)\subseteq\mathbb{R}^n$ be critical, and $w=(w_1,w_2,\dots,w_n)$ a noninteger vertex of P. We define $\mathcal{A}:=\mathcal{A}_\leq\cup\mathcal{A}_\geq$. Remark that in case $\mathcal{A}_\geq$ is empty then P is *minimal imperfect*, and in case $\mathcal{A}_\leq$ is empty then P is *minimal nonideal.*

Note that P has vertices. It is not necessarily bounded, but the 'characteristic' cone (that can be added to any of its element so that the result is still contained in P) is very simple: it is the set of nonnegative combinations of some unit vectors. (More exactly of those unit vectors whose index is not contained in $\cup\mathcal{A}_\leq$.) Since P^v is integer, every $x\in P^v$ is the convex combination of integer points of P^v.

(0.3) *$w_i>0$ for all $i\in\{1,2,\dots,n\}$, and $n\geq 3$.*

Indeed, if $w_i=0$, then w^i is a noninteger vertex of $P\setminus i$.
If $n\leq 2$ then P is integer. ∎

Since w is a vertex, and by (0.3) there is no tight nonnegativity constraint:

(0.4) $r(\text{core}(\mathcal{A},w))=n.$ ∎

Denote $\mathcal{C}_v:=\{C\subseteq V:|C\cap A|=1,\ \text{for all}\ A\in\text{core}(\mathcal{A},w)-v\}$. (Recall $\text{core}(\mathcal{A},w)–v=\{A\in\text{core}(\mathcal{A},w):v\notin A\}$). The following property of $\text{core}(\mathcal{A},w)$ will be crucial. Since w is a vertex and (0.3) holds, we have:

(0.5) *For any $a\neq v\in V$ there exists $C\in\mathcal{C}_v$ such that $a\in C$, that is, $C\cap A=\{a\}$ if $a\in A\in\text{core}(\mathcal{A},w)-v$.*

Proof. Since $w^v\in P^v$, where P^v is integer, there exist integer points of P^v, $p_1,p_2,\dots,p_k$ on the minimal face of P^v containing w^v, such that $w^v=$

$\sum_{i=1}^{k} \lambda_i p_i$, where $\lambda_i \geq 0$, $\sum_{i=1}^{k} \lambda_i = 1$. Let $C_i := supp(p_i)$ $(i=1,\dots,k)$ (some coordinates of p_i may be bigger than 1).

Now for arbitrary $A \in \text{core}(\mathcal{A}, w)$, $v \notin A$: since A corresponds to a constraint of P^v for which $w^v(A)=1$ and p_i is a point of P^v on the minimal face of P^v containing w^v, we have $p_i(A)=1$ and so $|C_i \cap A|=1$; that is, $C_i \in \mathcal{C}_v$, $(i=1,\dots,n)$.

By (0.3) $w_a^v > 0$, and we have $w^v = \sum_{i=1}^{k} \lambda_i p_i$ so at least one of the C_i-s contains a. ∎

Note: for $C \subseteq V$, $v \in C$ we have $C \in \mathcal{C}_v$ if and only if $C \setminus \{v\} \in \mathcal{C}_v$ (immediate from the definition); it is therefore sufficient to study $\mathcal{C}_v - v$.

(0.6) *Let $v \in V$ and let $A \in \text{core}(\mathcal{A}, w) - v$. Then $n - r(\text{core}(\mathcal{A}, w) - v) \geq r(\mathcal{C}_v - v) \geq |A|$.*

Proof. First, we prove $r(\mathcal{C}_v - v) \geq |A|$: for every $a \in A$ consider $C_a \in \mathcal{C}_v - v$, $C_a \cap A = \{a\}$ (it exists according to (0.5)); since all of the sets $\{C_a : a \in A\} \subseteq \mathcal{C}_v - v$ have a different unique common element with A, $r(\mathcal{C}_v - v) \geq |A|$ follows.

