Introduction to Combinatorial Optimization

I. Bird's Eyes View and Tour d'horizon

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support for a course at XRCE

What is combinatorial optimization ?

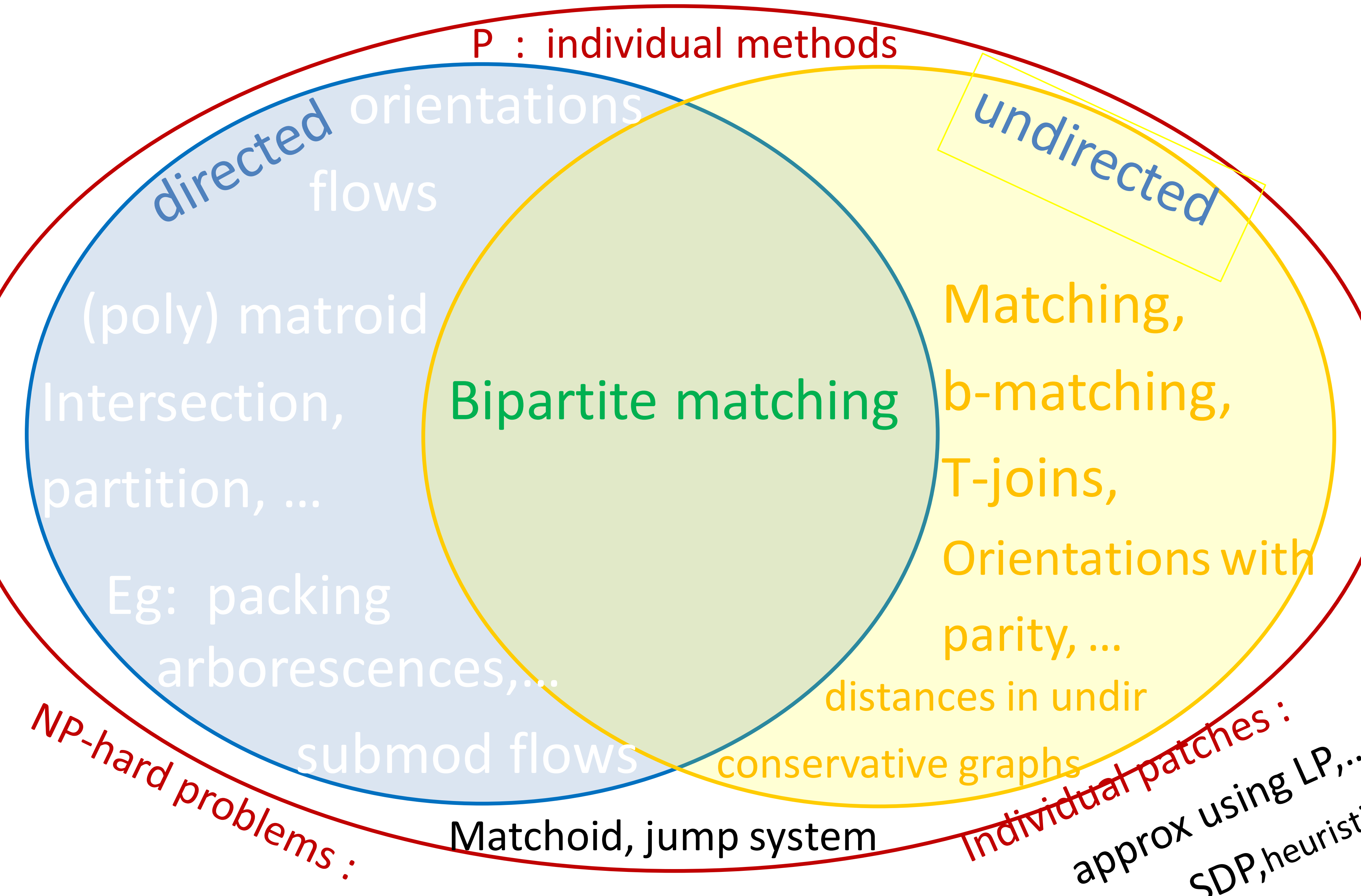
Given $f : 2^S \rightarrow \mathbb{R}$, find $X \subseteq S$ that minimizes f,

that is, such that $f(X) \leq f(Y)$ for all $Y \subseteq S$.

