Combinatorial Optimization :

Matchings, Matroids and the Travelling Salesman

On the crossroad of the postman and the salesman

András Sebő,

CNRS (G-SCOP) Grenoble

•

support for an advanced course in Buenos Aires

Exercises to revise for the first course : series 4 and 5.

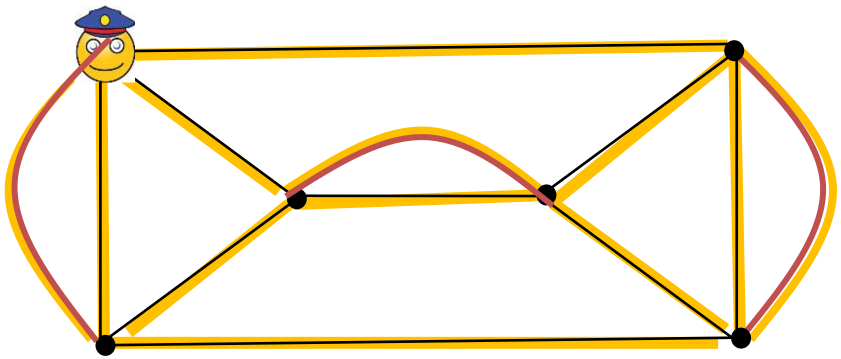
What is combinatorial optimization ?

Given $f : 2^S \to \mathrm{IR}$, find $X \subseteq S$ that minimizes f ,
that is, such that $f(X) \leq f(Y)$ for all $Y \subseteq S$.

TOO GENERAL, NOT EXACT, IRRELEVANT, NOT TRUE, DRY, BORING, IGNORING IMPORTANT ASPECTS LIKE COMPLEXITY ISSUES …

We have to go through more specific, structured examples !

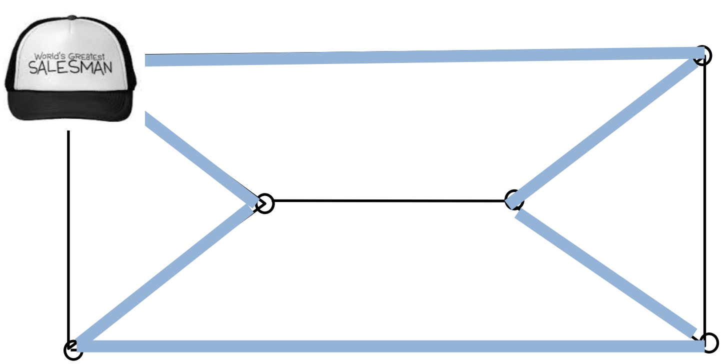
The postman



Edges = streets
Do all the streets
and come back !

In P
(Edmonds,Johnson73)

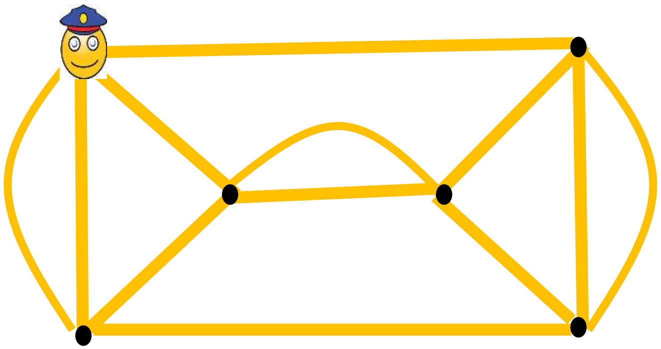
The Travelling Salesman



Vertices = Cities
Do all the cities
and come back !

NP-hard
(Karp, 1972)

The (Chinese) postman problem



Euler's theorem : G= (V,E), E : streets

One can go through all the streets exactly once ⇔

∀ Degree is even & G is connected, i.e. Eulerian)

min « Eulerian replication » ?

