Another proof of optimality for greedy

(If you can do it simple, make it complicated and sketch !)

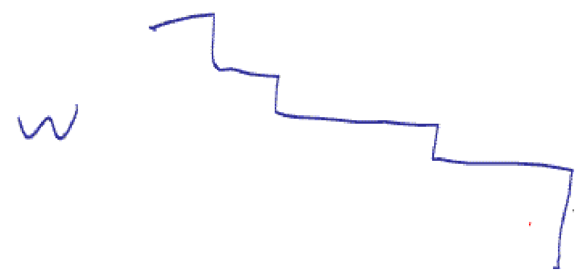
Thm (Edmonds): $M=(S,\mathcal{F})$ mat.

$$\{x : x(A) \le r(A),\ x \ge 0\} = \mathrm{conv}(\chi_F : F \in \mathcal{F})$$

$$x(A) = \sum_{a \in A} x_a$$

Proof: $w_1 \ge \dots \ge w_n$

$U_i = \{1, \dots, i\}$



Submodularity => Sets A with positive dual variables form a chain !

The F that we find satisfies: $|F \cap U_i| = r(U_i)$

$$w(F) = (w_1 - w_2)|F \cap U_1| + (w_2 - w_3)|F \cap U_2| + \dots + w_n |F \cap U_n|$$

dual solution

The inverse of the duality theorem

Theorem (Edmonds) : M= (S, $\mathcal{F}$) matroid. Then
conv (χ_F : F$\in\mathcal{F}_i$) = { x $\in$ IR^S : x (A) $\leq$ r (A) for all A $\subseteq$ S , x $\geq$ 0 }

Proof : $\subseteq$: Clear !

For = show $\forall$ w $\in$ IR^S max w^Tx for x on the left =
max w^Tx for x on the right

This suffices , since if not =, then $\subset$ and the hyperplane $c^Tx=b$ separating **some** x on the right from **all** on the left , shows that the max of c^Tx is larger on the right (choosing the sign of c appropriately).

But max of c^Tx on the right is equal, by the duality theorem to the min of its dual so the latter is larger then the max of c^Tx on the left, contradicting Edmonds' minimax theorem (previous transparency).

Matroid Intersection

Edmonds (1979)

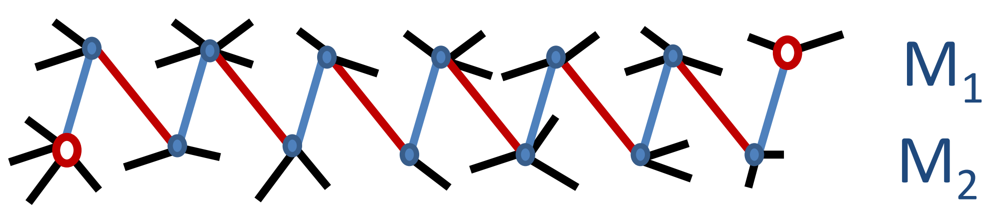
Let M_1 and M_2 be two matroids,
(S,r_1) and (S,r_2)
$(S,\mathcal{F}_1)$ and $(S,\mathcal{F}_2)$
$c: S \rightarrow \mathbb{R}_+$

maximize $\{ c(F) : F \in \mathcal{F}_1 \cap \mathcal{F}_2 \}$

Two examples :

2 disjoint spanning trees : M_1 and $M_2 := M_1^*$, $c = 1$ everywhere;
actually arbitrary number of disjoint spanning trees (network design)

Bipartite matchings :



Both M_1, M_2 are partition matroids: sums of uniform matroids on stars

Matroid Intersection Theorem

How to conjecture a « good characterization » ?

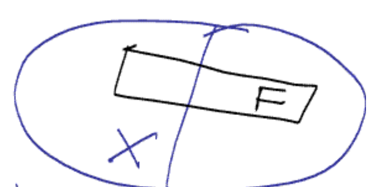
We know : $x \in \text{conv}\,(\chi_F : F \in \mathcal{F}_i) \Leftrightarrow x(A) \le r_i(A)$ for all $A \subseteq S$

maximize $\{\, |F| : F \in \mathcal{F}_1 \cap \mathcal{F}_2 \,\} = ?$ conv $(\chi_F : F \in \mathcal{F}_1 \cap \mathcal{F}_2) = ?$

$\{x(A) \le r_i(A)\ \ (i=1, 2)$ for all $A \subseteq S\}$

Theorem (Edmonds 1979): $\max_{F \in \mathcal{F}_1 \cap \mathcal{F}_2} |F| = \min_{X \subseteq S} r_1(X) + r_2(S \setminus X)$

Proof: $\le$



$\in \mathcal{F}_1 \cap \mathcal{F}_2$

$F : |F| = |F \cap X| + |F \setminus X| \le$

$\le r_1(X) + r_2(S \setminus X)$

If $|F| = r_1(M)$ define X !

Matroid Intersection Theorem

Generalization of bipartite matching
(of the alternating paths in the « Hungarian method »)

Proof of ≥ : that is, there is F and X with $|F| = r_1(X) + r_2(S \setminus X)$.

