Submodular Functions

Def : $f : 2^S \rightarrow IR$ is *submodular on* 2^S , if

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$$

monoton submodular $\Leftrightarrow \forall A \subseteq B , \quad x \in S :$

$$f(A \cup \{x\}) - f(A) \geq f(B \cup \{x\}) - f(B)$$

1.) occurs often 2.) useful 3.) ‘can be played with’

MIN $\in \mathscr{P}$

MAX $\mathscr{NP}$ - hard

versions: **for machine learning,** f(0)=0, mon, size k

Examples, special cases, connexions

rank of vectors in any vector space

In a graph the number of edges leaving a set of vertices

Minus the number of components of a set of edges

Maximum size of an acyclic graph (forest) on a given set of vertices

For k $\in$ IN and finite set S : min { k, the size of a subset }

Probability of the product of a subset of events

Total « Information in » a subset of random variables

Rank function of matroids

Many essential properties are reflected already in matroids:

Def: M=(S,r) *matroid*: r ($\emptyset$) =0,r monoton&submodular,r({s})=1,(s$\in$S)

Approx for submod max mon, size k, f(0)=0,

Algorithm (for sets of size k): (Nemhauser, Wolsey) Having X already,

WHILE $|X| < k$ choose x that maximizes

$$f(X \cup \{x\}) - f(X)$$

Lemma : $f(X \cup \{x\}) - f(X) \geq (f(OPT) - f(X)) / k$

Proof: Since mon: $f(OPT) \leq f(OPT \cup X) \leq$

$$\leq f(X) + k\,(f(X \cup \{x\}) - f(X))$$

Let X^i be what we found until step i. Then

$f(X^k) - f(X^{k-1}) \geq f(OPT) / k - f(X^{k-1}) / k$, so

$$f(X^k) \geq f(OPT) / k + (1 - 1/k)\, f(X^{k-1})$$

$$f(X^k) \geq f(OPT)\,(1 - (1 - 1/k)^k) \geq (1-1/e)\, f(OPT)$$

Matroids

$M = (S, \mathcal{F})$ is a *matroid* if

(i) $\emptyset \in \mathcal{F}$

that is, $\mathcal{F} \neq \emptyset$

(ii) $F \in \mathcal{F}$, $F' \subseteq F \Rightarrow F' \in \mathcal{F}$

(iii) $F_1, F_2 \in \mathcal{F}$, $|F_1| < |F_2| \Rightarrow \exists\, e \in F_2 \setminus F_1$:

$$F_1 \cup \{e\} \in \mathcal{F}$$

Def : $F \in \mathcal{F}$ is called an *independent set.*

The *rank function* of M is

$r : 2^S \to \mathbb{N}$ defined as $r(X) := \max\{|F| : F \subseteq X, F \in \mathcal{F}\}$

Exercise : Prove the equivalence with the previous def with rank functions! **Hint** : This means that submodularity etc have to be proved, and conversely $\mathcal{F}$ should be defined from r and (i)-(iii) be proved.

Examples

representable

S = finite set of vectors over a field (IR or extensions orGF(q)).
$\mathcal{F}$ family of linearly independent subsets of S.

graphic   M(G):=

Let G=(V,E) be a graph, and S := E
$\mathcal{F}$:= edge-sets of forests

uniform $U_{n,r}$

|S|=n, $\mathcal{F}$:= subsets of S of size at most r

Transversal matroids, Gammoids, …

Operations

Contraction, deletion, dual ;   Nashwilliams sum :

$M_1 = (S_1, \mathcal{F}_1)$, $M_2 = (S_2, \mathcal{F}_2)$:

M_1 **NW** M_2 is defined with $\{ F_1 \cup F_2 : F_1 \in \mathcal{F}_1 , \; F_2 \in \mathcal{F}_2 \}$

partition matroid : NW sum of uniform matroids;
often of rank 1

Circuits

Def: $\mathcal{C}$ family of (inclusionwise) minimal sets that are not independent

Proposition: (i) $C_1, C_2 \in \mathcal{C}$, $C_1 \not\subset C_2$
(ii) $C_1 \neq C_2 \in \mathcal{C}$, $x \in C_1 \cap C_2$, $\exists\, C_3 \in \mathcal{C} : C_3 \subseteq C_1 \cap C_2 \setminus \{x\}$

Proof: $r(C_1) + r(C_2) - r(C_1 \cap C_2) = |C_1| - 1 + |C_2| - 1 \;\; |C_1 \cap C_2| =$

$$= |C_1 \cup C_2| - 2$$

Exercise : Prove the other direction ! That is, define the independent sets from circuits and prove their axioms (i)-(iii) from the above axioms (i) – (ii).