= min "Eulerian duplication"

= min cardinality of a set F with $d_F(v) \equiv d_G(v)$ mod =2 $\forall v \in V$

postman set

Exercise 4.1

Proposition : A postman set P is minimum ⇔ $w(e) := \begin{cases} -1 \; if \; e \in P \\ 1 \; if \; e \notin P \end{cases}$ has no circruit of negarive total weight

Exercise 4.2.

TSP

TSP PATH

INPUT : V cities, s , t $\in$ V, c: $V \times V \to IR_+$ métrique, cad

$$c(uv) + c(vw) \geq c(uw) \qquad \forall\, u, v, w \in V$$

OUTPUT: shortest Hamiltonian path between *s* and *t* .

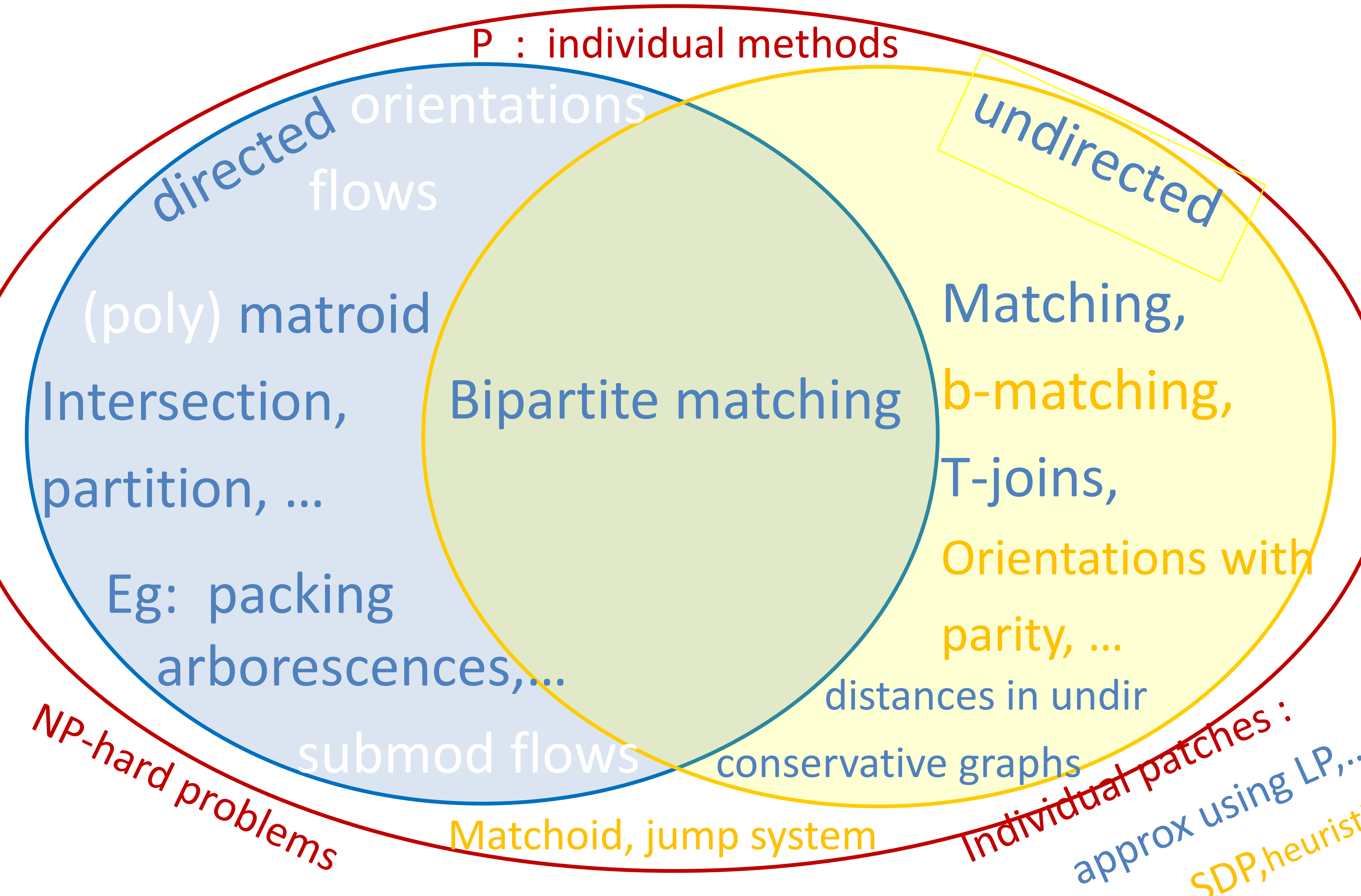
GRAPH-TSP PATH:

c (uv) := minimum cardinality of an (u,v)-path in INPUT graph G=(V,E).

GRAPHE- TSP : s=t

$c \in \{1,\infty\}$ any approx $\supseteq$ HAM ; the metric condition usually holds

Directions from bird's eyes …



Our program

1. Bin packing (cutting stock, scheduling), how to look at it ?

   LP, Total Dual Integrality, Integer Decomposition and Hilbert Bases

   Paths (GPS, PERT, …), what can be solved ?

   Cuts (routing, clustering) various problems, different ways …

2. Matching (mariages, scheduling) undirected shortest paths

   Tours : the salesman and the postman

3. Submodular functions (machine learning, network design), matroids

4. Matroid intersection, context and applications

5. Recent progress in approximating the TSP (using what we learnt)

Bin packing

BIN PACKING

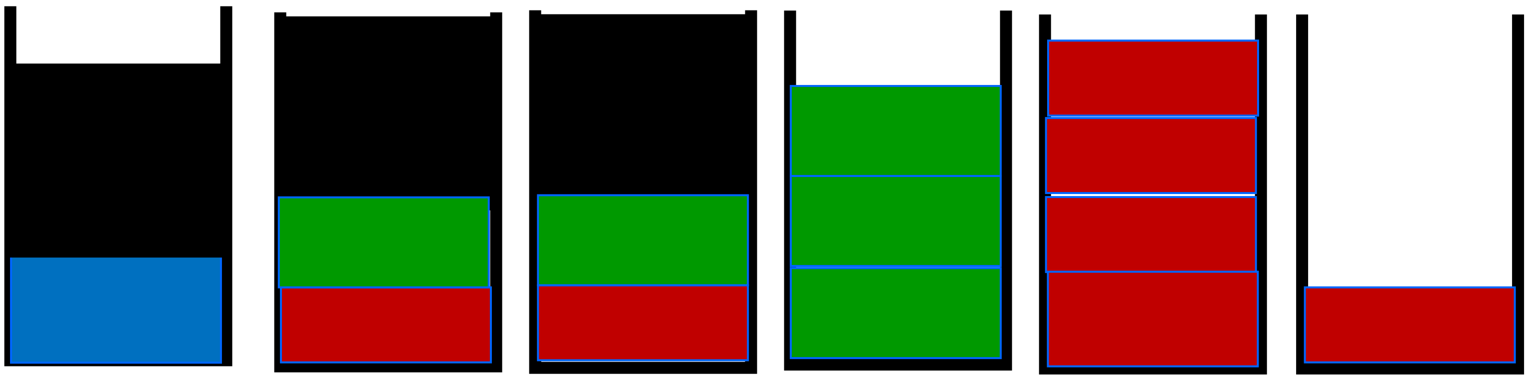
Input : $0 \leq s_1, \ldots, s_n \leq 1$ item *sizes,*

Task : Minimize the number of bins (capacity 1)

PARTITION : Are 2 bins enough ?