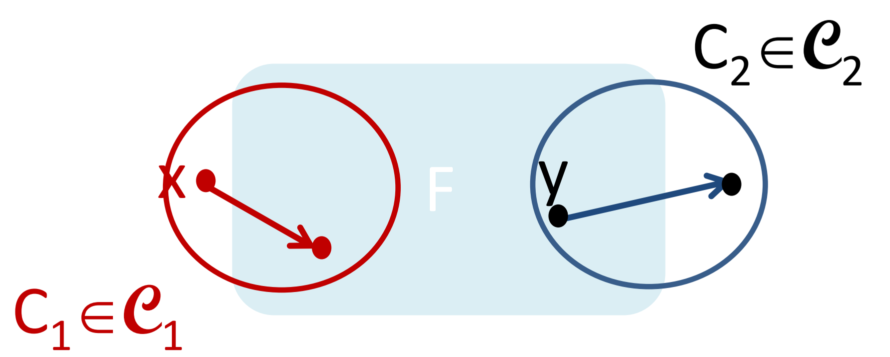
We prove that the following algorithm terminates with such an F and X.

Intersection algorithm

What is the INPUT ? S and → ORACLE - rank, independence, etc

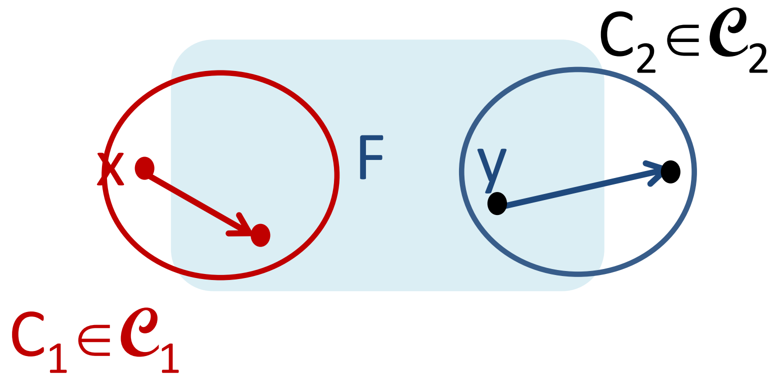
0.) Let : $F \in \mathcal{F}_1 \cap \mathcal{F}_2$ maximal by inclusion (greedily)

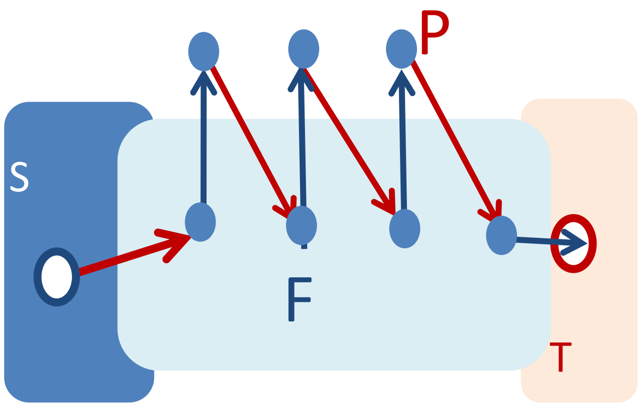
1.) Define arcs from unique cycles
Between S\F and S :



Matroid Intersection Theorem

Algorithmic proof



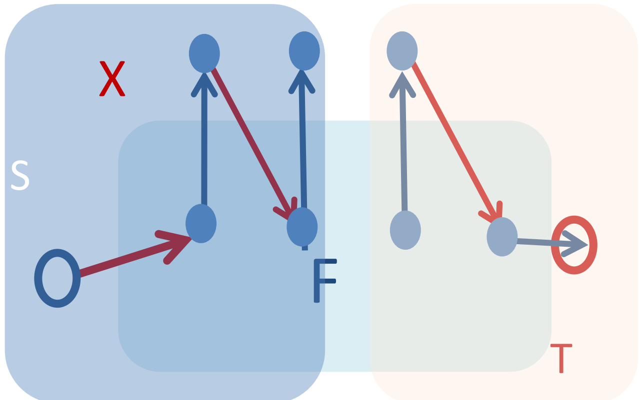


3.) Sources $S := \{x \in S \setminus F,\ F \cup \{x\} \in \mathcal{F}_2\}$ Sinks $T := \{x \in S \setminus F,\ F \cup \{x\} \in \mathcal{F}_1\}$

If S or T is empty ?

Find an (S,T)-path.

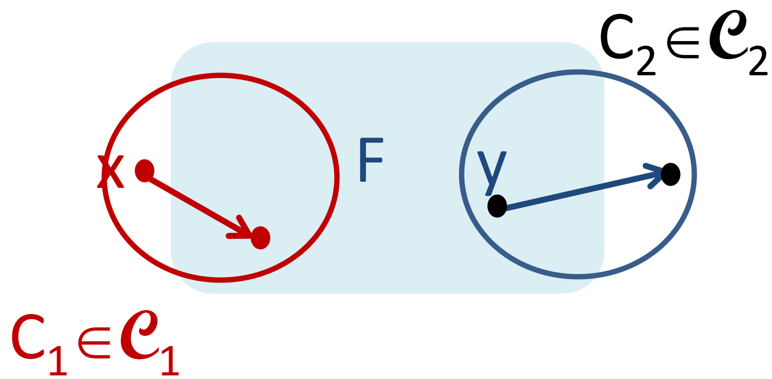
a.) If there exists one, let P be one with inclusionwise minimal vertex-set (equivalently, P is chordless).