To prove $n - r(\text{core}(\mathcal{A}, w) - v) \geq r(\mathcal{C}_v - v)$ note that the set $\{C \in \mathcal{C}_v : v \in C\}$ (of rank at least $r(\mathcal{C}_v - v)$) is orthogonal to $\{\chi_A - e_v : A \in \text{core}(\mathcal{A}, w) - v\}$ (of rank as least $r(\text{core}(\mathcal{A}, w) - v)$). Consequently, $r(\mathcal{C}_v - v) + r(\text{core}(\mathcal{A}, w) - v) \leq n$. ∎

Lemma 0.4. *The family* $\text{core}(\mathcal{A}, w)$ *is linearly independent,* $|\text{core}(\mathcal{A}, w)| = n$*, and for $v \in V$ and $A \in \text{core}(\mathcal{A}, w) - v$, the degree $d(v)$ of v in* $\text{core}(\mathcal{A}, w)$ *satisfies $d(v) = r(\mathcal{C}_v - v) = |A|$.*

Proof. We apply the Erdős-de-Bruijn-Lemma 0.1 to the graph $G := G_{\mathcal{H}}$ where $\mathcal{H} \subseteq \text{core}(\mathcal{A}, w)$ is an arbitrary maximum linearly independent subset of $\text{core}(\mathcal{A}, w)$. The conditions of the Erdős-de-Bruijn-Lemma are satisfied: we have $|X| = |Y|$ by (0.4); if $xy \in E$, that is, $x = A \in \mathcal{H}$ and $y \in V \setminus A$, then by (0.6),

$$n - d_G(y) = n - |\mathcal{H} - y| \geq n - r(\text{core}(\mathcal{A}, w) - y) \geq r(\mathcal{C}_y - y) \geq |A| = n - d_G(x).$$

Hence the equality holds here, that is,

$$n - |\mathcal{H} - v| = r(\mathcal{C}_v - v) = |A| \text{ for all } v \in V.$$

All that remains to be proved is $\mathcal{H} = \text{core}(\mathcal{A}, w)$: then $\mathcal{H}$ is a basis of $\mathbb{R}^n$, and $|\mathcal{H} - v| = n - d(v)$, so the just proven equality reads

$$d(v) = r(\mathcal{C}_v - v) = |A| \text{ for all } A \in \text{core}(\mathcal{A}, w), \text{ and } v \in V \setminus A.$$

So let us prove $\mathcal{H} = \text{core}(\mathcal{A}, w)$. If not, let $A \in \mathcal{H}$, $A' \in \text{core}(\mathcal{A}, w) \setminus \mathcal{H}$ be such that $\mathcal{H}' := (\mathcal{H} \setminus \{A\}) \cup \{A'\}$ is also linearly independent. By the Erdős-de-Bruijn-Lemma, every component of the corresponding graph $G' = G_{\mathcal{H}'}$ is regular, as well as the components of G. Furthermore we notice that there is no isolated vertex in G. Indeed an isolated vertex in G would mean that V is in $\mathrm{core}(\mathcal{A},w)$, but then since $w_i>0$ for $i=1,2,\ldots,n$ there is no other element in $\mathrm{core}(\mathcal{A},w)$ an so $n=1$, a contradiction to (0.3). By the Unicity Lemma we get that $A=A'$ which is impossible since by definition $\mathrm{core}(\mathcal{A},w)$ is a set of distinct elements. ∎

The following corollary was explicitly stated in the above proof:

Corollary 0.1. *Every component of the graph $G_{\mathrm{core}(\mathcal{A},w)}$ is regular and contains at least two vertices. No two elements of* $\mathrm{core}(\mathcal{A},w)$ *have the same neighborhood in $G_{\mathrm{core}(\mathcal{A},w)}$.* ∎

We would also like to emphasize the reformulation of the fact that $\mathrm{core}(\mathcal{A},w)$ is a basis of $\mathbb{R}^n$ (see Lemma 0.4) in terms of polyhedral structure. Indeed, recall that w is a vertex of P, and note that the facets containing w are exactly those defined by $\mathrm{core}(\mathcal{A},w)$.

