Matching and Routing: Structures and Algorithms

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Abstract

We will see how to find various kinds of paths, or maximum sets of disjoint paths in a graph. The algorithmic method we will use will be that of “canonical structures”, which consists in finding several primal solutions to a problem in order to either find 'the canonical dual solution' or a better primal solution. The maximum matching problem and its generalizations will be treated as particular cases of the search for certain types of paths; some particular kinds of multiflow problems (planar problems, Mader's path packings) will also be touched.

The course will be mainly based on exercises or “group discovery”.

1 Explanations

The course will be self-contained. However, if the audience is familiarized with the basic definitions and some fundamental simple facts, that helps:

Ford and Fulkerson's algorithm for max flow; Menger's theorems, also their vertex-versions, and variants; alternating paths (Hungarian method) for maximum matchings in bipartite graphs; the (2-connected) block structure of graphs; Eulerian graphs; definitions and basic facts about matroids. If this knowledge is missing to anybody, these can be recalled from any introductory book of graph theory or combinatorial optimization - or some selection of chapters 4, 5, 6 of Lovász's ‘Combinatorial Problems and Exercises' ; for matroids, see some basics in Recski's or Oxley's book.

We will not use any knowledge about non-bipartite matchings or generalizations (studied with different methods in [2]), but include an approach to these. However, if the audience has already heard about the classical approaches that will help her (his) intuition. (You could revise Edmonds' algorithm and Tutte's theorem.)

It is not necessary to read the documents of the bibliography below before the course: they contain some relevant explanations about the principles, but nothing yet is available about the precise subject of the course. We included some first exercises below in order to customize the style and for a warm-up. (I wrote the bibliography merely for introductive explanations and guidance.)

The course has three main objectives:

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– To provide a key to matchings and their generalizations, or problems reducible to matchings. The goal is not to learn all about them but to get a good feeling of some facts and algorithms that apply to them and of other derived problems.

– To get acquainted with the idea of 'structural algorithms'. A first, guiding example is the one in Section 9.3 of Lovasz and Plummer's book [1]. Don't read the details just the philosophy. Or the introductory explanations of [3], (for Hungarian readers better to see the introduction of Chapter 5 of my Candidates' thesis) *without details*. For the details we will use more recent material and just enough to learn a way of constructing algorithms from minimax theorems in new situations.

– To get a training and fun in solving exercises and problems. To apply the results and method for reaching some more recent results (jump systems, paths in planar graphs), and acquiring ability in using structures for algorithms; to enlarge the toolbox with this technique.

Thanks are due to Bernhard Fuchs (Köln), Dániel Marx (Budapest) and Peter Malkin (Louvain) for pointing out a flaw in the statement of Exercise 3 of the previous version.

2 Exercises

We will discuss solutions of exercises whenever they are needed in the course of the course. I will try to state the exercise latest the day before the use ... The day before the first day is starting today !

Even if there is not enough time to solve all exercises, having a look and little thinking about them in advance may make it easier to discover a solution. The exercises I state all contain an idea that may be useful to know at least for the course.

Let G be an undirected graph; V(G) denotes the set of its vertices, E(G) is its edge-set; $\nu(G)$ denotes the matching number of G, that is, the maximum size of a set of disjoint edges in G; $\tau(G)$ is the minimum vertex cover of G, that is, a set of vertices of minimum cardinality that meet every edge of G.

If $F \subseteq E(G)$, the vector $\{d_F(v) : v \in V(G)\} \in \mathbb{N}^{V(G)}$ is called a degree sequence.

——**The Berge-Tutte theorem and Edmonds' algorithm**——

Exercise 1 Let G be a bipartite graph, and $uv \in E(G)$. Then either $\nu(G-u) < \nu(G)$, or $\nu(G-v) < \nu(G)$. Deduce from this a simple inductive proof of König's theorem $\nu(G) = \tau(G)$ for every graph G.

Exercise 2 Let G be a graph, and $uv \in E(G)$. Then either $\nu(G-u) < \nu(G)$, or $\nu(G-v) < \nu(G)$, or for any maximum matching M_u of $G-u$, and M_v of $G-v$: $M_u \cup M_v$ contains an (u,v)-path P alternating between M_u and M_v.