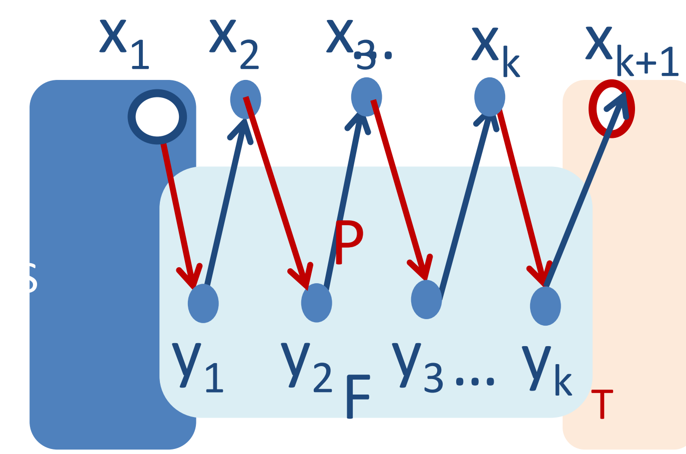


b.) If there exists none, $T \cap X = \emptyset$, where $X := \{x \in S : x \text{ is reachable from } S\}$

Matroid Intersection Theorem

exchange along an improving path





a.) If $P = \{x_1, y_1, x_2, \ldots x_k, y_k, x_{k+1}\}$ is a chordless path, then $F \Delta P \in \mathcal{F}_1 \cap \mathcal{F}_2$

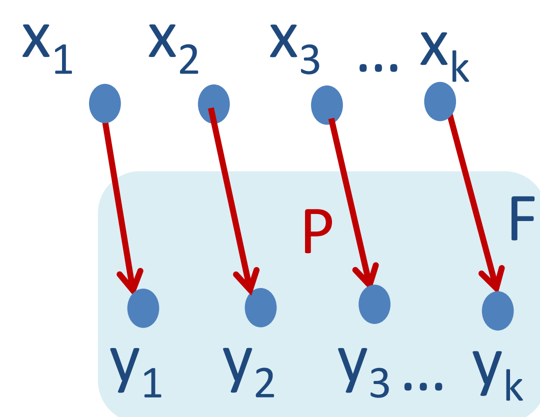
To prove this, apply the following to $F \cup \{x_1\} \in \mathcal{F}_2$, and $F \cup \{x_{k+1}\} \in \mathcal{F}_1$

Lemma : $M = (S, \mathcal{F})$ matroid, $F \in \mathcal{F}$, $x_1, \ldots, x_k \notin F$

If y_i is in the unique cycle of $F_i \cup x_i$,

but y_j, $j = i+1, \ldots k$ is not, then

$$(F \setminus \{ y_1, \ldots, y_k \}) \cup \{ x_1, \ldots, x_k \} \in \mathcal{F}$$



Proof: For k= 1 true, and then use it by induction to $(F \setminus \{y_k\}) \cup \{x_k\}$.

Matroid Intersection Theorem

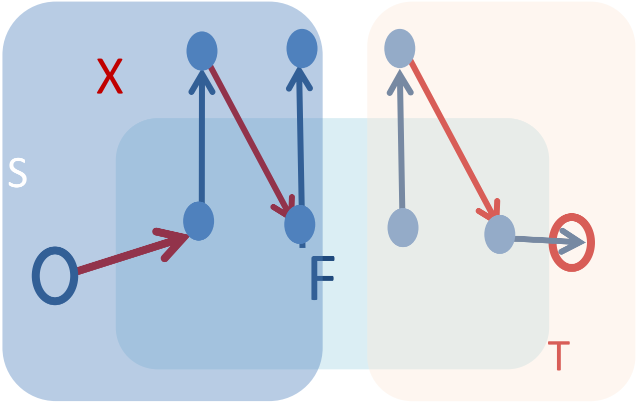
No improving path : show that the solution is optimal

Let X := {x ∈ S : x is reachable from S}

Lemma : Suppose b.) : X ∩ T = ∅ , where

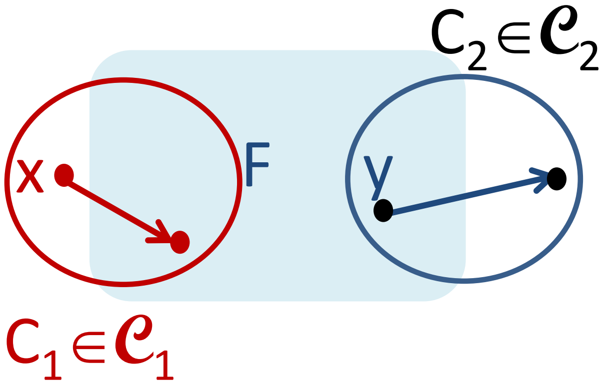
X := {x ∈ S : x is reachable from S}

Then |F| = r_1 (X) + r_2 (S \ X)



Proof : r_1 (X) = |F∩X|, because X ⊆ sp_1 (F∩X) .

r_2 (S \ X) = |F \ X |, because S \ X ⊆ sp_2(F \ X) .



Corollaries

Conversely these can be deduced with a similar algorithm and imply matroid intersection.

matchability to an independent set

Matroid union (partition)

Minimum number of independent sets covering every element

Maximum number of disjoint bases

Theorem (Nashwilliams) : In a graph there exist k disjoint spanning trees, if and only if for any partition $\mathcal{P}$ of the vertex-set there exist at least $|\mathcal{P}|$ - 1 edges with endpoints in different classes.