Corollary 0.2. *If the polyhedron P is full dimensional, any fractional vertex of P is simplicial and has n neighbors that are integer.*

Proof. Let w be a fractional vertex of P. By Lemma 0.4 the facets containing w are linearly independent, so, since P is full dimensional, w is a simplicial vertex of P. To show that w has n neighbors it remains to check that every one dimensional face that contains w is an edge. Notice first that no element of V belongs to exactly one element of $\mathrm{core}(\mathcal{A},w)$, since then w^v would be a non integer vertex of P^v (w has at least two fractional components, since if w_a was the only one, that would contradict $w_a(A)=1$ for $a\in A\in\mathrm{core}(\mathcal{A},w)$). Any ray containing w should satisfy $x(A)=0$ for all but one element of $core(\mathcal{A},w)$, say $A_1,A_2,\ldots,A_{n-1}$. This is not possible since $A_1\cup A_2\cup\ldots\cup A_{n-1}=V$ and P is contained in the nonnegative orthant.

Suppose now that w has a noninteger neighbor w'.

We can replace now w by w' in every proved statement, in particular every component of $G'=G_{\mathrm{core}(\mathcal{A},w')}$ is regular; w' is a simplicial vertex; since w' and w are neighbors on P, they share all but one of the incident facets of P, that is, $\mathrm{core}(\mathcal{A},w')=(\mathrm{core}(\mathcal{A},w)\setminus\{A\})\cup\{A'\}$, $(A\in\mathrm{core}(\mathcal{A},w), A'\in\mathrm{core}(\mathcal{A},w'), A\neq A')$.

Therefore, the conditions of the Unicity Lemma 0.2 hold to $G:=G_{\mathrm{core}(\mathcal{A},w)}$ and $G':=G_{\mathrm{core}(\mathcal{A},w')}$, $v=A$, $v'=A'$, but then $A=A'$, a contradiction. ∎

1. Imperfect Clutters

Here we consider the case where $\mathcal{A}_{\geq}$ is empty and so P is minimal imperfect. To use Corollary 0.2 we need first:

Lemma 1.1. *The polyhedron P is full dimensional.*

Proof. Since $e_v \in P$ ($v \in P$) we have $r(P) = n$. ∎

By Lemma 0.4, core$(\mathcal{A}, w)$ is a basis of $\mathbb{R}^n$, let us list its elements: A_1, $A_2, \dots,$ A_n; by Corollary 0.2 and the preceding lemma, w has n integer neighbors, let B_i be the (unique) neighbor of w which is not on the facet $\{x \in P : x(A_i) = 1\}$ $(i = 1, \dots, n)$.

Lemma 1.2. $|A_i \cap B_i| = 0$, $|A_i \cap B_j| = 1$ $(i \neq j \in \{1, \dots, n\})$.

Proof. Since $B_i \in P_{\leq}(\mathcal{A})$, we have $|A_i \cap B_j| \leq 1$ $i, j \in \{1, 2, \dots, n\}$; $|A_i \cap B_j| = 1$ if and only if B_j is on the face A_i, that is, if and only if $i \neq j$, as claimed. ∎

We have arrived now at the 'finish':

Theorem 1. *If $\mathcal{A}$ is a minimal imperfect clutter, then it is $\leq$-partitionable, and the unique fractional vertex of $P_{\leq}(\mathcal{A})$ is $\frac{1}{r}\underline{1}$, with $r = \max\{|A| : A \in \mathcal{A}\}$.*

Proof. Let w be a fractional vertex of $P_{\leq}(\mathcal{A})$, and let the rows of the matrix X be the (characteristic) vectors of the members of core$(\mathcal{A}, w)$. By Lemma 0.4 X is an $n \times n$ matrix. By Corollary 0.2 w is a simplicial vertex, and clearly, its neighbors on $P_{\leq}(\mathcal{A})$ are in the antiblocker of $\mathcal{A}$. Let Y be the $n \times n$ matrix whose columns are these neighbors, such that the i-th column of Y is the associate of the i-th row of X, that is, these rows and columns are the A_i's and B_i's of Lemma 1.2.

