Part B : TSP

1. **Classical TSP**

   s=t, General metric

2. **Graph metric**

   ear theorems `graph TSP' , s=t (S., Vygen) 2014

   Submodular functions, matroids

   matroid intersection and approx. of submod max

3. **General s,t path TSP**

   Zenklusen's 3/2 approx algorithm (April 2018)

Exercices series 6 Approximation : constant ratio

Optimal orders

TSP : s=t

Metric: triangle inequality, satisfied by reasonnable applications, without it: even approx is hard

s-t-Path Travelling Salesman Problem

INPUT : V «cities», s , t $\in$ V, c: $V \times V \to IR_+$ metric

OUTPUT: shortest s-t -Hamiltonian path

OPT(c)

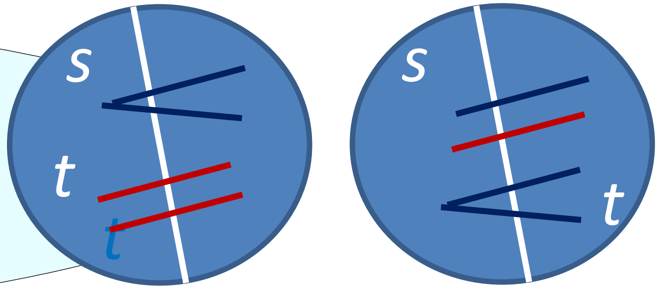
$P(V,s,t) = \{ x \in IR_+^E : x(\delta(W)) \geq 2, \emptyset \neq W \subset V$, s, t $\in$ W or $\notin$

1, if s,t separated by W

= on vertices (1 for s, t ; else 2)}

min $c^T x$

$OPT_{LP}(c)$



Approximation and Integrality ratio

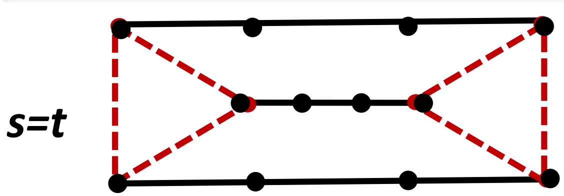
For a minimization problem

- *the approximation ratio is at most* ρ if for any input a solution of value at most ρOPT can be found in polynomial time.

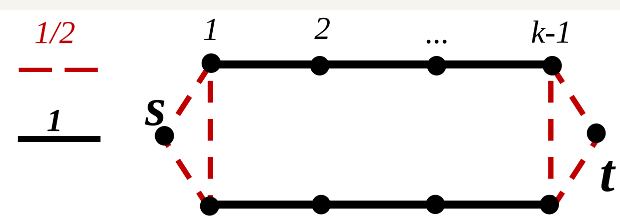
- *the integrality gap is at most* ρ if for any input

$$OPT \,/\, \mathrm{OPT}_{\mathrm{LP}} \le \rho \,.$$

Lower bound for integrality gap : graph metrics



$\geq \frac{4}{3}$



$\geq \frac{3}{2}$

Lower bound for approximation ratio: 123/122 NP-hard (Karpinsky Lampis, Schmied)

Famous Conjectures: integrality gap and approximation ratio $\leq \frac{4}{3}$ resp $\frac{3}{2}$

1. Classical TSP

s=t

TSP

INPUT : V cities, $c: V \times V \rightarrow IR_+$ metric

OUTPUT: shortest Hamiltonian circuit

Without it no constant ratio (easy from HAM)

NP-hard (Karp, 1972)

Christofides (1976)

Determine: a minimum weight spanning tree

Add : Add a minimum T_F - join J_F to make it Eulerian

Shortcut the Eulerian tour

A proof of ratio 2 and two proofs of $\frac{3}{2}$

Approximation ratio 2 : Double a min cost spanning tree F and shortcut.

Approximation ratio $\frac{3}{2}$: $F + J_F$, where $c(F) \leq OPT$, $c(J_F) \leq \frac{1}{2} OPT$, since connected, Eulerian $\Rightarrow$ has two disjoint T-joins for all T

$OPT_{LP} := \{\min c(x) : x \in IR_+^E,\ x(\delta(W)) \geq 2,$ for all $\emptyset \neq W \subset V$, = for vertices$\}$

Theorem (Wolsey '80, Cunningham 1984) G=(V,E) graph.