On the crossroad of the postman and the salesman

Polyhedra for the postman and the salesman

For the postman apply to $T:=T_G$:

Theorem Edmonds, Johnson (1973) : conv (T-joins) + IR_+^n =

$Q_+(G,T) := \{x \in IR_+^E \; x(\delta(W)) \geq 1,\ \delta(W)$ is a T-cut, i.e. $|W \cap T|$ is odd$\}$

Fractional relaxation of the TSP (subtour elimination « Held-Karp »):

$P(V,s,t)=\{x \in IR_+^E: x(\delta(W)) \geq 2,\ \emptyset \neq W \subset V,\ s,t \in W$ or $\notin$
1, if s,t separated by $W\}$

Integer points : Hamiltonian cycles

Objectifs : **Conjecture: OPT $\leq$ 4/3 LIN** **Conjecture (s,t) : OPT $\leq$ 3/2 LIN**

OPT := c-min Ham

Relaxation: LIN := min $\{ c^T x : x \in P(V,s,t) \} \rightarrow$ **x^***

Tours

A *tour* in G=(V,E) is a « spanning Eulerian subgraph of 2G »,
that is, H = (V, F) such that

- the elements of F are in E and with 1 or 2 parallel copies
- all degrees of H are even
- H is connected

min c-weight of a tour = OPT of TSP (min of metric HAM)

Tour = 'Graphical TSP tour ' of
Cornuéjols, Fonlupt, Naddef (1986)

= TSP

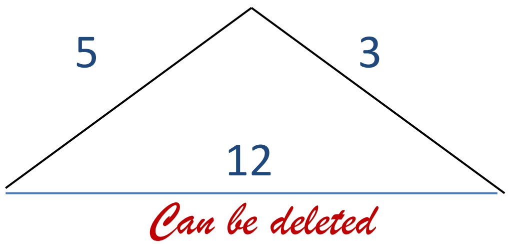
min cardinality of a tour = OPT of graph - TSP

Reformulations to tours

TSP
INPUT: G graphe, c: E(G) $\rightarrow$ IR_+
OUTPUT: c-min tour (dans 2G, degrés pairs, connexe)

TSP PATH
INPUT: G graph, s,t $\in$ V(G), c: E(G) $\rightarrow$ IR_+
OUTPUT: min (s,t) – tour (in 2G, s,t: odd, otherwise even, connected)

Advantages :
- No restriction on c , - no more necessarily a metric !
- Even degrees, relaxed comparing to 2
- equivalence with a less dense graph
- **has a cardinality case c ≡ 1**
- becomes graph theory with combinatorial methods



graph-TSP : minimum cardinality of a tour

The last results

cycle or path / cardinality or weights	cycle	(s,t)-path
cardinality	Sebő, Vygen SV12, Jan 2012 **1.4** 4	Gao: preuve simple, mars 2013 Sebő, Vygen SV12, Jan 2012 **1,5** 2
general	1 Christofides CHR, 1976 **1,5**	3 Sebő S12, Sept 2012 **1,6**

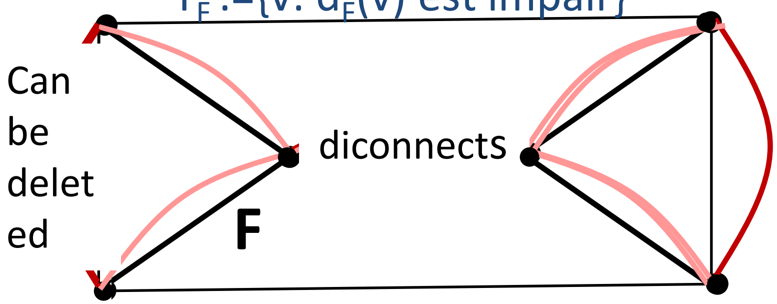
Clermont-Ferrand, 21 juin 2013

Christofides : connectivity & parity correction

Christofides Tour : c-min spannicng tree F + parity correction (pc)

T_F-joins, où

T_F :={v: d_F(v) est impair}



tour \ T_F-join is a T_F-join => pc ≤ 1/2

for (s,t)-tours 2/3

Trick : If $x \in Q_+(G,T)$, then c(modifying the parith on T) ≤ $c^T x$

Wolsey '80 : $x^* \in P(G,s=t)$, so $x^*/2 \in Q_+(G,T)$ $\forall$ T, apply to $T=T_F$