According to Lemma 1.2, $XY = J - I$.

Case 1: X is uniform.

The condition of Lemma 0.3 is true, and (i) is satisfied, so (iv) is also satisfied, that is, (X, Y) is partitionable.

By definition, the rows of X are in $\mathcal{A}$ and the columns of Y are in the antiblocker of $\mathcal{A}$; $\mu = 0$; so $\mathcal{A}$ is $\leq$-partitionable, as claimed. Now we conclude by noting that w is the unique solution of the equation $Xx = 1$; since X is r-uniform ($r \in \mathbb{N}$), $w = \frac{1}{r}\underline{1}$ follows; because of $w(A) \leq 1$ for all $A \in \mathcal{A}$, $r = \max\{|A| : A \in \mathcal{A}\}$).

Case 2: X is not uniform.

By Corollary 0.1 the connected components of the bipartite graph $G_{\text{core}(\mathcal{A},w)}$ are regular but our assumption implies that the whole graph is not regular. Hence by (0.2): for every $u,v \in V$ there exists $i \in \{1,\dots,n\}$ such that $A_i \supseteq \{u,v\}$. But then $|B_k|=1$ $(k=1,\dots,n)$ follows from Lemma 1.2, and therefore (iii) of Lemma 0.3 is satisfied. So (i) is also satisfied, contradicting our assumption. ∎

A graph $G=(V,E)$ is said to be *partitionable*, if it has $n=\alpha\omega+1$ vertices $(\alpha,\omega \in \mathbb{N})$, and for all $v \in V(G)$, $G-v$ can be partitioned both into α cliques and into ω stable-sets. Lovász [8] proved that minimal imperfect graphs are partitionable and Padberg [9] deduced further properties of minimal imperfect polyhedra. All of these properties have already been stated above, or obviously follow from the results we have proved (using lemmas and remarks of Section 0). The following theorem states the most important property, Lovász's coNP characterization of perfectness:

Corollary 1.1. *If $\mathcal{A}$ is a minimal imperfect clutter, then it is either a $\mathcal{H}_n^{n-1}$ $(n = 3,4,\dots)$ clutter, or the clutter of maximal cliques of a partitionable graph.*

Indeed, by Theorem 1 $\mathcal{A}$ is $\leq$-partitionable. If $r = n-1$, then clearly, $\mathcal{A}=\mathcal{H}_n^{n-1}$. If $r \leq n-2$, then since $\mathcal{A}$ is minimal imperfect, it does not contain $\mathcal{H}_k^{k-1}$ for any $k \in \mathbb{N}$ as a minor, so it is the set of maximal cliques of a graph G. From the fact that $\mathcal{A}$ is $\leq$-partitionable it is easy to see that G is a partitionable graph. ∎

2. Nonideal Clutters

Here we consider the case where $\mathcal{A}_{\leq}$ is empty and so P is minimal non ideal. To use Corollary 0.2 we need first:

Lemma 2.1. *The polyhedron P is full dimensional.*

Proof. Since $\underline{1}+e_v \in P$ $(v \in V)$, we have $r(P)=n$. ∎

By Lemma 0.4, core$(\mathcal{A},w)$ is a basis of $\mathbb{R}^n$, let us list its elements: A_1, $A_2,\dots,$ A_n; by Corollary 0.2 and the preceding lemma, w has n integer neighbors, let B_i be the (unique) neighbor of w which is not on the facet $\{x \in P : x(A_i)=1\}$ $(i=1,\dots,n)$.

Lemma 2.2. $|A_i \cap B_i| \geq 2$, $|A_i \cap B_j|=1$ $(i \neq j \in \{1,\dots,n\})$.

Proof. Since $B_i \in P_{\geq}(\mathcal{A})$, we have $|A_i \cap B_j| \geq 1$ $i,j \in \{1,2,\ldots,n\}$; $|A_i \cap B_j| = 1$ if and only if B_j is on the face A_i, that is, if and only if $i \neq j$, as claimed. ∎

The 'finish' is now a few lines longer, because of the exception of the degenerate projective planes.

