To come :

Matchings,

Undirected Shortest Paths,

T-joins

…

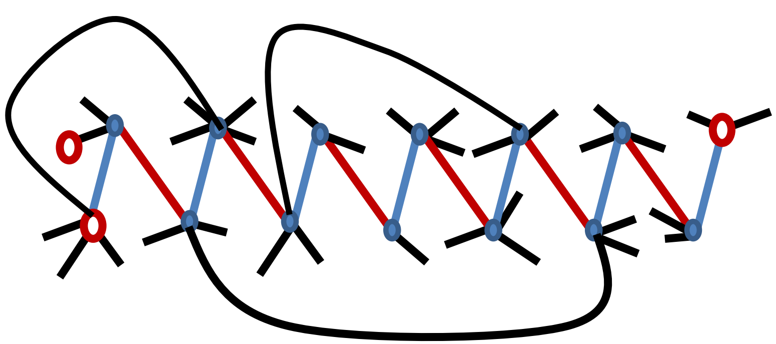
Exercises to revise for the second course: series 3 and 6

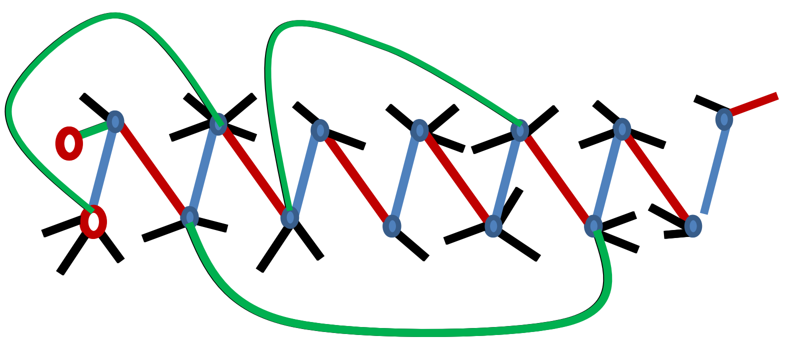
Matching

matching : a set M $\subseteq$ E of vertex-disjoint edges

INPUT : G=(V,E) graph.
TASK : Find a matching of maximum size

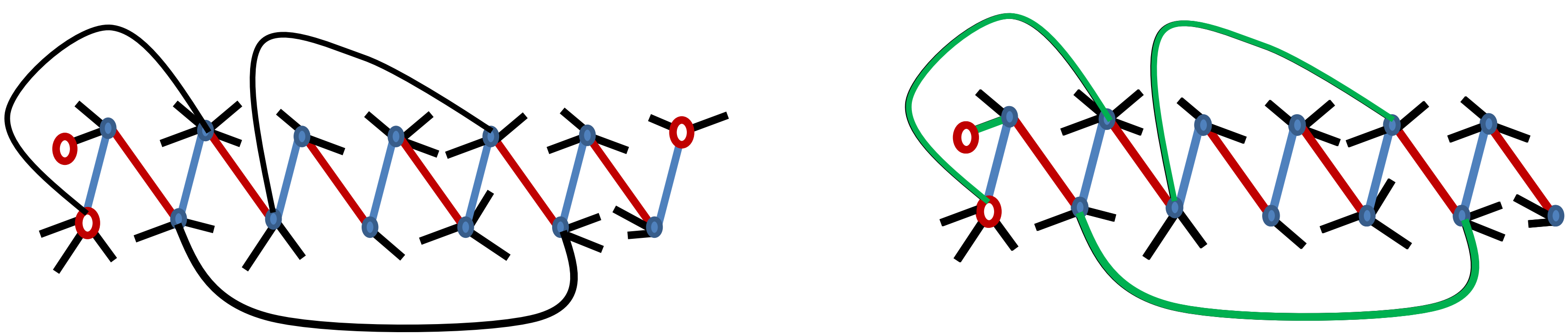
Do the red edges form a maximum matching ?





Augmenting Paths

augmenting path with respect to matching M : path alternating between M and E \ M with the 2 endpoints uncovered by M.