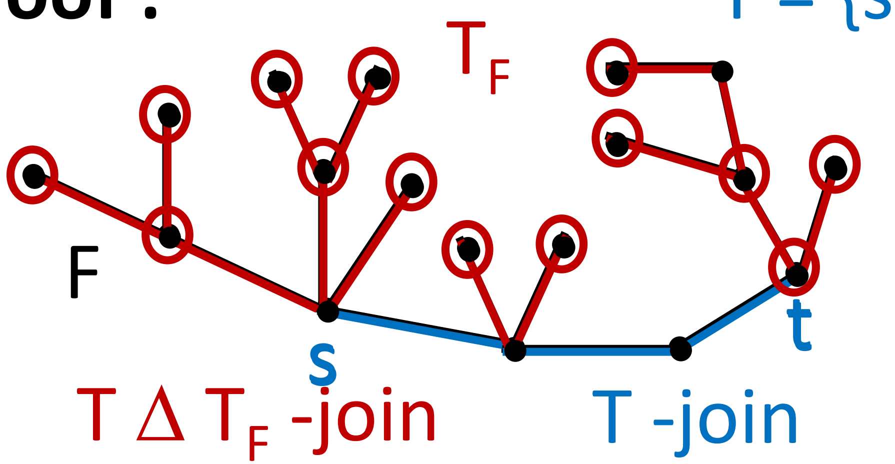
5/3 for T-tours: another proof

INPUT : G graph, $T \subseteq V(G)$, $c: E(G) \to IR_+$
OUTPUT: shortest T-tour

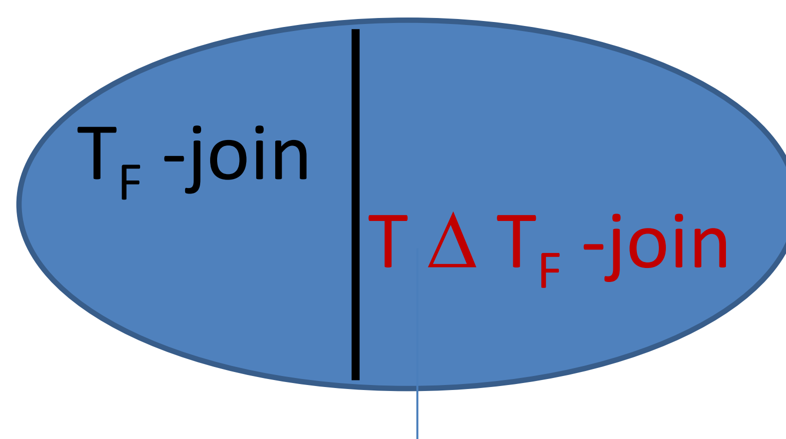
Theorem: (Hoogeveen 1991) : Christofides-type alg is 5/3-approx

'Christofides type' : c-min spanning tree F + parity correction

Proof : $T = \{s,t\}$



opt connected T-join



Graph-TSP paths = {s,t}-tours, cardinality

Theorem (SV12) : 3/2 approximation for graph-TSP paths

$$\text{OPT} \leq 3/2 \text{ LIN'}$$

Theorem (Gao, mars 2013) : $\text{OPT} \leq 3/2 \text{ LIN}$

Proof: AKS: $x^*/2$ can be << 1 on T-cuts, no more good for parity corr !

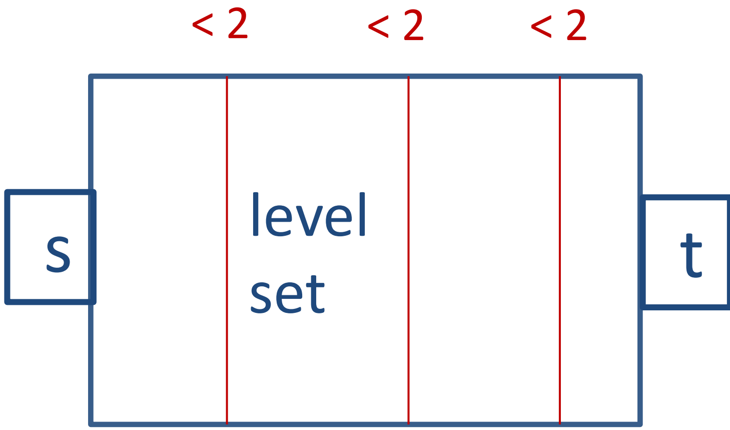
AKS: $\mathcal{Q} := \{ Q \text{ is a cut}, x^*(Q) < 2 \}$ *narrow {s,t}-cuts*

By submodularity, belongs to a chain of vertex-sets !

Gao : The «level-sets » of $\mathcal{Q}$ are connected :

$\exists$ spanning tree F st $|F \cap Q| = 1$ for all $Q \in \mathcal{Q}$

$x^*/2 \in Q_+(G,T_F)$: good for parity correction



{s,t}-tours arbitrary weights

Theorem (S 12) : 8/5 approximation for TSP paths

OPT $\leq$ 8/5 LIN

'**Classical**' part : « Random sampling » derived from x*, where
x is (≥) a conv combination of a pol number $\mathcal{F}$ of spanning trees*

Used by Gharan, Saberi, Singh ('12) for « random sampling » .

Not just matroid partition! Cunningham(1984), Barahona(1995), Gabow, Manu (1998)

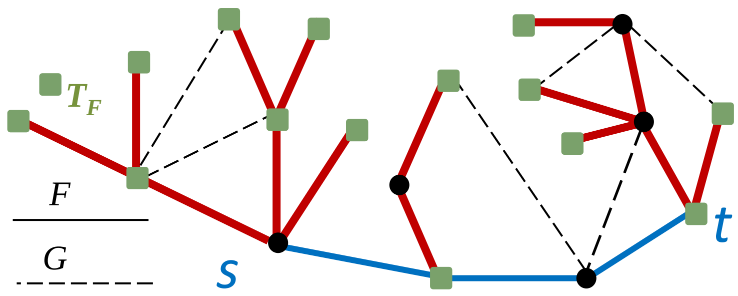
Best of Many (BOM) Algorithm: (AKS11) Output **$F + J_F$** ,where
$F \in \mathcal{F}$ minimizes $c(F)+ c(J_F)$, J_F is a (c-min) **$T_F \Delta$ {s,t}** –join in G

Complete x* / 2 with some correcting vector (AKS11)


World's Greatest
SALESMAN

{s,t}-tours arbitrary weights

New part (S 12) :