We find at most $\frac{3}{2} OPT_{LP}$ since $c(F) \leq OPT_{LP}$, $c(J_F) \leq \frac{1}{2} OPT_{LP}$

Proof. $x \in P$: $E[\mathcal{F}] \leq x$, $E[J_{\mathcal{F}}] \leq x/2$, $E[\mathcal{F} + J_{\mathcal{F}}] = E[\mathcal{F}] + E[J_{\mathcal{F}}]$

2. Classical TSP with graph metric, and min size Two-edge-connected spanning subgraph

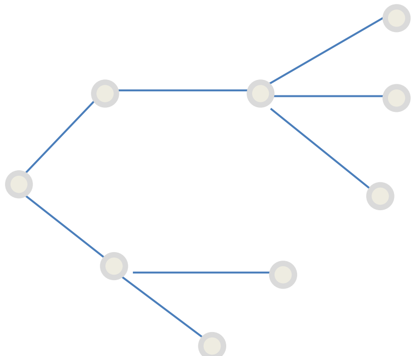
‘Network reliability’

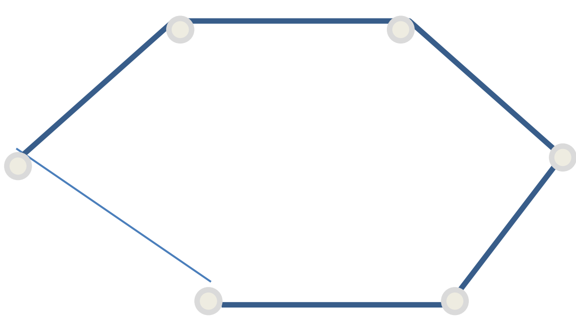
2-Edge Connected Spanning Subgraph, 2ECSS

graph-TSP, graph-TSP paths

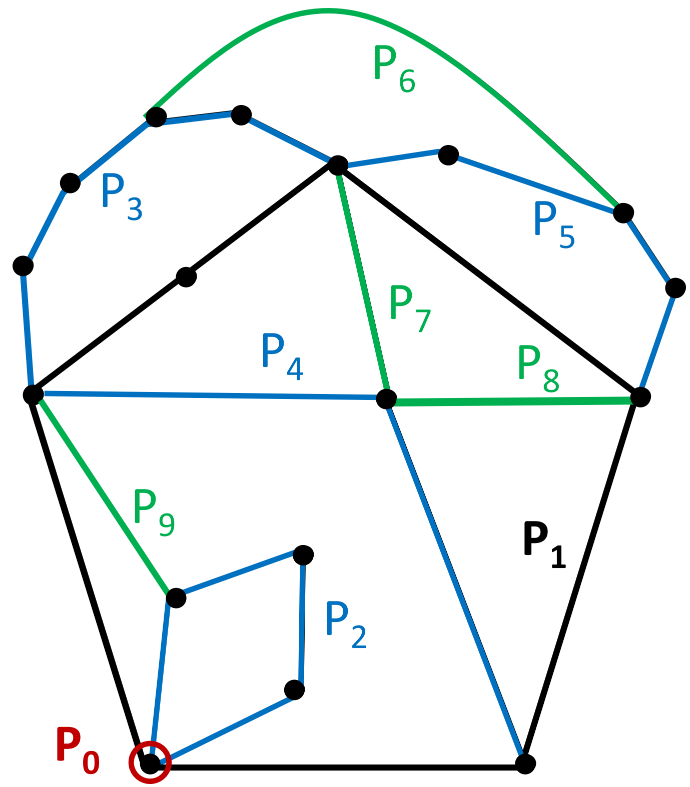
Minimum cardinality 2-edge-connected spanning subgraph .

Def: A graph G=(V,E) is *2-edge-connected*,
if (V, E \ e) is connected for all e $\in$ E .





Ears



$G = P_0 + P_1 + P_2 + \ldots + P_k$

2-approx for 2ECSS: delete 1-ears!