TOO GENERAL, NOT EXACT, IRRELEVANT, NOT TRUE, BORING,...

we have to go through more specific examples !

Directions from bird's eyes …



Tour d’horizon: 6 fundamental benchmarks

Bin packing (cutting stock, scheduling)

Shortest paths (trafic, PERT)

Matching (mariages)

Tours (travelling, postman)

Cuts (routing, clustering)

Submodular functions (machine learning)

Tour d'horizon I : Bin packing

BIN PACKING

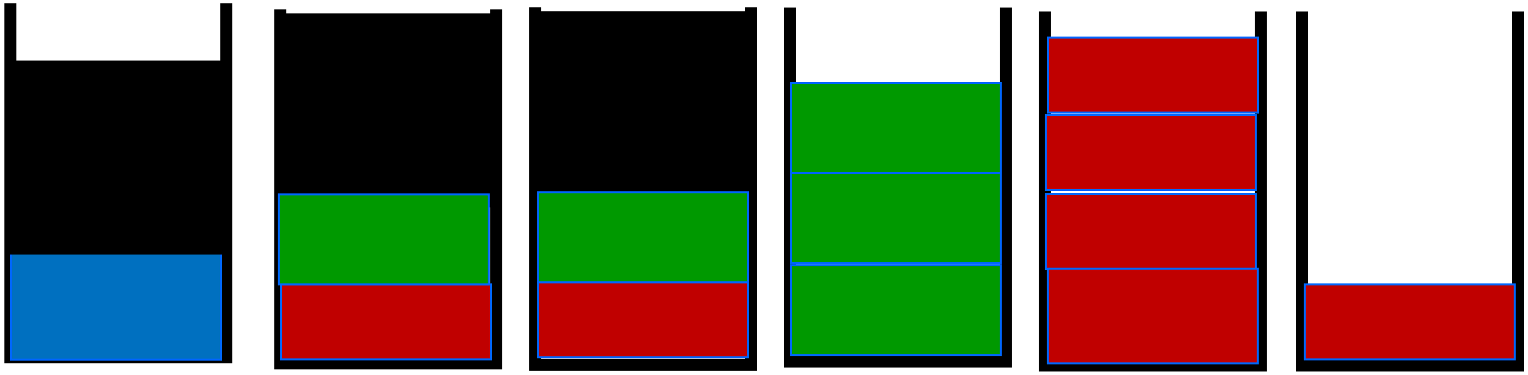
Input : $0 \leq s_1, \ldots, s_n \leq 1$ item *sizes*,

Task : Minimize the number of bins (capacity 1)

PARTITION : Are 2 bins enough ?

NP-hard

Tour d'horizon I : Bin packing cont'd (example)



Tour d'horizon I: cont'd - bin packing (heuristics)

BIN PACKING

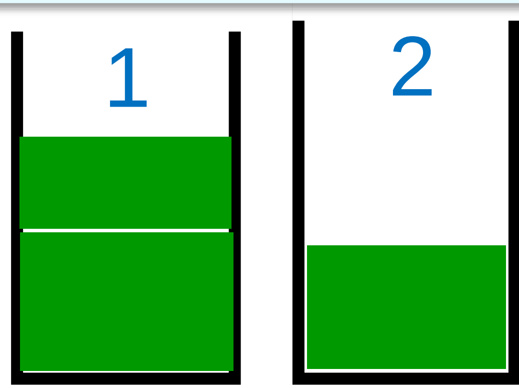
Input : $0 \le s_1, \ldots, s_n \le 1$ item *sizes*,

Task : Minimize the number of bins (capacity 1)

Heuristics :	NF,	FF,	NFD,	FFD
	2	17/10		11/9 OPT +1

Proposition : NF $\le$ 2 OPT

Proof :



> 1 ...