If G is a graph, and $X \subseteq E(G)$, then G/X denotes the graph we get from G by identifying the endpoints of the edges in X.

Exercise 3 Let G be a graph, and M a maximum matching in G, moreover $uv \in E(G)$, $\nu(G-u) = \nu(G)$, $\nu(G-v) = \nu(G)$. Then for the alternating path P of the previous exercise $\nu(G/P) = \nu(G) - k$, where $2k$ is the number of edges of P.

Exercise 4 Deduce by induction, using exercises 2 and 3 the theorem of Berge-Tutte: the minimum number of vertices not covered by a matching is equal to the maximum of $q(X) - |X|$, where $q(X)$ denotes the number of odd components of $G - X$. If you know Edmonds' algorithm deduce also a proof of its correctness.

If for a set X the value of $q(X) - |X|$ is maximum, then it is called a Tutte-set.

Exercise 5 If $v \in V(G)$ is contained in some Tutte-set then it is covered by every maximum matching of G

——Alternating Paths——

Exercise 6 Recall one of the easiest graph theory results you know: Given a directed graph $D = (V, A)$ and $x_0, x \in V$, either there exists a directed (x_0, x)-path in D or a certificate $C \subseteq V$, $x_0 \in C$, $x \notin C$ so that no arc has its tail in C and head outside C. *Observe (and prove) that there exists a unique certificate $C_0 \subseteq V$ suitable for all those $x \in C$ for which there exists no directed (x_0, x)-path.*

If some of the edges of G are colored (say red and the other edges are blue), an *alternating walk* is defined to be a walk that does not contain a repetition of edges, and any two consecutive edges are of different color. Recall Berge's statement (and prove it): a matching M is not maximum if and only if painting the edges of M red there exists no alternating path between two points that are not covered by M.

Alternating walks contain not only augmenting paths for matchings or their generalizations, but include also undirected paths in graphs (replace every edge with two parallel edges, one blue and one red), and directed paths as well. To show the latter, subdivide a directed edge into two edges in series (that is a path of length two by putting a new vertex on it), and painting the part incident to the 'arrow vertex' in red, and the one incident to the tail vertex in blue.

An alternating walk is red-blue, red-red, blue-red, blue-blue if its first and last edge (one of the two possible orders is fixed) is red, blue, etc respectively.

Exercise 7 (generalizing transitivity of access by directed paths) Let G be any graph whose edges are painted red-blue. If there exists a red-blue alternating walk between a and b, a red-blue alternating walk between b and c, then

- either there exists a red-blue alternating walk between a and c,
- or there exist two alternating walks, a red-red one between a and b and a blue-blue one between b and c.

Show an example where the second alternative holds.

Let M be a maximum matching of G. Paint the edges of M red, and add a vertex x_0 to G joined to all other vertices of G. Paint these additional edges red exactly if the other endpoint is not covered by M (otherwise let them be blue). In the following exercises the starting point of paths is assumed to be x_0 unless said otherwise.

The following two exercises will turn out to generalize Exercise 6.

Exercise 8 Let G be a graph and $A \subseteq V(G)$ a Tutte-set. Paint the edges of G as indicated above. Then (from x_0)

- to $x \in A$ there is no red-red alternating path
- to $x \in C$ where C is an even component of $G - A$ there is neither a red-red, nor a red-blue alternating path.

Exercise 9 Let $C(G) := \{x \in V(G) :$ there is neither red-blue nor red-red alternating path to $x\}$. $A(G) := \{x \in V(G) :$ there is no red-red alternating path to $x\} \setminus C(G)$. Show that $A(G)$ and $C(G)$ do not depend on the choice of the maximum matching M.

An equivalent form of the 'Gallai-Edmonds structure theorem' that we will use and prove algorithmically is the following:

$A(G)$ is a Tutte-set and $C(G)$ is the union of the even components of $G - A(G)$.

We will prove this for some generalizations of matchings and deduce algorithmic consequences for b-matchings. According to Exercise 8, $(A(G), C(G))$ is the most exclusive among the Tutte-set–even-component pairs. Let us make this more precise and provide a feeling through an (easy) exercise.