So we can now take (i), (ii) as the definition of matroids with their

Bases

Let $\mathcal{M}=(\mathcal{S}, \mathcal{F})$ be a matroid. B is a *base* if $B \in \mathcal{F}$, $|B| = r(S)$.

Set of bases : $\mathcal{B}$

Fact : $\forall\, B_1 , B_2 \in \mathcal{B}$, $\forall\, x \in B_1 \setminus B_2$
$\exists\, y \in B_2 \setminus B_1$: $(B_1 \setminus x) \cup \{y\} \in \mathcal{B}$ } Basis axiom

Proposition: $\mathcal{B} \neq \emptyset$ is the set of bases of a matr $\Leftrightarrow$ the Fact holds.

Proof : 1.) => The stated property holds. <= :

2.) There is unique *possible matroid* with base-set $\mathcal{B}$.

3.) The uniquely defined set system is indeed a matroid

axiom (iii) to
$F_1 = B_1 \setminus x$, $F_2 = B_2$

$\mathcal{F} := \{F \subseteq B : B \in \mathcal{B}\}$

use the fact

So we can now take « Fact » as the definition of matroids !

Rank again and Span

Bases, continuation

Fact : $\forall\, B_1, B_2 \in \mathcal{B}$, $\forall\, x \in B_2 \setminus B_1$
$\exists\, y \in B_1 \setminus B_2$: $(B_1 \setminus y) \cup \{x\} \in \mathcal{B}$

Proposition: $\mathcal{B} \neq \emptyset$ is the set of bases of a matr $\Leftrightarrow$ the Fact holds.

Proof : => : Through the following property from the circuit-axiom:

Proposition : M= (S, $\mathcal{F}$) matroid, $F \in \mathcal{F}$, $e \in S \setminus F$. Then :
either $F \cup \{e\} \in \mathcal{F}$
or $F \cup \{e\}$ contains a unique circuit of M.

So we can now take « Fact » as the definition !

Corollary : $\{S \setminus B : B \in \mathcal{B}\}$ also satisfies the basis axioms.

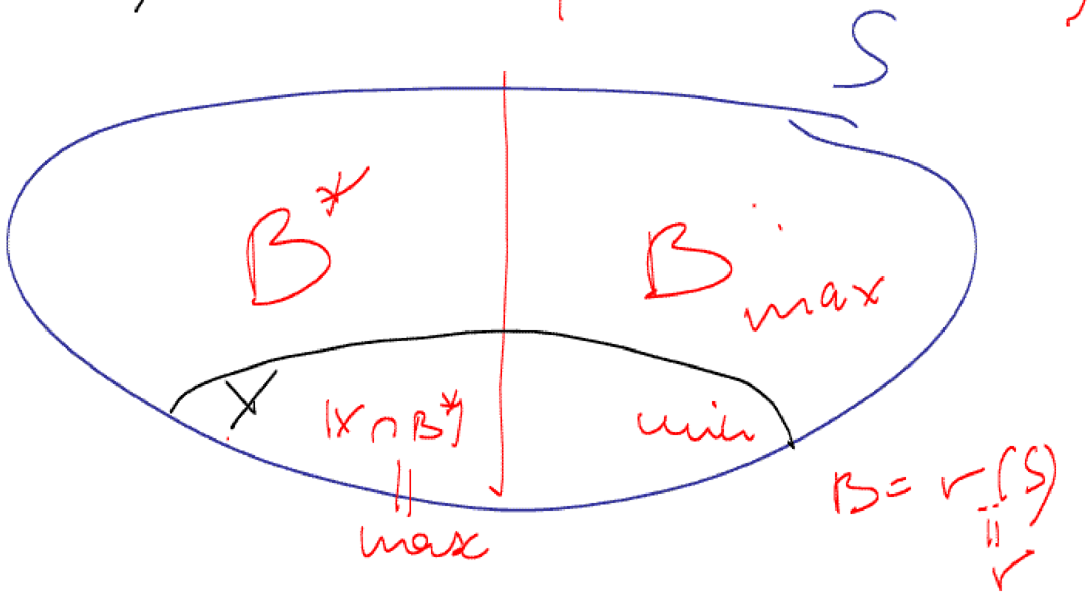
Dual Matroid

Def: dual : $M^* = (S, \mathcal{B}^*)$ dual de $M = (S, \mathcal{B})$

$\mathcal{B}^* = \{ S \setminus B : B \in \mathcal{B} \}$

Fact: $r^*(X) = |X| - \left(r(S) - r(S \setminus X) \right)$

Proof:



Def: ~~cocircuit~~ coupe d'un matroïde : circ. du dual

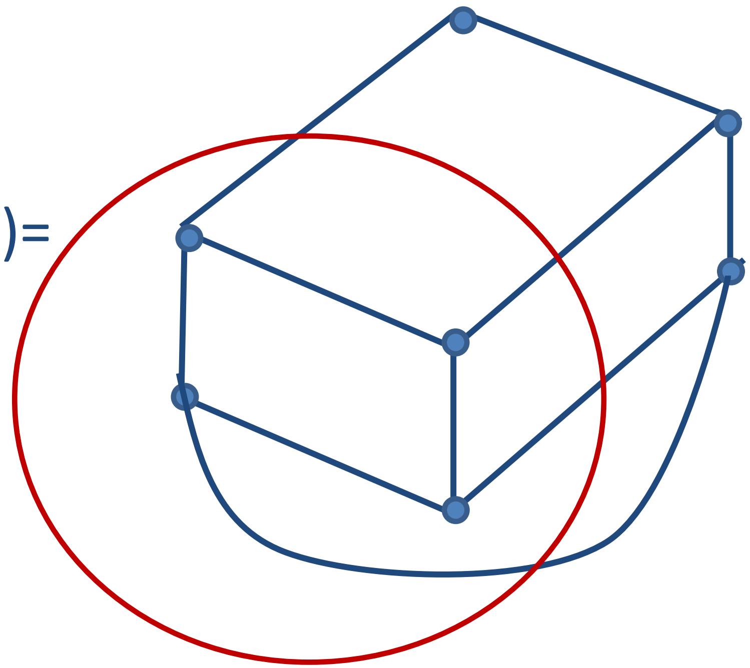
Planarity and Duality

circuits of G = circuits of M(G)

cocircuits of G = cocircuits of M(G)=

Inclusionwise min cuts of G*

M*(G) = M(G*)



Equivalently : F is a spanning tree ⇔
E \ F is a spanning tree of the dual graph

Euler's formula : $n - 1 + f - 1 = m$

Greedy alg for max weight indep

Greedy algorithm for a family of sets $\mathcal{H} \subseteq 2^S$:

If $x_1, \dots, x_i$ have been chosen,

let x_{i+1} be such that $\{x_1, \dots, x_{i+1}\} \in \mathcal{H}$, $c(x_{i+1})$ max

Theorem If $\mathcal{H}$ is hereditary, then the greedy algorithm finds the optimum for any nonnegative objective function $\Leftrightarrow$ $\mathcal{H}$ is a matroid.

Proof: =>

<= :

We find : $c(x_1) \geq \dots \geq c(x_i) \geq \dots$

The opt: $c(x'_1) \geq \dots \geq c(x'_i)$

The independence axiom (iii) contradicts the choice of x_i

If you can do it simple, make it complicated!

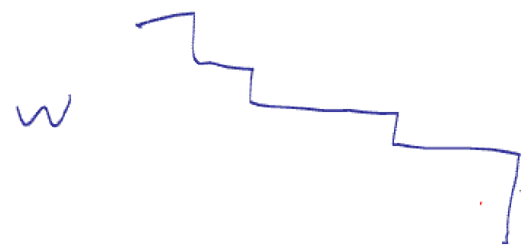
Thm (Edmonds): $M=(S,\mathcal{F})$

$$\left\{ x : \begin{array}{l} x(A) \le r(A) \\ x \ge 0 \end{array} \right\} = \operatorname{conv}(\chi_F : F \in \mathcal{F})$$

$x(A) = \sum_{a \in A} x_a$

Proof: $w_1 \ge \dots \ge w_n$

$U_i = \{1, \dots, i\}$



Submodularity => Sets A with positive dual variables form a chain !