OPT $\ge \lceil \text{size} \rceil$

NF < 2 size $+1$

Tour d'horizon I : cont'd - bin packing (patterns)

INPUT : $0 \leq s_1, \ldots, s_d \leq 1$ item *sizes*,
$b_1, \ldots, b_d \in$ IN item *multiplicities*

Pack them to a min number of bins of capacity 1

pattern : $p \in \mathbb{Z}_+^d$ such that $p_1 s_1 + \ldots + p_d s_d \leq 1$

P : = the columns are the incl max patterns

Tour d'horizon I : Bin packing cont'd (examples)

$d=3$

$s=(1/2, 1/3, 1/5)$ $b=(1, 2, 4)$

$$P = \begin{pmatrix} 2 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 5 & 0 & 2 & 1 & 3 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

SIZE = 59/30 LP = ½ + 2/3 + 4/5 = 59/30

Exercise : OPT= 2 or 3 ?

Tour d'horizon I : cont'd - bin packing (LP)

pattern : $p \in \mathbb{Z}_+^d$ such that $p_1 s_1 + \ldots + p_d s_d \leq 1$

Gilmore-Gomory LP :

$$\mathbf{Px \geq b} \quad (P \in \mathbb{Z}_+^{d \times big}) \qquad \mathbf{yP \leq 1}$$

$$\mathbf{x \geq 0} \qquad \mathbf{y \geq 0}$$

$$\min\ 1^T x \quad (b \in \mathbb{Z}_+^d) \quad = \quad \max 1^T y$$

Conjecture (Scheithauer,Terno): $OPT \leq \lceil LP \rceil + 1$

(not better for restricted patterns)

Tour d'horizon II: Paths in Graphs

Directed, nonnegative weights (Dijkstra)

Directed -1 weights NP-hard (HAM)

Conservative (no circuit of neg total weight): P

Undirected: nonnegative ? , -1 ?, conservative ?

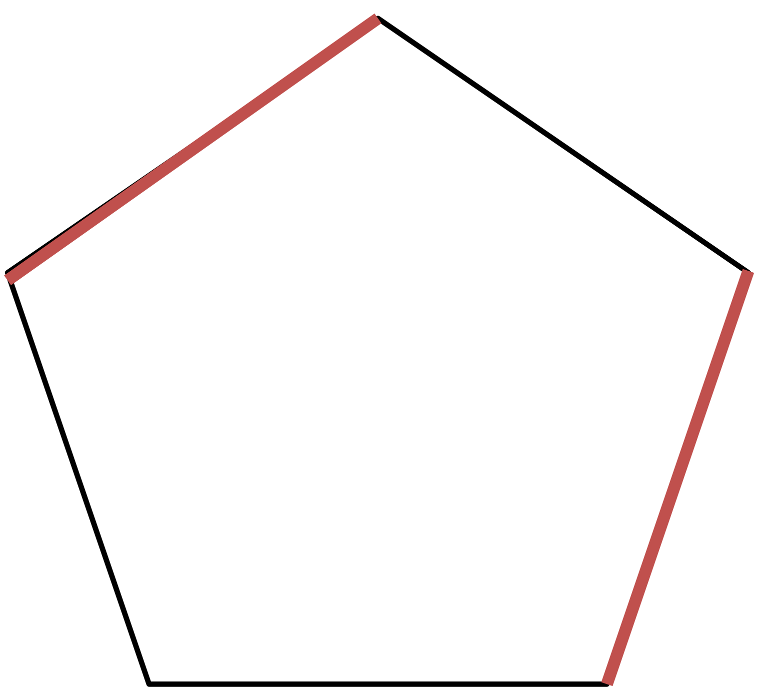
Exercise :   Does the triangle inequality hold in the undirected case ? Are subpaths of shortest paths shortest ?