Theorem 2. *If $\mathcal{A}$ is a minimal nonideal clutter on n vertices, then it is $\geq$-partitionable, or $\mathcal{A} = \mathcal{F}_n$.*

In the former case the unique fractional vertex of $P_{\geq}(\mathcal{A})$ is $\frac{1}{r}\underline{1}$, with $r = \min\{|A| : A \in \mathcal{A}\}$; in the latter case, $\mathcal{A}_{\geq} = \mathcal{F}_n$, $w = (\frac{1}{n-1}, \ldots, \frac{1}{n-1}, \frac{n-2}{n-1})$ (after possible permutation of the coordinates).

Proof. Let w be a fractional vertex of $P_{\geq}(\mathcal{A})$, and let X be the matrix whose rows are the (characteristic) vectors of the members of core$(\mathcal{A}, w)$. By Lemma 0.4 X is an $n \times n$ matrix. By Corollary 0.2, w is a simplicial vertex, and clearly, its neighbors on $P_{\geq}(\mathcal{A})$ are in the blocker of $\mathcal{A}$. Let Y be the $n \times n$ matrix whose columns are these neighbors, such that the i-th column of Y is the associate of the i-th row of X, that is, these rows and columns are the A_i's and Y_i's of Lemma 1.2.

Case 1: X is uniform.

The condition of Lemma 0.3 is true, and (i) is satisfied, so (iv) is also satisfied, that is, (X, Y) is partitionable.

By definition, the rows of X are in $\mathcal{A}$ and the columns of Y are in the blocker of $\mathcal{A}$; by Lemma 1.2 $\mu \geq 2$; so $\mathcal{A}$ is $\geq$-partitionable, as claimed. Now we conclude by noting that w is the unique solution of the equation $Xx = 1$; since X is r-uniform ($r \in \mathbb{N}$), $w = \frac{1}{r}\underline{1}$ follows; because of $w(A) \geq 1$ for all $A \in \mathcal{A}$, $r = \min\{|A| : A \in \mathcal{A}\}$.

Case 2: X is not uniform.

Then the bipartite graph $G_{\text{core}(\mathcal{A},w)}$ is not regular but by Corollary 0.1 its connected components are regular. Hence:

If no element of core$(\mathcal{A}, w)$ has cardinality $n-1$ then, by (0.2), for every $u, v \in V$ there exists $i, j \in \{1, \ldots, n\}$ such that $A_i, A_j \supseteq \{u, v\}$. Each B_k ($k = 1, \ldots, n$) meets at least one of these in at most one element, so $|B_i| = 1$ follows, contradicting Lemma 2.2 ($|A_i \cap B_i| \geq 2$).

Otherwise we can suppose $A_n := \{1, 2, \ldots, n-1\}$. Since for each $i \in \{1, \ldots, n-1\}$, $|B_i \cap A_n| = 1$, and $|B_i \cap A_i| \geq 2$, we have that $n \in B_i$, $n \in A_i$. Since the B_i are all distinct, one can set $B_i = \{n, i\}$ ($i = 1, \ldots, n-1$), and now $\{A_1, \ldots, A_n\} = \mathcal{F}_n$ follows by Lemma 2.2.

Since $\mathcal{A}$ is a clutter, $\mathcal{A} = \mathcal{F}_n$ follows. ∎

Note that the proofs of Theorem 1 and 2 can be shortcut by using the symmetry between vertices and facets (X and Y). However, we avoided using this in view of Section 3 where this symmetry is lost.