The F that we find satisfies: $|F \cap U_i| = r(U_i)$

$$w(F) = (w_1 - w_2)|F \cap U_1| + (w_2 - w_3)|F \cap U_2| + \dots + w_n |F \cap U_n|$$

} dual solution

The inverse of the duality theorem

Thm (Edmonds): $M=(S,\mathcal{F})$ matr.

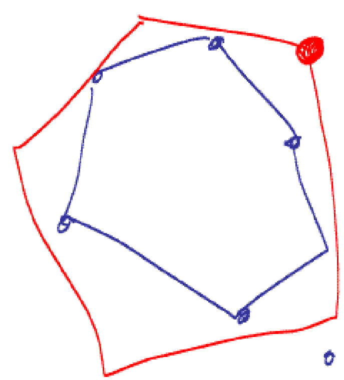
$$\mathrm{conv}(\chi_F : F \in \mathcal{F}) =$$
$$= \{x \in \mathbb{R}^S : x(A) \le r(A),\ x \ge 0\}$$

$\subset$: clear !

Montrer que $\forall w$

$$\max_{x \in \text{gauche}} w^T x = \max_{x \in \text{droite}} w^T x$$

SUFFIT :



car sinon $\exists\, x_0 \in$ droite $\setminus$ gauche

$\{x : c^T x = b\}$ $\subset$ hyperplan séparateur qui sépare x_0 de gauche

Farkas' Lemma

$c^T x_0 > b$

$c^T x \le b \quad \forall x \in$ gauche

Matroid Intersection
Edmonds (1979)

Let M_1 and M_2 be two matroids, c:

(S,r_1) and (S,r_2)
$(S,\mathcal{F}_1)$ and $(S,\mathcal{F}_2)$

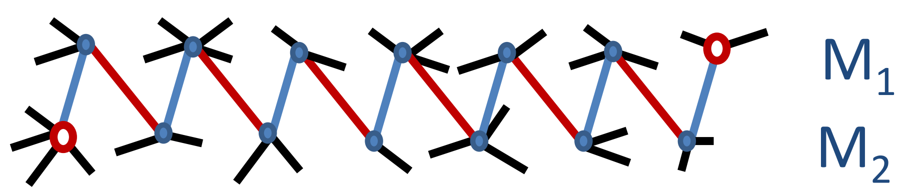
maximize $\{ c(F) \; : F \in \mathcal{F}_1 \cap \mathcal{F}_2 \}$

2 disjoint spanning trees : M_1 and $M_2 = M_1$

Two examples of cases :

2 disjoint spanning trees : M_1 and $M_2 := M_1^*$

Bipartite matching



Both are partition matroids: sums of uniform matroids on stars

Matroid Intersection Theorem

How to conjecture a « good characterization » ?

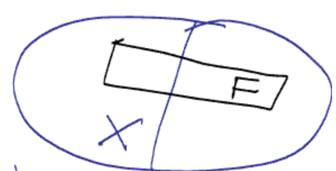
We know : $x \in \text{conv}(\chi_F : F \in \mathcal{F}_i) \Leftrightarrow x(A) \leq r_i(A)$ for all $A \subseteq S$

maximize $\{ |F| : F \in \mathcal{F}_1 \cap \mathcal{F}_2 \}$ =? $\text{conv}(\chi_F : F \in \mathcal{F}_1 \cap \mathcal{F}_2)$

max $\{ 1^T x : \quad x(A) \leq r_i(A) \ (i=1, 2)$ for all $A \subseteq S \}$

Theorem (Edmonds 1979): $\max_{F \in \mathcal{F}_1 \cap \mathcal{F}_2} |F| = \min_{X \subseteq S} r_1(X) + r_2(S \setminus X)$

Proof: $\leq$



$F \in \mathcal{F}_1 \cap \mathcal{F}_2$: $|F| = |F \cap X| + |F \setminus X| \leq$

$\leq r_1(X) + r_2(S \setminus X)$

If $|F| = r_1(M)$?

Matroid Intersection Theorem

Generalization of bipartite matching
(of the alternating paths in the « Hungarian method »)

Proof of $\geq$: that is, there is F and X with $|F| = r_1(X) + r_2(S \setminus X)$.