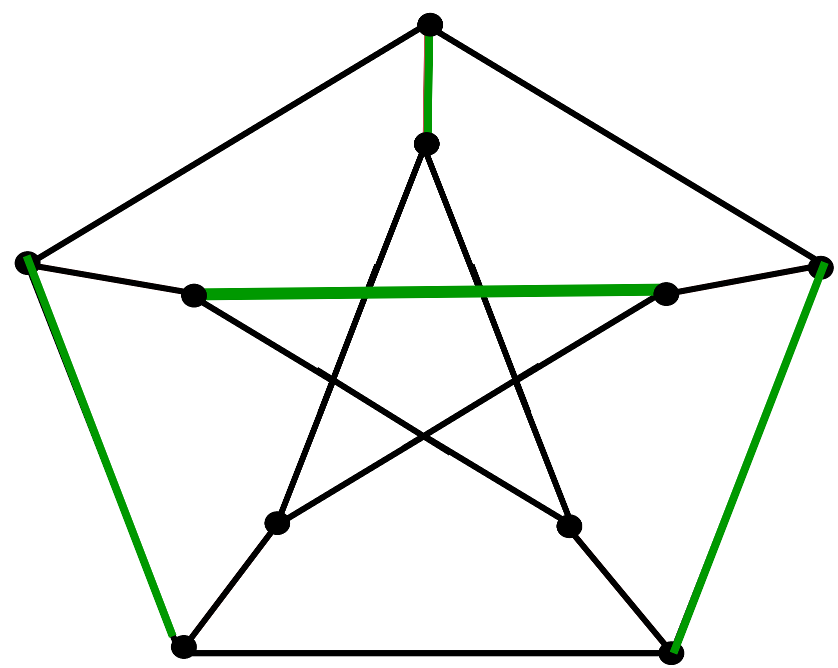
Tour d'horizon III : matching

INPUT : G=(V,E) graph.

matching : a set $M \subseteq E$ of vertex-disjoint edges

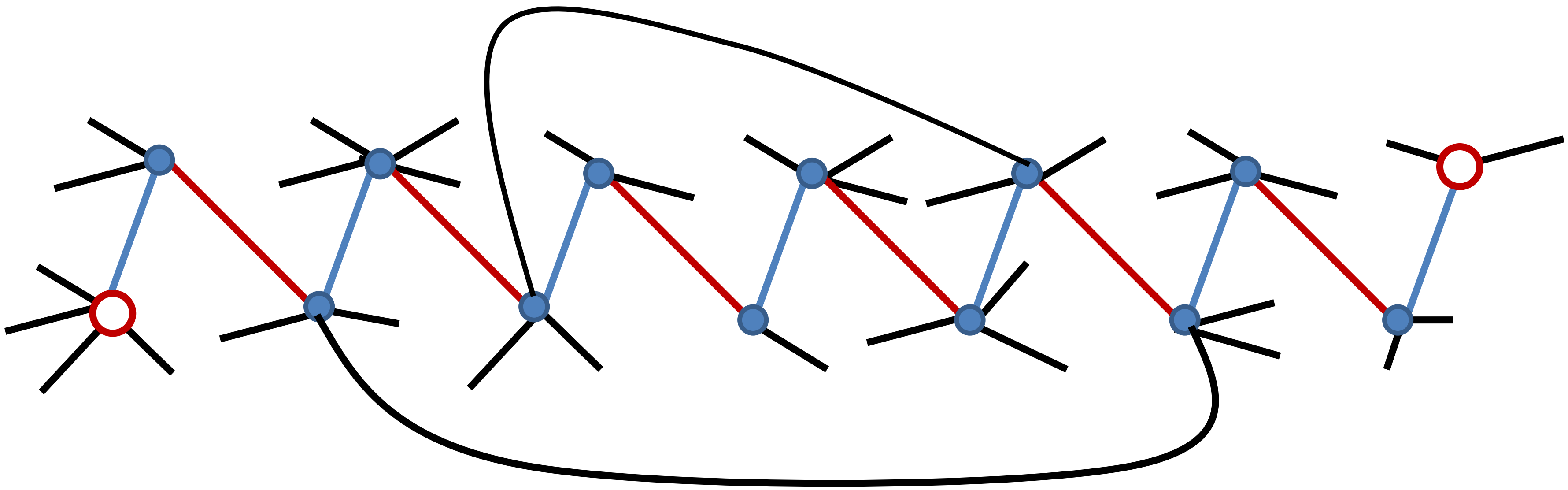
TASK : Find a matching of maximum size





Tour d'horizon III : matching cont'd (augmenting paths)

augmenting path with respect to matching M : path alternating between M and E \ M with the 2 endpoints uncovered by M



Proposition (Berge) : G graph, M matching in G.