NP-hard

Bin packing (picture)



Bin packing (heuristics)

BIN PACKING

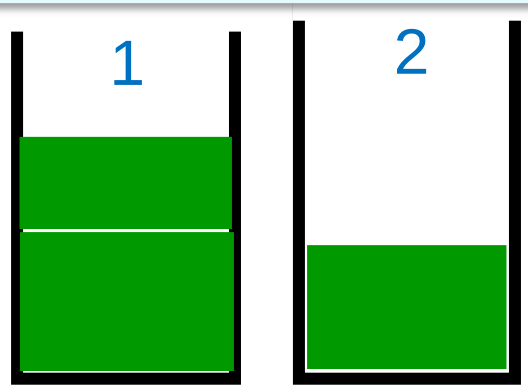
Input : $0 \le s_1, \ldots, s_n \le 1$ item *sizes*,

Task : Minimize the number of bins (capacity 1)

Heuristics : NF,	FF,	NFD,	FFD
2	17/10	11/9 OPT	+1

Proposition : NF ≤ 2 OPT -1 ? **Exercise**

Proof :



> 1 ...

OPT ≥ ⌈size⌉

NF < 2 size +1

Bin packing (patterns)

INPUT : $0 \leq s_1, \ldots , s_d \leq 1$ item *sizes,*

$b_1, \ldots , b_d \in \mathbb{N}$ item *multiplicities*

Pack them to a min number of bins of capacity 1

pattern : $p \in \mathbb{N}_+^d$ such that $p_1 s_1 + \ldots + p_d s_d \leq 1$

P : = the columns are the inclusinwise max patterns

Bin packing (example)

d=3

s=(1/2, 1/3, 1/5)   b=(1, 2, 4)

$$P = \begin{pmatrix} 2 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 5 & 0 & 2 & 1 & 3 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

SIZE = 59/30   LP = ½ + 2/3 + 4/5 = 59/30

Exercise :   OPT= 2 or 3 ?

Bin packing (LP)

pattern : $p \in \mathbb{N}_+^d$ such that $p_1 s_1 + \ldots + p_d s_d \leq 1$

Gilmore-Gomory LP :

$$\mathbf{Px \geq b} \quad (P \in \mathbb{N}_+^{d \times big}) \qquad\qquad \mathbf{yP \leq 1}$$

$$\mathbf{x \geq 0} \qquad\qquad \mathbf{y \geq 0}$$

$$\min\ 1^T x \quad (b \in \mathbb{N}_+^d) \qquad = \qquad \max\ 1^T y$$

Conjecture (Scheithauer,Terno): $\quad OPT \leq \lceil LP \rceil + 1$

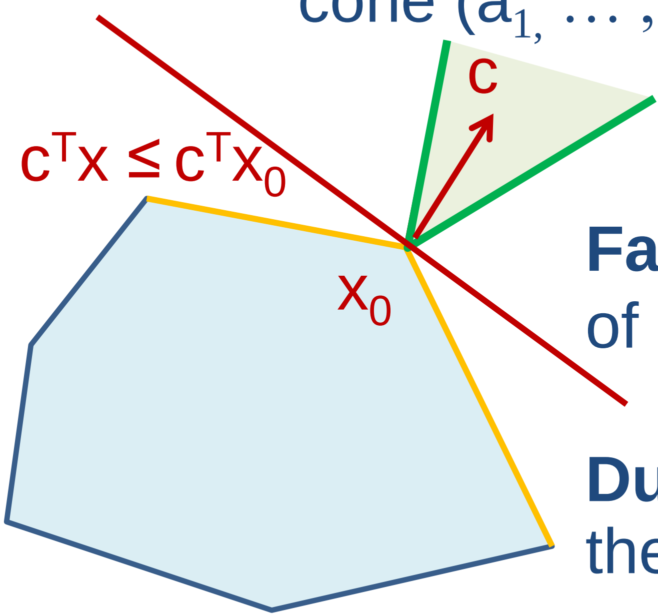
(not better for restricted patterns)

Linear Programming

Duality Theorem

$Ax \leq b$		**$yA = c$**
$(A \in \mathbf{Q}^{m\times n}, b, c \in \mathbf{Q}^n)$	dual:	**$y \geq 0$**
max c^Tx	=	min y^Tb

cone $(a_1, \ldots, a_m) := \{ \sum \lambda_i a_i \ : \ \lambda_i \geq 0, i=1, \ldots, m \}$



Farkas Lemma: Every 'tight' consequence of linear inequalities is their nonneg.lin.comb.