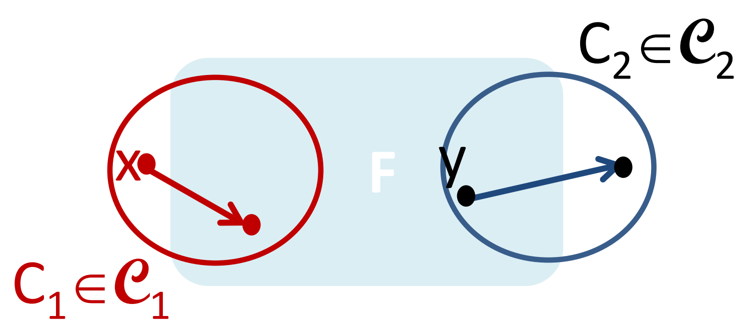
We prove that the following algorithm terminates with such an F and X.

Algorithme d'intersection

What is the INPUT ? → ORACLE - rank, independence, etc

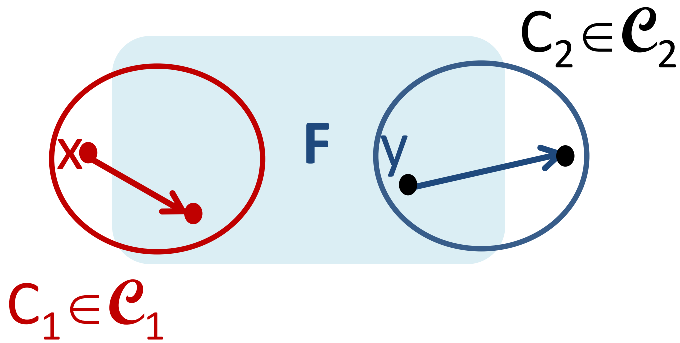
0.) Let : $F \in \mathcal{F}_1 \cap \mathcal{F}_2$ maximal by inclusion (greedily)

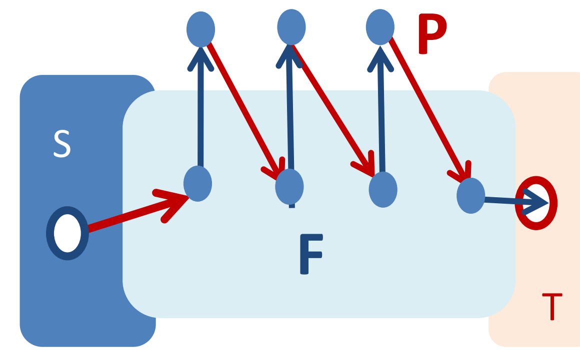
1.) Define arcs from unique cycles :



Matroid Intersection Theorem

Algorithmic proof of the matroid intersection theorem





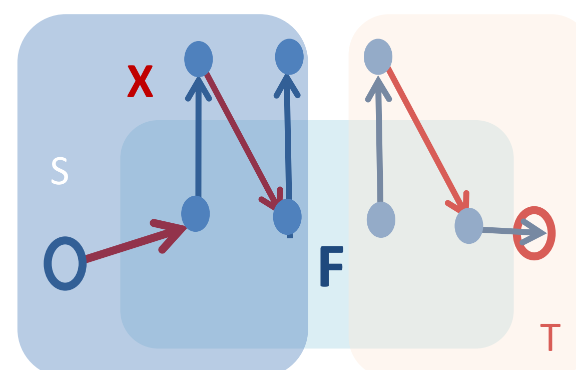
3.) Sources $S := \{x \in S \setminus F,\ F \cup \{x\} \in \mathcal{F}_2\}$ **Sinks T:=**$\{x \in S \setminus F,\ F \cup \{x\} \in \mathcal{F}_1\}$

If S or T is empty ?

Find an (S,T)-path.

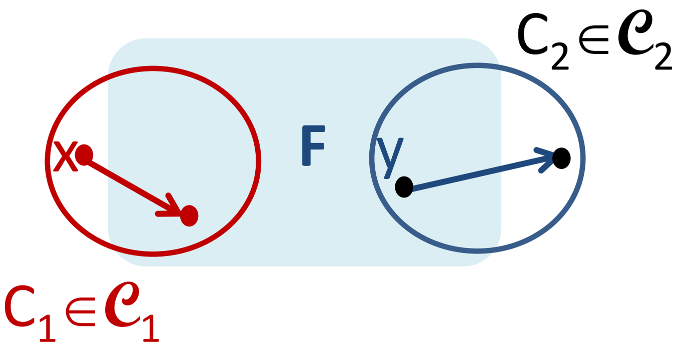
a.) If there exists one, let P be one with inclusionwise minimal vertex-set (equivalently, P is chordless).