$p^* := E[\mathcal{F}(s,t)]$

$q^* := E[\mathcal{F} \setminus \mathcal{F}(s,t)]$

$x^* = p^* + q^* = E[\mathcal{F}]$

$(x^* + p^*)/2$ is in Q_+, i.e. dominates parity correction

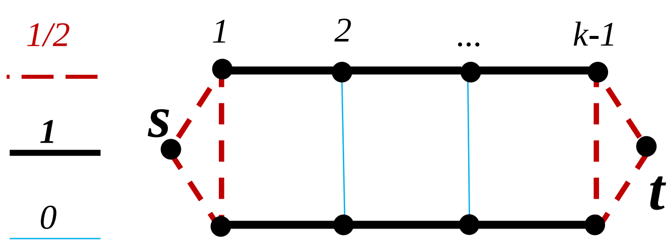
$\mathcal{F} \setminus \mathcal{F}(s,t)$ corrects the parity of $\mathcal{F}$

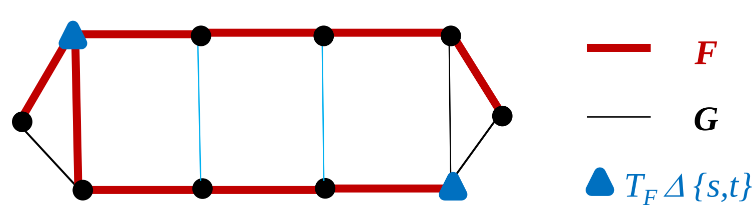
$c^Tx^*/2 + c^Tp^*/2 = X/2 + Y/2$

$c^Tq^* = c^Tx^* - c^Tp^* = X - Y$

E[c(parity correction)] ≤

≤ 2/3 c^Tx^* = 2/3X



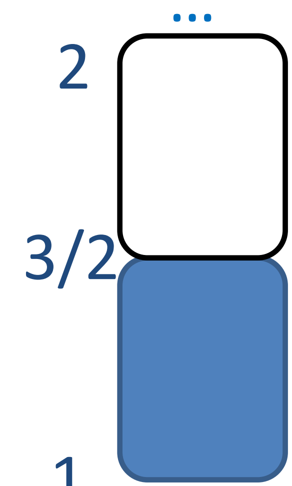


OPT=LIN

BOM=3/2 OPT

Key idea (for pc$\leq$ 3/5LIN) in the worst case

Suppose the worst : $x^*(Q) = 3/2$ for all $Q \in \mathcal{Q}$



$1 \leq x^*(Q) \leq 2$

The complement : $x^Q(e) :=$ Pr ($\{e\} = Q \cap \mathcal{F}$)

Add x^Q only when necessary.

Pr ($|Q \cap \mathcal{F}| =$ 1 and then we don't add x^Q) $\geq$ ½

Pr (we add x^Q) $\leq$ ½

Events '$Q \cap \mathcal{F} = \{ e \}$' :

- mutually exclusive for different Q ,
- $\subseteq$ '$e \in \mathcal{F}(s,t)$'

In expectation we add: $\frac{1}{2} \sum x^Q \leq \frac{1}{2} p^*$

Cost of the parity correction : $c^T \frac{1}{2} (x^* + p^* / 2)$

X - Y

X/2 +Y/4

$\leq$ 0.6X

Graph-TSP = min cardinality tours

Theorem : (SV12) ∃ T-tour of cardinality ≤ 3/2 LIN - π

Corollary 1 : (SV12) ∃ tour of cardinality ≤ 7/5 LIN

Proof : Applying an ingenious lemma of Mömke & Svensson :

∃ tour of cardinality ≤ 4/3 |V| + 2/3 π

Relaxation : 2-Edge-Connected Spanning Subgraph (2ECSS)

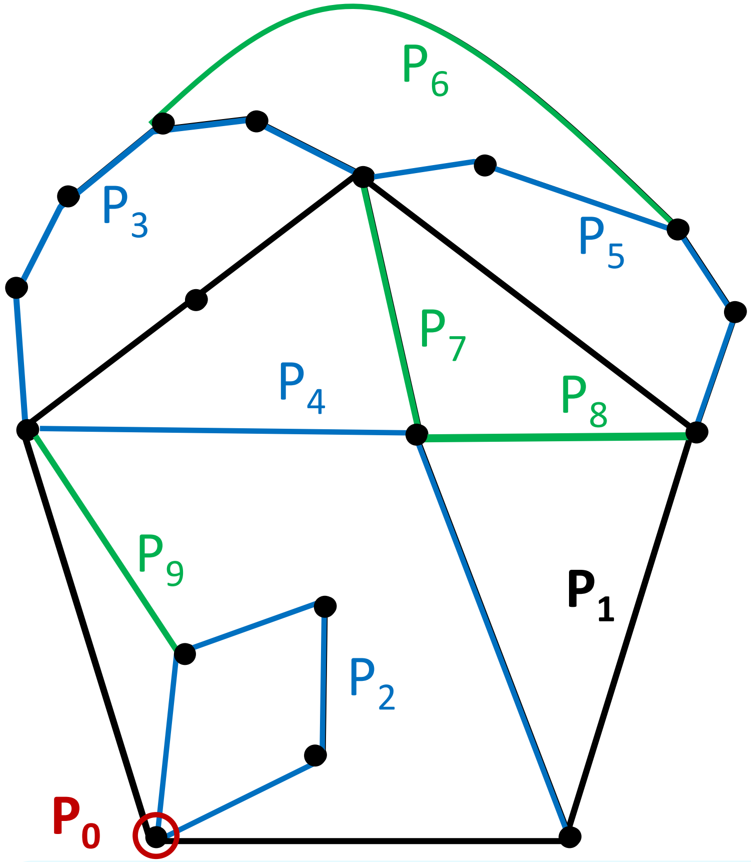
Corollary 2 : (SV12) ∃ 2ECSS of size ≤ 4/3 LIN

