4. Complements
to the first 3 series

The postman polyhedron

Def : $\delta(W) \subseteq E(G)$ $(W \subseteq V)$ is a *T-cut,* if $|W \cap T|$ is odd

Proposition : F T-join, $\delta(W)$ T-cut $\Rightarrow$ $| F \cap \delta(W) | \geq 1$

Theorem Edmonds,Johnson (1973) : $Q_+(G,T)$:= conv (T-joins) + IR_+^n =

$\{x \in IR_+^E \ x(\delta(W)) \geq 1, \delta(W)$ is a T-cut, i.e. $|W \cap T|$ is odd$\}$

Proof : Through the following slides.

Minmax bipartite

$\tau(G,T)$:= $\min\{\ |F|\ :\ F \subseteq E,\ F \text{ is a T-join }\}$

$\nu(G,T)$:= $\max\{\ |\mathcal{C}|\ :\ \mathcal{C} \text{ disjoint T-cuts }\}$

Easy : $\tau(G,T) \geq \nu(G,T)$

Theorem (Seymour '81) If G is bipartite,

$$\tau(G,T) = \nu(G,T)$$

Minmax nonbipartite

$\nu_2(G,T)$:= max{ $|\mathcal{C}|$: $\mathcal{C}$ 2-packing of T-cuts }, where
a *2-packing* is a family covering every element $\leq$ twice

Easy : $\tau(G,T) \geq \nu_2(G,T)/2$

Proof : Let F be a T-join, and $\mathcal{C}$ a 2-packing of T-cuts.

Then $2\,|F| = \sum_{C \text{ in } \mathcal{C}} |F \cap C| \geq \nu_2(G,T)$

Theorem (Edmonds-Johnson '73) If G is arbitrary,

$$\tau(G,T) = \nu_2(G,T)/2$$

Packing

A *packing* is a family covering every element $\leq$ once

A 2-packing is a family covering every element $\leq$ twice

$\nu_2(G,T)/2 \geq \nu(G,T)$

(Possibly) fractional : coefficients y_C ($C \in \mathcal{C}$) whose sum has to be maximized : ν^* for packings .

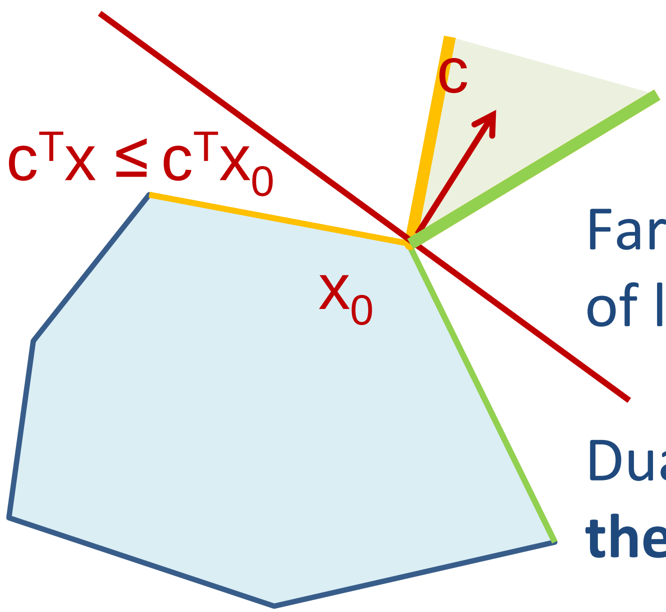
For $c: E \rightarrow IR_+$: $\nu(G,T,c)$, $\nu_2(G,T,c)$, $\nu^*(G,T,c)$

Linear Programming

Duality Theorem

$Ax \le b$		$yA = c$
$(A \in Q^{m\times n}, b, c \in Q^n)$	dual:	$y \ge 0$
$\max\ c^T x$	$=$	$\min\ y^T b$

Weak duality $\le$: every primal x is feasible for all nonnegative combinations



Farkas Lemma: Every 'tight' consequence of linear inequalities is their nonneg.lin.comb.

Duality Theorem ⇔ If the max for c is $c^T x_0$,

then $cx \le c^T x_0$ is a nonneg.lin.comb of $a_i^T x_0 \le b_i$

Weak duality for the T-join polyhedron

Let F be a T-join, and $\mathcal{C}$ a 2-packing of T-cuts.

Then $2\,|F| = \sum_{C \text{ in } \mathcal{C}} |F \cap C| \geq \nu_2(G,T)$

Let F be a T-join, and $\mathcal{C}$ a (possibly fractional) 1-packing of T-cuts with coefficients y_C ($C \in \mathcal{C}$)

Then $|F| = \sum_{C \text{ in } \mathcal{C}} y_C |F \cap C| \geq \nu^*(G,T)$

Let F be a T-join, and $\mathcal{C}$ a (possibly fractional) c-packing of T-cuts with coefficients y_C ($C \in \mathcal{C}$)

Then $|F| = \sum_{C \text{ in } \mathcal{C}} y_C |F \cap C| \geq \nu^*(G,T,c)$ (or $\nu(G,T,c)$)

Linear Programming

Hilbert bases (normal semigroups)

$H \subseteq Z^n$ is a *Hilbert basis* if any integer vector which is a nonneg comb is also a nonneg integer comb

Example n=2 :

pointed cone : cone { (-2,3), (5,-1) } |det| = 13

adding (-1,2), (0,1), (1,0) } : Hilbert basis

Integer Caratheodory property (+'partition' into unimodular cones)

Proving the T-join polyhedron Thm

$Q_+(G,T)= \{x \in IR^E : x(W) \geq 1 , W \text{ is a T-cut}, x \geq 0 \}$

Edmonds-Johnson: ½ TDI, vertices: T-joins

$\Leftrightarrow$

$\tau(G,T, c)= \nu(G,T,2c) /2$

$\Leftrightarrow$

Seymour (81): If G arbitrary, $\tau(G,T) = \nu_2(G,T) /2$

$\Leftrightarrow$

Edmonds, Johnson(73): If G is bipartite, $\tau(G,T) = \nu(G,T)$

Metatheorem : Polyhedron the same as weighted minmax theorem

If negative weights are allowed ?

$c\ (F) = |c|\ (F \setminus E_-) - |c|\ (F \cap E_-) = |c|\ (F \,\Delta\, E_-) - c\ (E_-)$

(So if (G,w) is conservative, $\lambda_w(x,y) := \min\{w(P) :\ P \text{ path}\} =$

$\min\{w(P) :\ P\ \{x,y\}\text{-join}\}$

Is reducible to min weight perfect matchings.)

This reduction leads to the

T-join *polytope*

Another application

SCHEDULING IDENTICAL JOBS ON 2 IDENTICAL MACHINES

Input: Partially ordered set of tasks of unit length.
Output: Schedule of min completion time T

Theorem : (Fujii & als) : $T = n - \upsilon\,(G_{input})$

Solutions for max (weighted) matchings:
with Edmonds' algorithm (1965)
Grötschel, Lovász, Schrijver
with Padberg-Rao (1979)

To come : matroids

Exercises to revise for the third course : series 7.