The longer the ears, the smaller the quotient n. of edges / vertices

Exploited by Cheriyan, S., Szigeti (1998) for a 17/12 -approx

Matroids

$C = (S, \mathcal{F})$, $\mathcal{F} \subseteq \mathcal{P}(S)$ is a *matroid* if

(i) $\emptyset \in \mathcal{F}$

that is, $\mathcal{F} \neq \emptyset$

(ii) $F \in \mathcal{F}$, $F' \subseteq F \Rightarrow F' \in \mathcal{F}$

(iii) $F_1, F_2 \in \mathcal{F}$, $|F_1| < |F_2| \Rightarrow \exists\, e \in F_2 \setminus F_1 :\ F_1 \cup \{e\} \in \mathcal{F}$

$F \in \mathcal{F}$ is called an *independent set.*

The *rank function* of M is

$r : 2^S \to \mathbb{N}$ defined as $r(X) := \max\{|F| : F \subseteq X, F \in \mathcal{F}\}$

Examples: Forests in graphs, Linearly independent sets , partition matr.

Matroid Intersection Theorem

$M = (S, \mathbf{F})$ matroid

$\mathrm{conv}\,(\chi_F : F \in \mathbf{F}) = \{x \in \mathrm{IR}^S : x(A) \le r(A) \text{ for all } A \subseteq S\}$ (Edmonds)

maximize $\{\,|F| : F \in \mathbf{F}_1 \cap \mathbf{F}_2\,\} = ?$

$\max \{ 1^T x : x(A) \le r_i(A) \ (i=1, 2) \text{ for all } A \subseteq S \}$

Theorem (Edmonds 1979): $\max_{F \in \mathbf{F}_1 \cap \mathbf{F}_2} |F| = \min_{X \subseteq S} r_1(X) + r_2(S \setminus X)$

Polynomial algorithm for both and also if weights are given.

Matroid Intersection Algorithm

Generalization of bipartite matching
(of the alternating paths in the « Hungarian method »)

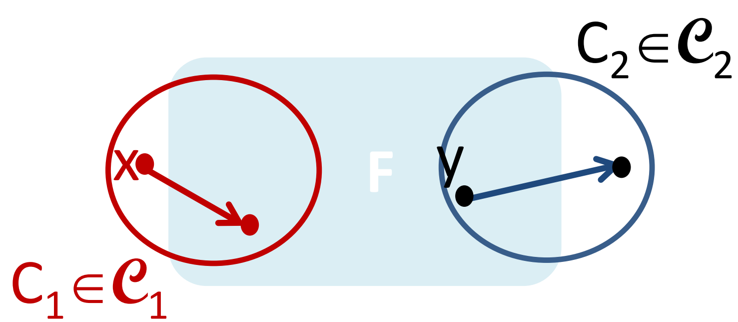
Proof of $\geq$: that is, there is F and X with $|F| = r_1(X) + r_2(S \setminus X)$.

We prove that the following algorithm terminates with such an F and X.

What is the INPUT ? → ORACLE - rank, independence, etc

0.) Let : $F \in \mathcal{F}_1 \cap \mathcal{F}_2$ maximal by inclusion (greedily)

1.) Define arcs from unique cycles :



Approx for submod max mon, size k, f(0)=0,

Algorithm (for sets of size k): (Nemhauser, Wolsey) Having X already,

WHILE $|X| < k$ choose x that maximizes

$$f(X \cup \{x\}) - f(X)$$

Lemma : $f(X \cup \{x\}) - f(X) \geq (f(OPT) - f(X)) / k$

Proof: Since mon: $f(OPT) \leq f(OPT \cup X) \leq$

$$\leq f(X) + k (f(X \cup \{x\}) - f(X))$$

Let X^i be what we found until step i. Then

$f(X^k) - f(X^{k-1}) \geq f(OPT) / k - f(X^{k-1}) / k$, so

$$f(X^k) \geq f(OPT) / k + (1 - 1/k) f(X^{k-1})$$

$$f(X^k) \geq f(OPT) (1 - (1 - 1/k)^k) \geq (1-1/e) f(OPT)$$