Proof : Simple recursion + result of A. Frank : 5/4 LIN + ½ π

The future ? : Boyd, Iwata, Takazawa ('11) for 3-EC cubic: **6/5**|V|

Ears

For understanding π + matroid idea +useful if you don't know :



$G = P_0 + P_1 + P_2 + \ldots + P_k$

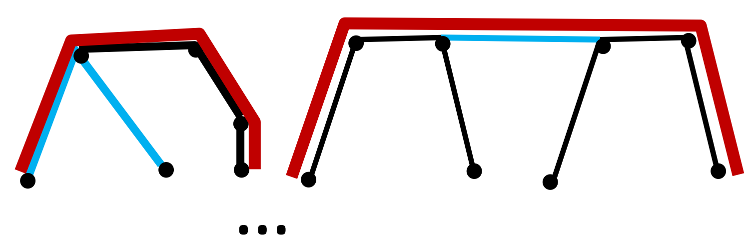
2-approx for 2ECSS: delete 1-ears!

You get : $\leq 5/4$ OPT + $1/2\pi_3$

The longer the ears, the smaller the quotient n. of edges / vertices

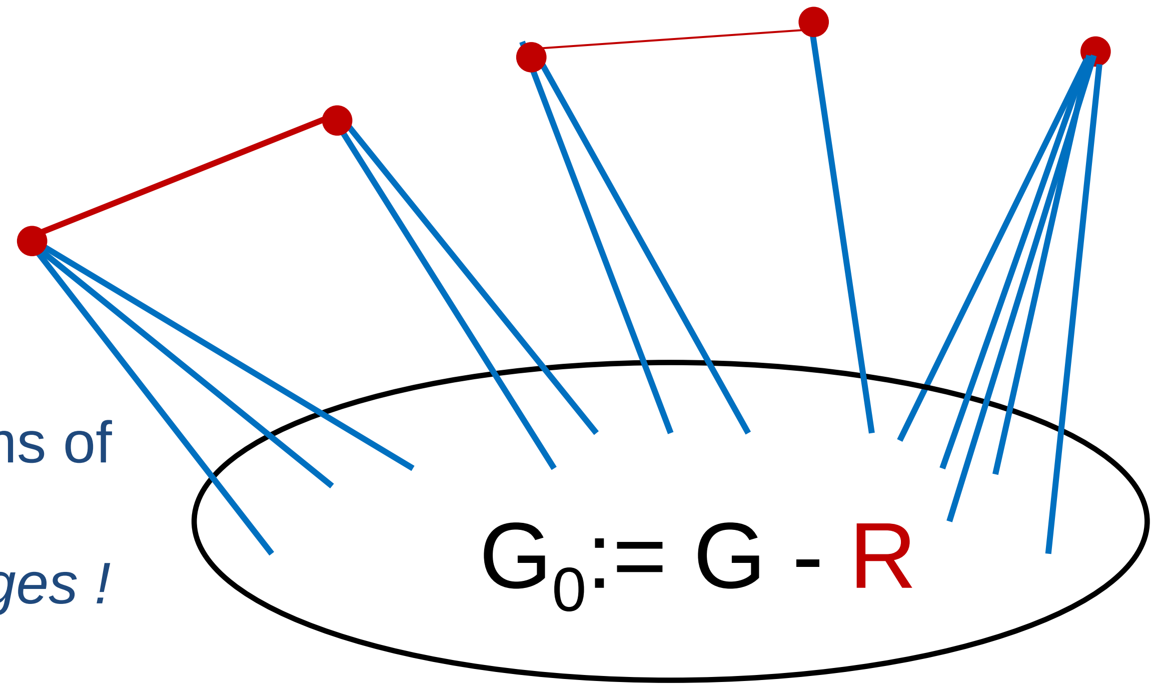
Theorem : (Whitney, Cheriyan, Sebő, Szigeti, Vygen, 1932-2012)
If G is 2-connected, then there exists a nice open ear-decomposition, i.e.

- 1-ears last, 2-ears, 3-ears « before the last »
- no edges between their inner vertices,
- min number of even ears