Proposition (Berge) : G graph, M matching in G.
M is a maximum matching in G iff there is no augmenting path

Matching and vertex cover

matching : M set of vertex-disjoint edges

Max |M| : ν

vertex cover : T set of vertices so that G-T has no edges

Min |T| : τ

$\nu \leq \tau$

Minmax for bipartite graphs

Theorem (Kőnig) : If $G=(V,E)$ is bipartite, then $\upsilon = \tau$

Proof: $\leq$ is the proven 'easy part'; $\geq$ is to be proved:

If for some $v \in V$: $\nu(G - v) = \nu(G) - 1$, by induction :

$$\nu(G) = \nu(G - v) + 1 = \tau(G - v) + 1 \geq \tau(G).$$

If $uv \in E$ then either u or v satisfy this condition !

Exercise 3.1, 3.2

Q.E.D.

LP for bipartite matchings

MATCHING POLYTOPE for G=(V,E) bipartite

$$x \in \mathbb{R}^E :$$

$$x(\delta(v)) \leq 1 , \forall v \in V$$

$$x \geq 0$$

dual:

VERTEX COVER for G=(V,E) bipartite

$$x \in \mathbb{R}^V :$$

$$x_i + x_j \geq 1 , \forall ij \in E$$

$$x \geq 0$$

TDI (TU+Cramer, or no odd circuit)

The method of variables

G = (A, B, E) bipartite, |A|=|B|. M := $(x_{ij} \text{ if } ij \in E, \text{ else } 0)_{n \times n}$:

Proposition : M is a nonzero polynomial ⇔ ∃ perfect matching

Proof : All terms of M are different. (There is no cancellation.)

n! Terms, but determinants can be computed in polynomial time :
randomized algorithm: substitute values and then compute !

Questions : If then the det is nonzero can we conclude ?
If it is zero ?
What to do for nonbipartite graphs ?

The method of variables

The probability of error, precisely

Lemma: (Schwartz, Zippel) Let q be a nonzero polynomial of n variables $x_1,\ldots, x_n$, and let it be of degree d ; $S \subseteq$ IN is finite, $s:=|S|$. Moreover, let $X_1,\ldots, X_n$ be random variables taken independently and uniformly from S.

Then $\Pr(q(X_1,\ldots, X_n)=0) \leq d/s$.

Proof: For n=1 obvious. Le $p \in Q[x_1,\ldots, x_{n-1}]$ the coefficient of the highest exponent to power μ of x_n , and let π be the degree of p.

$$\Pr(q(X_1,\ldots, X_n)=0) \leq \Pr(p(X_1,\ldots, X_n)=0) + \Pr(q(X_1,\ldots, X_n)=0 \mid p(X_1,\ldots, X_n) \neq 0)$$

$$\leq \pi/s + \mu/s \leq d/s$$

The method of variables

The Randomized Algorithm

Oracle Algorithm :

An oracle tells the substitution values of a polynomial in pol(deg) time.

1. Let S = {1,…,2n}.

2. Make independent uniform choices in S for each variable.

3. Compute the polynomial (oracle call) for the chosen values.

   If $\neq 0$: the polynomial is nonzero ($\exists$ perfect matching)

   If =0 ? We decide: no perfect matching: Pr (error) =½

Why not bigger S ? Better to choose |S| = const x deg and repeat !

Proposition : After O(log 1/ε) repetitions Pr (error) $\leq$ ε

The complexity class $P \subseteq RP \subseteq NP$

Σ alphabet

$L \subseteq \Sigma^*$

$L \in NP \iff \exists R_L : \Sigma^* \times \Sigma^* \to \{0,1\}$

$x \notin L: \qquad R(x,y)=0 \quad \forall y \in \Sigma^*$

$x \in L: \exists y \in \Sigma^* : R(x,y)=1$

Imagine : x= a graph, y the certificate (eg a substitution with ≠0 polynomial value)

$L \in RP \iff$ — 〃 — $\quad x \to Y_x$ |

— 〃 — — 〃 —

$x \in L: \quad \{y \in \Sigma^* : R(x,y)=1\} \geq \frac{|Y_x|}{2}$

The same def as NP but there are many certificates : constant proportion

Tutte-Berge theorem

Theorem : Let $G=(V,E)$ be a graph. Then the minimum, over all matchings M of the number of uncovered vertices of V =

$$\max \{ q(X) - |X| \ : \ X \subseteq V \}$$

Def : $q(X)$ is the n. of comps of $G-X$ having an odd number of vertices

Proof : $\geq$ easy.