Duality Theorem ⇔ If the max for c is c^Tx_0, then c is a nonneg.lin.comb of a_i with $a_i{}^Tx_0=b_i$

Linear Programming

Carathéodory's theorem

Fact : (Carathéodory) $v \in$ cone $(a_1, \ldots, a_m) \subseteq \mathbb{R}^n$ =>
v is also a nonneg. comb of a linearly independent subset

Proof : $v = \lambda_1 a_1 + \ldots + \lambda_k a_k$: $\lambda_i > 0$ $(i=1, \ldots k)$. If lin. dep:

$0 = \alpha_1 a_1 + \ldots + \alpha_k a_k$ not all α_i are nonnegative.

Add the right multiple of the second to the first.

Exercise: It is possible to do this so as at the same time

- to have one less nonzero coefficient
- to maintain the nonnegativity of all coefficients.

Q.E.D. « ∃ solution => ∃ basic solution »

Linear Programming

Integer Solutions

$\mathbf{Ax \le b}$ $\quad$ dual: $\quad$ $\mathbf{yA = c}$

$(A \in \mathbf{Q}^{m \times n}, b, c \in \mathbf{Q}^n)$ $\quad$ $\mathbf{y \ge 0}$

$\max\ c^T x \quad = \quad \min y^T b \quad =: \mathrm{LIN}$

integer polyhedron $\mathbf{Ax \le b}$ if $\forall$ c the LIN is integer $\Leftrightarrow$ vertices (if any) are integer.

Totally dual integral (TDI) : $\mathbf{Ax \le b}$, if LIN= $\{\min\ y^T b,\ yA = c,\ y \ge 0,\ y \text{ integer}\}$

integer rounding, if $\forall c : \{\min\ y^T b,\ yA = c,\ y \ge 0,\ y \text{ integer}\} = \lceil \mathrm{LIN}(c) \rceil$

TDI system = IR system & integer polyhedron

Linear Programming

Hilbert bases (normal semigroups)

$H \subseteq Z^n$ is a *Hilbert basis* if any nonneg comb which is also an integer comb is also a nonneg integer comb

Example n=2 :

pointed cone : cone { (-2,3), (5,-1) } |det| = 13

adding (-1,2), (0,1), (1,0) } : Hilbert basis

Integer Caratheodory property (+'partition' into unimodular cones)

Linear programming
Normal semigroups

Schrijver : **Ax ≤ b** TDI ⇔ if $\forall x_0$ the equalities for x_0 form a H.b.
Full dim => unique minimal TDI System (Schrijver sys.)

s,t paths and cuts,matching polytope,spanning trees,arborescences,
bin packing, matroids and submodular polyhedra,

Gomory-Chvátal procedure, integer hull … :
rounding down the Schrijver system.

Integer Caratheodory property holds in 2 and 3 dim.

General Integer Caratheodory bound in n dim: 2n – 2, open in gen.

General Integer Programming :
Gomory-Chvátal, Lovász-Schrijver, Balas, Ceria, etc …

Bin packing (LP)

Theorem (McCormick, Smallwood, Spieksma 1990) : For two different item sizes $OPT \leq \lceil LP \rceil$ and can be found in poly. time.

Exercise* : Suppose d=2, and prove that { (p,1) : p is a pattern} is a Hilbert basis

Hint: Show that any three linearly independent vectors among these not containing a fourth one, form a Hb.

Theorem (Sebő, Shmonin 2006-) : For at most 7 different item sizes, $OPT \leq \lceil LP \rceil + 1$ and can be "easily" found (Conj. True)

Theorem (Jansen, Solis-Oba 2011) : For any fixed number of item sizes OPT+1 can be found in polynomial time.