3. The mixed case

We are now in the case where both $\mathcal{A}_{\leq}$ and $\mathcal{A}_{\geq}$ are non empty.
In this case the fact that P is full dimensional requires a different proof; we also add a property that does not occur in the two previous cases:

Lemma 3.1. *The polyhedron P is full dimensional. If w is a fractional vertex, then $\text{core}(\mathcal{A},w)\subseteq\mathcal{A}_{\leq}$ or $\text{core}(\mathcal{A},w)\subseteq\mathcal{A}_{\geq}$.*

Proof. Let w be a fractional vertex of P. Since P is critical, w^i is a nontrivial convex combination of points of P^i ($i=1,\dots,n$), that is, $w^i=\sum_{s\in S}\lambda(s)s^i$, where $S=S(w,i)\subseteq P$ is a finite subset of P, $\lambda(s)\in\mathbb{R}^+$ ($s\in S$), and $\sum_{s\in S}\lambda(s)=1$. Define $w[i]:=\sum_{s\in S}\lambda(s)s$. Clearly, $w[i]-w=te_i$, for some $t=t(i)\in\mathbb{R}$.

If $w[i]=w$, then w is a nontrivial convex combination of points of P, contradicting the fact that it is a vertex. So $t\neq 0$. Therefore $e_i=(w[i]-w)/t$ is in the linear space generated by P for all $i=1,\dots,n$, proving that P is full dimensional.

It also follows now that $\text{core}(\mathcal{A},w)\subseteq\mathcal{A}_{\leq}$ or $\text{core}(\mathcal{A},w)\subseteq\mathcal{A}_{\geq}$, that is, $w(A_1)=1=w(A_2)$ cannot hold for $A_1\in\mathcal{A}_{\leq}$ and $A_2\in\mathcal{A}_{\geq}$. Indeed, the existence of $i\in A_1\cap A_2$ would contradict $w+te_i=w[i]\in P$; so $A_1\cap A_2=\emptyset$. But then V can be partitioned into nonempty sets I and J so that the members of $\text{core}(\mathcal{A}_{\leq},w)$ are subsets of I, and the members of $\text{core}(\mathcal{A}_{\geq},w)$ are subsets of J. In particular among the n inequalities that w satisfies with equality there are at most $|I|$ in $\text{core}(\mathcal{A}_{\leq},w)$, and at most $|J|$ in $\text{core}(\mathcal{A}_{\geq},w)$ and the equality follows throughout.

It follows that w^I is a vertex of P^I, w^J is a vertex of P^J, and at least one of them is fractional, contradicting the critical property of P. ∎

By Lemma 0.4 $\text{core}(\mathcal{A},w)$ is a basis of $\mathbb{R}^n$, let us list its elements: A_1, $A_2,\dots,$ A_n; by Corollary 0.2 and the preceding lemma, w has n integer neighbors, let B_i be the (unique) neighbor of w which is not on the facet $\{x\in P: x(A_i)=1\}$ ($i=1,\dots,n$).

Lemma 3.2. *Either $|A_i\cap B_i|=0$ for all ($i\in\{1,\dots,n\}$), or $|A_i\cap B_i|\geq 2$ for all ($i\in\{1,\dots,n\}$); $|A_i\cap B_j|=1$ ($i\neq j\in\{1,\dots,n\}$).*

Proof. By Lemma 3.1, $\{A_1,\dots,A_n\} \subseteq \mathcal{A}_\leq$, or $\{A_1,\dots,A_n\} \subseteq \mathcal{A}_\geq$, and the assertion follows, since $|A_i \cap B_j| = 1$ if and only if B_j is on the face A_i, that is, if and only if $i \neq j$. ∎

Denote $r_\leq := \max\{|A| : A \in \mathcal{A}_\leq\}$, $r_\geq := \min\{|A| : A \in \mathcal{A}_\geq\}$.