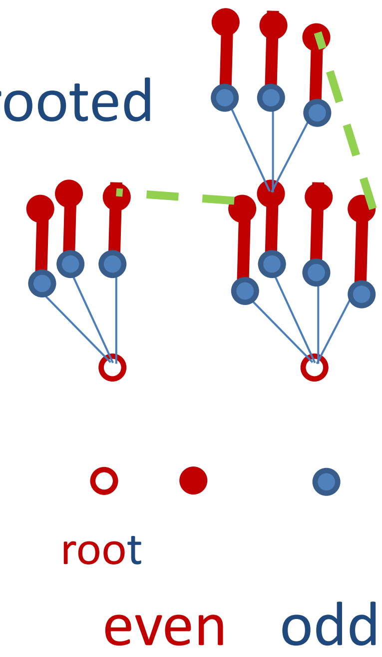
$\leq$: We can adapt the proof of Kőnig's theorem:
If $\nu(G - v) = \nu(G) - 1$, induction is easy, else apply the exercises.

Exercise 3.3

Exercise 3.5

Hint : In which part of the theorem are the vertices uncovered by matchings : in X ? An even comp of $G-X$? An odd comp ?

Edmonds' algorithm



1. Grow an (inclusionwise max) alternating forest F rooted in uncovered vertices

2. If two even vertices are adjacent
   a.) between 2 different components : augment
   b.) in the same component
      Generalize Exercise 3.3 to this case.
      Heureka you shrink ! (Edmonds)

   In both cases **GOTO 1** (possibly using the actual forest).

3. If there is no edge between the even vertices **STOP**
   X:= odd vertices

Theorem : X is a Tutte-set and F is a maximum matching

Summary of algorithms for matchings

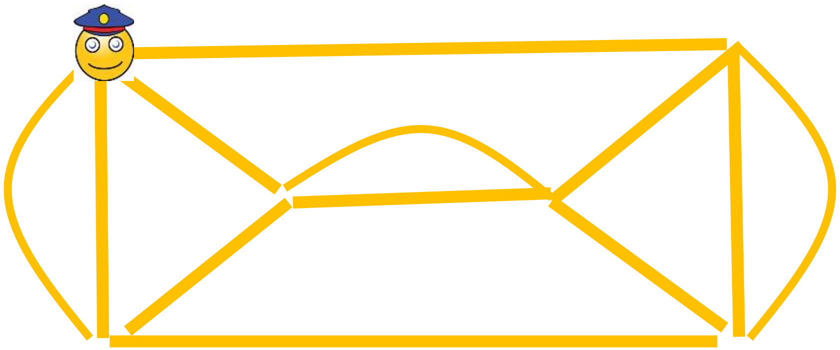
Unweighted :

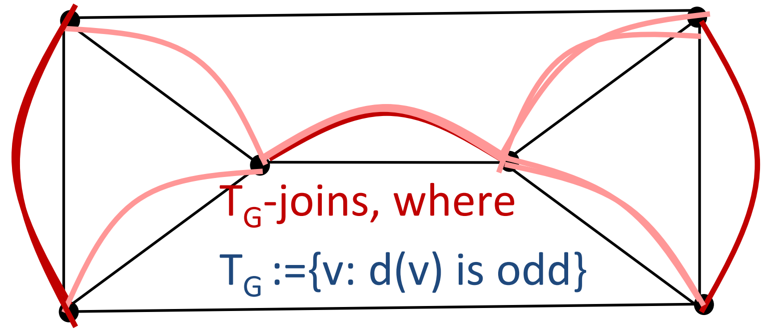
- Algorithms for bipartite graphs: paths in digraphs;
- Method of variables
- Edmonds'algorithm;
- Structural algorithms (for matchings by Lovász, T-joins, b-match: S.)