Paths in Graphs

Directed, nonnegative weights (Dijkstra)

-1 weights NP-hard (HAM)

Conservative (no circuit of neg total weight): $\in$ P

Undirected shortest paths with nonnegative weights ?

With -1 weights ?

With a conservative weighting ?

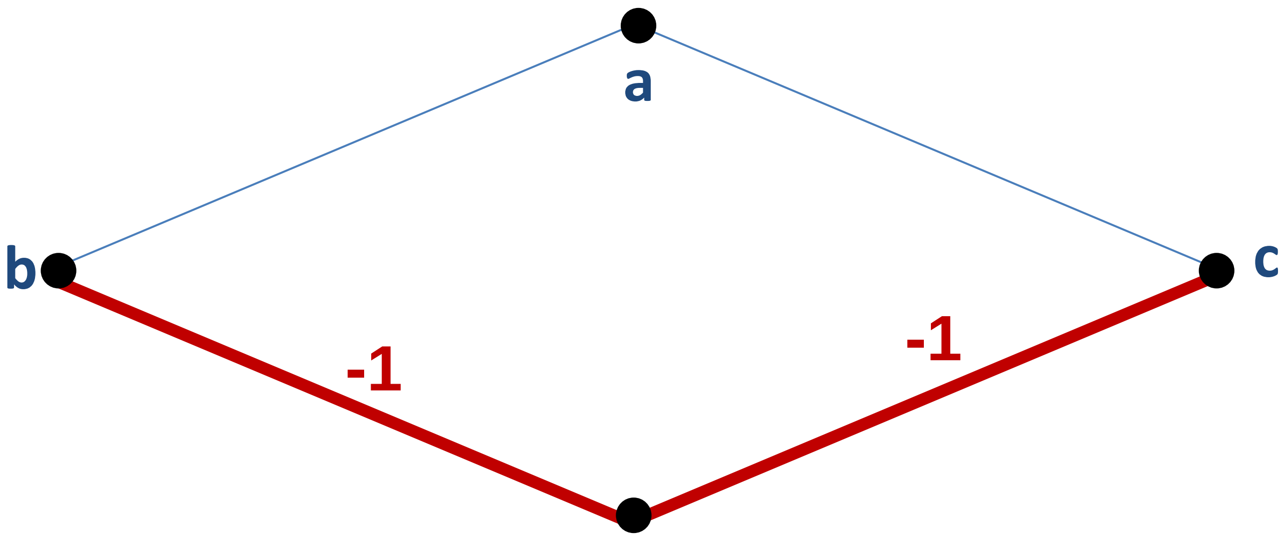
Exercise : Does the triangle inequality hold in the undirected case ? Are subpaths of shortest paths shortest ?

Can we solve undirected shortest path problems in the same way as directed ones ? Or reduce one to the other ?

Conservativeness

Def: (G,w) where G is a graph, w: E(G) → Z is *conservative*, if for every circuit C of G : $w(C) \geq 0$.

$\lambda(x,y) := \lambda_w(x,y) := \min\{w(P) : P \text{ path}\} =$
$\min\{w(P) : P\ \{x,y\}\text{-join}\}$



$\lambda(a,b) = \lambda(a,c) =$ **-1** ; $\lambda(b,c) =$ **-2** ; $\lambda(a,b) + \lambda(b,c) < \lambda(a,c)$

A shortest (a,c)-path is not shortest between a and b.