Theorem 3. *Let $P := P_\leq(\mathcal{A}_\leq) \cap P_\geq(\mathcal{A}_\geq) \subseteq \mathbb{R}^n$ be a critical polyhedron.*

Then either $\mathcal{A}_\leq = \emptyset$, $\mathcal{A}_\geq = \mathcal{F}_n$ and $w = (\frac{1}{n-1},\dots,\frac{1}{n-1},\frac{n-2}{n-1})$ is a vertex of P (after possible permutation of the coordinates), or one or both of the following two statements hold:

- *$\mathcal{A}_\leq$ is $\leq$-partitionable, $r_\leq \leq r_\geq$, and $w = \frac{1}{r_\leq}\underline{1}$ is a vertex of P.*
- *$\mathcal{A}_\geq$ is $\geq$-partitionable, $r_\leq \leq r_\geq$, and $w = \frac{1}{r_\geq}\underline{1}$ is a vertex of P.*

These are the only possible fractional vertices of P.

Proof. Let w be a fractional vertex of P, and let the rows of the matrix X be the (characteristic) vectors of the members of $\mathrm{core}(\mathcal{A}, w)$. By Lemma 0.4 X is an $n \times n$ matrix. By Corollary 0.2, w is a simplicial vertex, and clearly, its neighbors on P are $0-1$ vectors. Let Y be the $n \times n$ matrix whose columns are these neighbors, such that the i-th column of Y is the associate of the i-th row of X, that is, these rows and columns are the A_i's and B_i's of Lemma 3.2.

According to the first statement of Lemma 3.2 *the diagonal elements of XY are either all* 0 *or all bigger than* 1.

Case 1: X is uniform.

Then the conditions and (i) of the Commutativity Lemma 0.3 hold, so (iv) is also satisfied.

Clearly, in case $\mu = 0$, the rows of X are in $\mathcal{A}_\leq$, the columns of Y are in the antiblocker of $\mathcal{A}_\leq$, so $\mathcal{A}_\leq$ is $\leq$-partitionable; in case $\mu \geq 2$, the rows of X are in $\mathcal{A}_\geq$, the columns of Y are in the blocker of $\mathcal{A}_\geq$, so $\mathcal{A}_\geq$ is $\geq$-partitionable.

Now we conclude by noting that w is the unique solution of the equation $Xx = 1$; since X is r-uniform ($r \in \mathbb{N}$), $w = \frac{1}{r}\underline{1}$ follows; because of $w(A) \leq 1$ for all $A \in \mathcal{A}_\leq$, and $w(A) \geq 1$ for all $A \in \mathcal{A}_\geq$, we have $r_\leq \leq r \leq r_\geq$. Since $Xw = 1$, if the rows of X are in $\mathcal{A}_\leq$, then $r_\leq = r$, if they are in $\mathcal{A}_\geq$, then $r = r_\geq$.

Case 2: X is not uniform.

Then Case 2 of the proofs of Theorem 1 or Theorem 2 can also be applied without change (with the trivial exception that $\mathcal{A}$ has to be replaced by $\mathcal{A}_\leq$, respectively by $\mathcal{A}_\geq$, and the references to Lemma 1.2, respectively Lemma 2.2 have to be changed to Lemma 3.2 respectively). ∎

In particular, a critical polyhedron of the form $P_{\le}(\mathcal{A}_{\le})\cap P_{\ge}(\mathcal{A}_{\ge})$ has at most two fractional vertices, a vertex of $P_{\le}(\mathcal{A}_{\le})$, a vertex of $P_{\ge}(\mathcal{A}_{\ge})$, and both can occur:

Example 3. Let $V:=\{1,2,3,4,5\}$, $\mathcal{A}_{\le}:=\{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,1\}\}$, $\mathcal{A}_{\ge}:=\{\{1,2,3\},\{2,3,4\}\{3,4,5\}\{4,5,1\}\{5,1,2\}\}$. The polyhedron $P_{\le}(\mathcal{A}_{\le})\cap P_{\ge}(\mathcal{A}_{\ge})$ has two fractional vertices, $(\frac{1}{2},\dots,\frac{1}{2})$ and $(\frac{1}{3},\dots,\frac{1}{3})$.

This polyhedron, like Example 1, also has projections that are not $0-1$-constrained. But unlike Example 1, this is a critical polyhedron.