Weighted :

- Primal-Dual framework with max cardinality subroutine
- Ellipsoid method

T-joins





Euler's theorem : G= (V,E), E : streets
One can go through all the streets exactly once $\Leftrightarrow$
$\forall$ Degree is even & G is connected

$F \subseteq E(G)$ is a *T-join,* if
T = vertices of odd degree of F.

Easy facts about T-joins : G connected, |T| even $\Rightarrow$ $\exists$ T-join ;
min weight « Eulerian replication » = duplication of a minimum T_G-join.

G=(V,E), w: E $\to$ IR, F is a minimum weight T-join $\Leftrightarrow$ (G, w[C]) is conservative, where $w(e) := \begin{cases} -1 \; if \; e \in F \\ 1 \; if \; e \notin F \end{cases}$

Exercise 4.2, 6.2

Polynomial algorithm

Input : G=(V,E), w: E $\rightarrow$ IR

Task : minimize the weight of a T-join

Proposition (Edmonds) : If the weights are nonnegative easy reduction tminimum weight matching of the complete graph on T where the Weights are the w-shortest paths in G between the vertices of T.

The postman polyhedron

Def : $\delta(W) \subseteq E(G)$ $(W \subseteq V)$ is a *T-cut,* if $|W \cap T|$ is odd

Proposition : F T-join, $\delta(W)$ T-cut $\Rightarrow$ $| F \cap \delta(W) | \geq 1$

Theorem Edmonds,Johnson (1973) : $Q_+(G,T)$:= conv (T-joins) + IR_+^n =

$\{x \in IR_+^E \; x(\delta(W)) \geq 1, \delta(W)$ is a T-cut, i.e. $|W \cap T|$ is odd$\}$

Minmax

$\tau(G,T)$:= $\min\{\ |F|\ : F \subseteq E,$ F is a T-join $\}$

$\nu(G,T)$:= $\max\{\ |\mathcal{C}| : \mathcal{C}$ dijoint T-cuts $\}$

Easy : $\tau(G,T) \geq \nu(G,T)$

Theorem (Seymour '81) If G is bipartite,

$$\tau(G,T) = \nu(G,T)$$

Proving the T-join polyhedron Thm

$Q_+(G,T)= \{x \in IR^E : x(W) \geq 1 ,$ W is a T-cut

$x \geq 0 \}$

Edmonds-Johnson: ½ TDI, vertices: T-joins

$\Leftrightarrow$

$\tau(G,T, c)= \nu(G,T,2c) /2$

$\Leftrightarrow$

Seymour: If G is bipartite, $\tau(G,T) = \nu(G,T)$

Metatheorem : Polyhedron the same as weighted minmax theorem

Connection to Shortest Paths

Guan (1962): J T-join w-min iff w[C] conservative

conservative : no negative weight circuit

$\lambda(x,y) := \lambda_w(x,y) := \min\{w(P) : P \text{ path}\} =$

$\min\{w(P) : P \ \{x,y\}\text{-join}\}$

Reformulation of Seymour's theorem (81)

G bipartite, $w : E(G) \to \{-1,1\}$;

Theorem : G conservative $\Leftrightarrow$ E_- can be covered by disjoint cuts C, with $|C \cap E_-| = 1$

If negative weights are allowed ?

$c(F) = |c|(F \setminus E_-) - |c|(F \cap E_-) = |c|(F \Delta E_-) - c(E_-)$

(So $\lambda_w(x,y) := \min\{w(P) : P \text{ path}\} =$

$\min\{w(P) : P \ \{x,y\}\text{-join}\}$

Is reducible to min weight perfect matchings.)

This reduction leads to the
T-join polytope

Another application

SCHEDULING IDENTICAL JOBS ON 2 IDENTICAL MACHINES

Input: Partially ordered set of tasks of unit length.
Output: Schedule of min completion time T

Theorem : (Fujii & als) : $T = n - \upsilon(G_{input})$

Solutions for max (weighted) matchings:
with Edmonds' algorithm (1965)
Grötschel, Lovász, Schrijver
with Padberg-Rao (1979)

To come : matroids

Exercises to revise for the third course : series 7.