A Quick Proof of Seymour's theorem

Theorem: G bipartite, $w:E(G) \rightarrow \{-1,1\}$, (G,w) conservative $\Leftrightarrow$

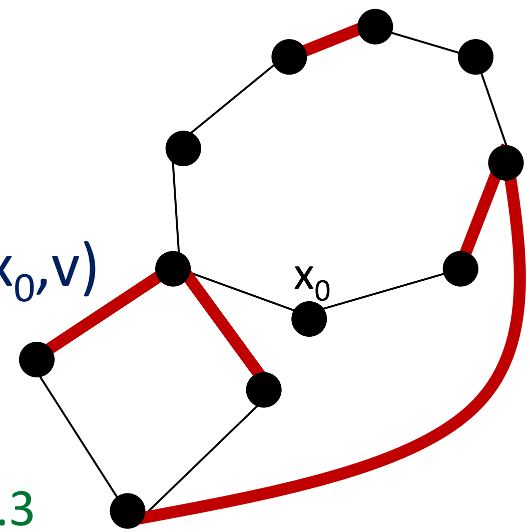
E_- can be covered by disj cuts meeting it in exactly one edge each.

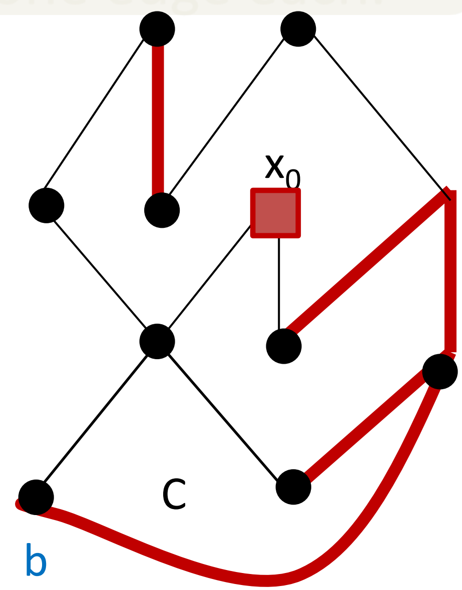
Proof (Sebő)

$x_0 \in V(G)$

Take $b \neq x_0$ such that

$\lambda_w(x_0,b)=\min_{v\in V(G)} \lambda_w(x_0,v)$





Claim 1 $|\, \delta(b) \cap E_- \,| = 1$

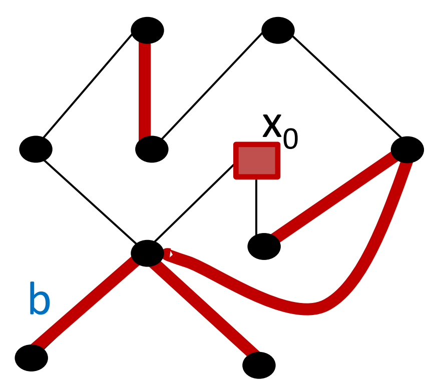
Exercise 5.3

Claim 2 : Switching on C, w(C)=0 :

a. Remains conservative

b. Distances don't change

Exercise 5.1



Claim 3: Contracting δ(b), a., (and b.) remain true!

Exercise 5.4

Cuts

Input : $G=(V,E)$, $c: E \to Q$
Output: Partition $\{X, Y\}$ of V that minimizes $\sum_{x\in X,\, y\in Y,\, xy\in E} c(xy)$

minimum cut : c non-negative $\in \mathscr{P}$

maximum cut : c non-positive $\mathscr{NP}$ - complete

Randomized 2-approx : Flip a coin !
2-approx : Derandomize !

Cuts

Short Summary

MIN CUT $\in \mathscr{P}$

MAX CUT $\mathscr{NP}$ - hard

Ford Fulkerson: algorithm and MFMC thm. (Improvments,analysis since then …) Menger’s theorems.

Goldberg-Tarjan : preflow push

Karger : uniform distribution on edges. Choose an edge, contract, stop if |V|=2.

Nagamochi-Ibarraki has been known (is maybe the derandomization of Karger)

NP-hard, even max | | see GJ.

In planar graphs = Chinese postman problem. $\in \mathscr{P}$

Exercise: why ? (Hint : a cut is max iff 1 on it and -1 else, Is conservative in the dual)

0.878-approx: Goemans-Williamson with Semidefinit Programming

To come :

Matchings,

Undirected Shortest Paths,

T-joins

…

Exercises to revise for the second course: series 3 and 6