Theorem 3 may appear a bit disappointing since it leads to fractional vertices similar to those of the two previous sections. Nevertheless Example 3 shows that it can happen that the members of the '$\le -$core' can intersect those of the '$\ge -$core', and integer vertices can be contained in both types of facets. In fact a similar approach [14] with a refined definition of minors allows to include 'mixed odd circuits' as well (see Example 1): in such 'mixed odd circuit polyhedra' a fractional vertex can be contained at the same time in blocking- and antiblocking-types of facets, and 'mixed odd circuits' turn out to be the only 'minimal noninteger' polyhedra having such 'mixed' vertices.

A noninteger polyhedron all of whose minors have only $0-1$ constraints has a critical minor, whence Theorem 3 can be applied. Therefore we get a characterization of the nonintegrality of such polyhedra. (This class includes both blocking and antiblocking. It is straightforward to see that a polyhedron $P_{\le}(\mathcal{A}_{\le})\cap P_{\ge}(\mathcal{A}_{\ge})$ together with all its minors is $0-1$-constrained if $A_1\in\mathcal{A}_{\le}$ and $A_2\in\mathcal{A}_{\ge}$ are either disjoint, or $A_1\subseteq A_2$.)

The definition of partitionable clutters contains all properties of minimally imperfect or minimally nonideal clutters that are usually stated, in particular, it is easy to see that partitionability certifies nonintegrality. Properties proved in statements of 'Padberg's theorem' or 'Lehman's theorem' are straightforward consequences of Theorem 1, Lemma 1.2 or Theorem 2, Lemma 2.2. We mention here only one more property that could be relevant from the viewpoint of clarity:

The definition of a partitionable clutter $\mathcal{A}$ with parameters r, s, μ does not include, and it is not necessary for it to include that r and s are the maxima (if $\mu=0$), or minima (if $\mu\ge 2$) of the cardinalities of the sets in $\mathcal{A}$.

Indeed, this statement *immediately follows* from the definition! This implication is important for certifying imperfectness or nonidealness. It is easy to prove, let us for instance check it for $\mu\ge 2$, $|A|\ge r$ for all $A\in\mathcal{A}$: $|A|=\frac{1}{s}\sum_{i=1}^{n}|A\cap B_i|\ge\lceil\frac{n}{s}\rceil=r$.

We finally wish to point at the relation of our proofs to previous work:

The main frame of the proofs is similar to the proof of Lehman's theorem in Seymour's 'reading' [15]. The key idea of our proofs that led to the common generalization is the following: instead of basing on the antiblocking or blocking relation, we work with the hypergraph $\mathcal{C}_v - v$, whose hyperedges do not necessarily correspond to vertices of P (and might not be in P at all). We hope that this is not too confusing: in order to apply it the role of vertices and facets had to be interchanged comparing to the usual treatment of the perfect and ideal special cases. The role of facets and vertices is no more symmetric. The subfamily of $\mathcal{C}_v - v$ that occurs in the proofs is in fact a set of points on the minimal face containing w^v in P^v, and an important fact simplifying the proof of Lehman's theorem and opening the way to the generalization is that *these points do not necessarily have to be vertices in* P. We mention another small problem occurring only in the general case: the full dimensionality of P, crucial for the solution. In the special cases it is just trivial, whereas in the general case it is somewhat subtle (see Corollary 3.2).

Note that there are several variants of presenting a common proof of Theorem 1, Theorem 2 and Theorem 3. Among the possible proofs our goal was to include the 'most common to all the three' simple one.

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Grigor Gasparyan

Yerevan State University,
Yerevan-49,
Armenia
grigor@mondes.com

Myriam Preissmann

CNRS,
Département de Mathématiques Discrètes,
Laboratoire LEIBNIZ-IMAG,
46 avenue Félix Viallet,
38000 Grenoble Cedex 1,
France
Myriam.Preissmann@imag.fr

András Sebő

CNRS,
Département de Mathématiques Discrètes,
Laboratoire LEIBNIZ-IMAG,
46 avenue Félix Viallet,
38000 Grenoble Cedex 1,
France
Andras.Sebo@imag.fr