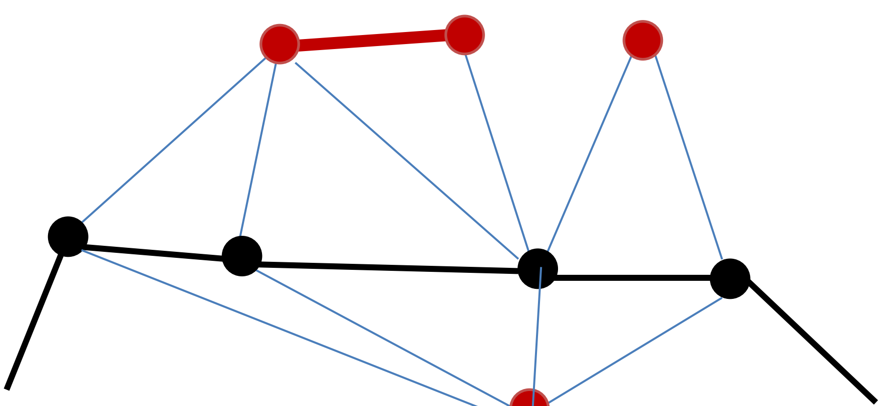


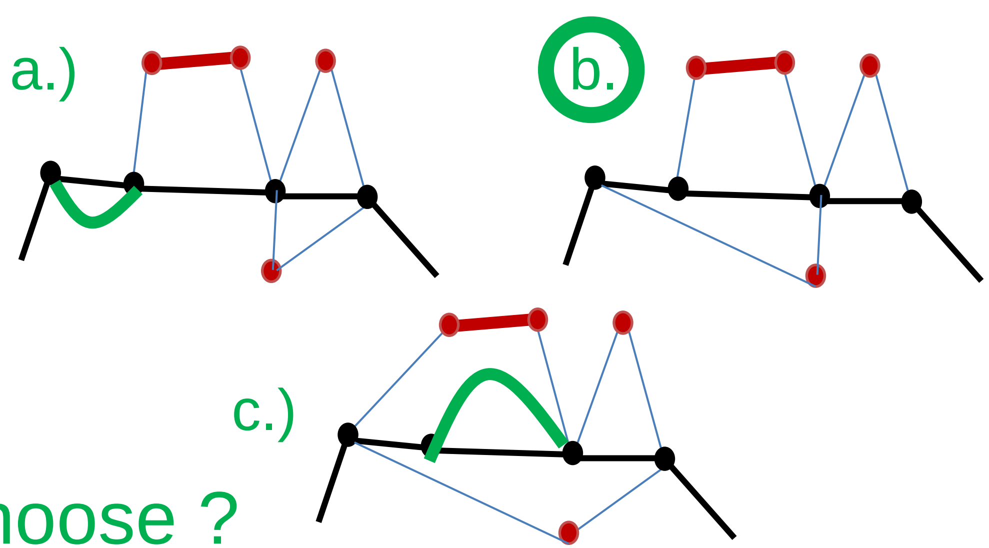
« Rerout » short ears

R: = internal vertices
of short ears
(2-, 3-oreilles)



Short ears are not efficient in terms of
n. of edges / n.of vertices
but they are very flexible for changes !

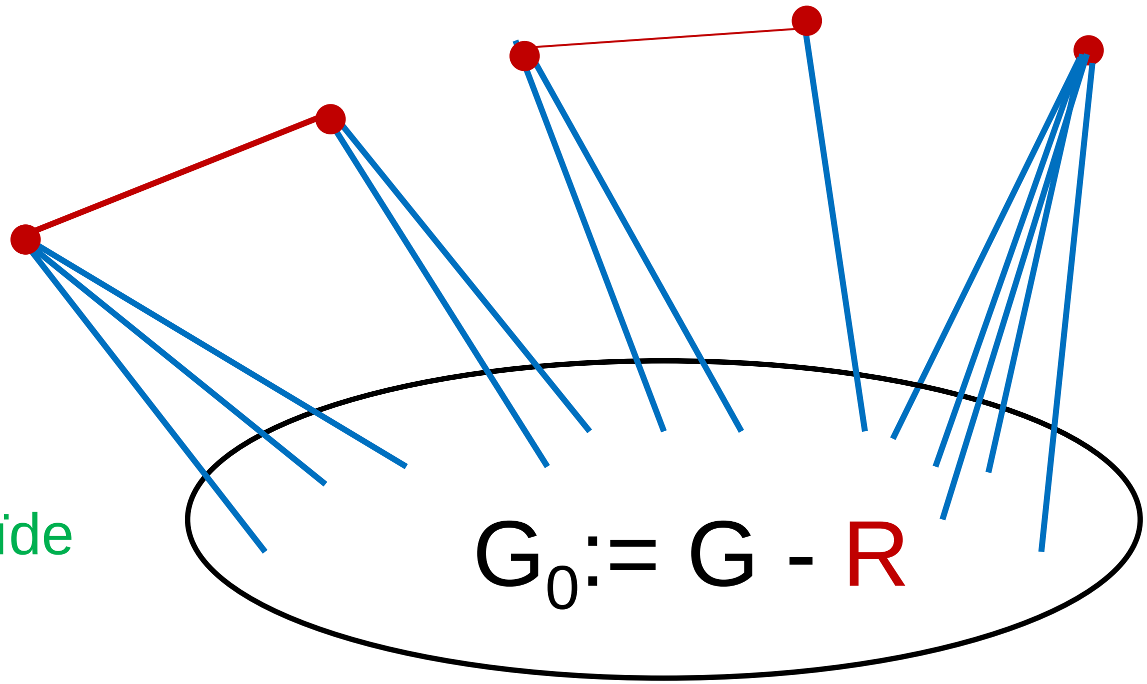




Which three ears would you choose ?

The 2 ingrediants of the alg and of the proof

a.) **1** ear for all edges
and vertices in R
(independent in a
partition matroid)



b.) acyclic
(independent in a cycle-matroïde

1.) Heureka, intersection of 2 matroides (Edmonds 1965) solves it !

2.) Heureka, the parity has to be corrected only in G_0 , whence $-\pi$


SALESMAN
THE APPY HEND

Lower bounds for best garantee unless P=NP (tours):

$$\frac{5381}{5380}\ ;\ \frac{3813}{3812}\ ;\ \frac{220}{219}\ (2000);\ \frac{185}{184}\ (2012)$$

Papadimitriou, Vempala; Lampis

For paths ?

Can the bounds be improved ?

Study of BOM for all variants !

.Directions from bird's eyes …