Given a graph define an order between Tutte-sets: $A_1 >_G A_2$ iff $C_1 \supseteq C_2$ and $A_1 \cup C_1 \supseteq A_2 \cup C_2$, where C_i is the union of the odd components of $G - A_i$ $(i = 1, 2)$.

Exercise 10 Show that $>_G$ is an order. Using the above formulation of the Gallai-Edmonds structure theorem show that $A(G)$ is the unique maximal Tutte-set with respect to the order $>_G$.

I add some exercises about two slightly different subjects, hoping that they can provide assistance for the first steps in these subjects for those who have a bit of time. (Studying these may facilitate the intuition about the extensions (that generalize matchings, orientations and other objects) we will study at the course.)

—**Degree Sequences**—

The first and last edges of walks will be called *terminal edges*.

Exercise 11 Let F and F' be subsets of edges of G. Paint the edges of $F \setminus F'$ red and those in $F' \setminus F$ blue (the rest black). Then $(F \setminus F') \cup (F' \setminus F)$ is the vertex-disjoint union of a set $\mathcal{W}$ of red-blue alternating walks so that for all $v \in V$ and walks $W \in \mathcal{W}$ that have an endpoint in v, all terminal edges of W incident to v have the same color.

Exercise 12 If b and d are degree sequences, then there exist $x_1, \dots, x_k \in \mathbb{Z}^{V(G)}$ $(k \in \mathbb{N})$ with sum of absolute values (l_1-norm) equal to 2 so that $d = b + x_1 + \dots x_k$ and adding the sum of any subset of $\{x_1, \dots, x_k\}$ to b we get a degree sequence again, moreover $|d - b| = |x_1| + \dots |x_k|$.

Hint: Let F and F' realize the degree sequences b and d. Observe how these change when the edges of an alternating path are switched (edges of F and F' interchanged), and use the result of the previous exercise.

—**Orientations**—

The following two exercises show the augmenting paths and minimax results related to a problem similar to matchings in spirit. They lead to a proof of a theorem of Frank and Gyárfás:

Suppose we are given a graph G and an integer $u(v)$ for every $v \in V(G)$. We are interested in an orientation where the number of outgoing edges is at most $u(v)$ for every $v \in V(G)$. Such an orientation will be called *feasible*.

Given an orientation of G, a vertex with outdegree at least $u(v)$ will be said to be *saturated*, one that has outdegree bigger than $u(v)$ will be said to be *over-saturated* (with respect to the orientation). The sum of $d_{\text{out}}(v) - u(v)$ on oversaturated vertices of G, where $d_{\text{out}}(v)$ is the outdegree of v, is called the *deficiency* of the orientation.

Exercise 13 Show that reversing the arcs on a path between an over-saturated and a not saturated vertex, decreases the deficiency of the orientation.

Exercise 14 Show that a necessary and sufficient condition for the existence of a feasible solution is that there exists no set X that induces more than $\sum_{x \in X} u(x)$ edges. *Hint:* Apply the previous exercise and Exercise 6.

Exercise 15 Derive the analogous result if there is a lower bound for every vertex. Can you generalize the result for the case when both upper and lower bounds are given on the vertices ? If there are both lower and upper bounds on the vertices derive that there exists an orientation respecting both in every vertex if and only if there exists one that respects the lower bound in every vertex and there also exists one respecting the upper bounds in every vertex.

3 Preliminary Research

Matchings and more general factorization problems (b-matchings, the Chinese Postman problem) come up often in applications.

It is not among the goals of the course to collect such applications, and I think everybody can believe for now that these exist. However, if some of you have time in the following two weeks to make a web-search and collect nice and simple applications, we could spend several times half an hour for this purpose.

References

[1] L. Lovász, Plummer, Matching Theory

[2] J.R. Edmonds, E. L. Johnson, *Matching: A Well-Solved Class of Integer Linear Programs*, republished in M. Jünger, G. Reinelt, G. Rinaldi, 'Combinatorial Optimization–Eureka, You Shrink!', Springer LNCS 2570, 2003.

[3] A. Sebő, *Finding the T-join structure of graphs*, Math. Programming **36** (1986), 123–134.