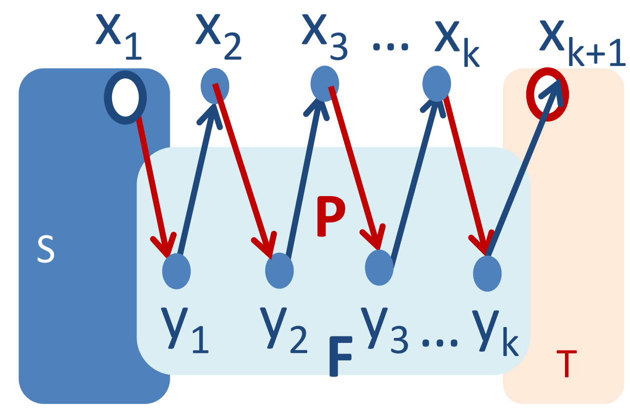
b.) If there exists none, $T \cap X = \emptyset$, where $X := \{x \in S : x \text{ is reachable from } S\}$



Matroid Intersection Theorem

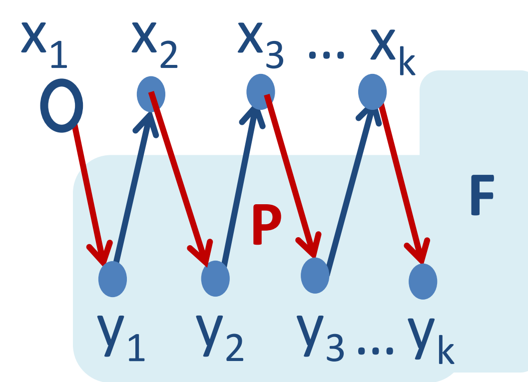
exchange along an improving path





a.) If $P = \{x_1, y_1, x_2, \ldots x_k, y_k, x_{k+1}\}$ is a chordless path, then $F \,\Delta\, P \in \mathscr{F}_1 \cap \mathscr{F}_2$

Apply the following to $F \cup \{x_1\} \in \mathscr{F}_2$, and $F \cup \{x_{k+1}\} \in \mathscr{F}_1$



Lemma : $M = (S, \mathscr{F})$ matroid, $F \in \mathscr{F}$, $x_1, \ldots, x_k \notin F$

If y_i is in the unique cycle of $F_i \cup x_i$,

but y_j, $j = i+1, \ldots k$ is not, then

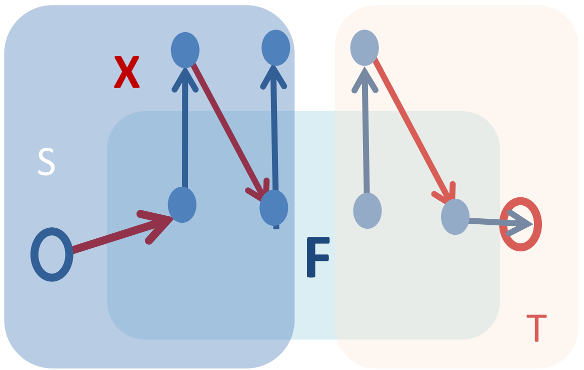
$$(F \setminus \{ x_1, \ldots, x_k \}) \cup \{ y_1, \ldots, y_k \} \in \mathscr{F}$$

Proof: For k= 1 true, and then use it by induction to $(F \setminus \{x_k\}) \cup \{y_k\}$.

Matroid Intersection Theorem

No improving path : show that the solution is optimal

Let **X** := {x ∈ S : x is reachable from S}



Lemma : Suppose b.) : X ∩ T = ∅ , where

X := {x ∈ S : x is reachable from S}

Then |F| = $r_1(X) + r_2(S \setminus X)$

Proof : $r_1(X) = |F \cap X|$, because $X \subseteq sp_1(F \cap X)$.

$r_2(S \setminus X) = |F \setminus X|$, because $S \setminus X \subseteq sp_2(F \setminus X)$.