M is a maximum matching in G iff there is no augmenting path

Tour d'horizon III : cont'd - matching (cover)

matching : M set of vertex-disjoint edges

Max |M| : υ

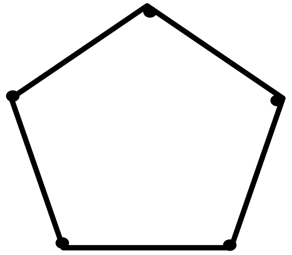
vertex cover : T set of vertices

so that G-T has no edges

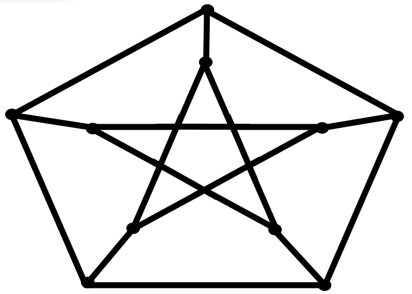
Min |T| : τ

$\upsilon \leq \tau$

Tour d'horizon III : cont'd - matching (minmax)



Theorem (Kőnig) : If G is bipartite $\upsilon \;=\; \tau$



≤ is *'the easy part';* ≥ is to be proved

1st Proof : If for some $v \in V$: $\nu(G-v) = \nu(G) - 1$

DONE !

If uv E then either u or v satisfy this condition !

Q.E.D.

Tour d'horizon III : cont'd - matching (submod)

Thm (Kőnig-Hall) : Let G= (A,B,E) be a bipartite graph
Then $\upsilon = \min \{ |A| - (|X| - |N(X)|) : X \subseteq A \}$

2nd Proof : $\delta(X)$ is *supermodular* .
Call X *tight*, if $\delta(X)$ is max =: ≥ 0

Claim: If X, Y are tight, $X \cap Y$, $X \cup Y$ too.

Are there disjoint tight sets ? How does the family of inclusionwise min tight sets look like ? How does an inclusionwise min graph of given δ_G look like ?

Exercise : Prove Kőnig and Kőnig-Hall from one another

Tour d'horizon III : matching cont'd (LP&RP)

VERTEX COVER for G=(V,E) bipartite

$x \in \mathbb{R}^V$:

$x_i + x_j \geq 1$, $\forall$ $ij \in E$

$x \geq 0$

MATCHING POLYTOPE for G=(V,E) bipartite

$x \in \mathbb{R}^E$:

$x(\delta(v)) \leq 1$, $\forall$ $v \in V$

$x \geq 0$

Integrality (TU+Cramer, no odd circuit)

$(x_{ij})_{n \times n}$ randomized algorithm, method of variables

Tour d'horizon III: matching cont'd (algorithms)

Proposition (Berge) : G bipartite, M matching .
M is a maximum matching iff there is no augmenting path

Algorithms for bipartite graphs: paths in digraphs;
Algorithmic proof of Kőnig-Hall ; Integer Linear Prog;
Ellipsoid method, method of variables

For non-bipartite ? Is it useful ?

Tour d'horizon III: matching cont'd (application)

SCHEDULING IDENTICAL JOBS ON 2 IDENTICAL MACHINES

Input: Partially ordered set of tasks of unit length.
Output: Schedule of min completion time T

Thm : (Fujii & als) : $T = n - \upsilon\,(G_{input})$

Solutions for max (weighted) matchings:
with Edmonds' algorithm (1965)
Grötschel, Lovász, Schrijver
with Padberg-Rao (1979)

Tour d'horizon IV: CPP versus TSP

Input : $G=(V,E)$, $w: E \to \mathbb{Z}_+$

Task : minimize the total weight :

CHINESE POSTMAN:

- of a closed walk through all edges

Graphic TRAVELLING SALESMAN:

- of a closed walk through all the vertices

Exercise: $G=(V,E)$ connected, $w: E \to \mathbb{Z}_+$, $T \subseteq$ even.
Find a minimum weight subgraph F with $d_F(v)$ odd $\Leftrightarrow v \in T$ in polynomial time. (Hint: use min weight perfect matching in (T,w-distances))

Tour d'horizon IV: CPP versus TSP cont'd (metric)

TRAVELLING SALESMAN: once through every vertex

Metric '': + w satisfies the triangle inequality

Theorem: (Christofides) Heuristic for graphic & for metric TSP which provides at most 3/2 OPT

Proof. Heuristic: Min weight spanning tree F + with $T = \{v : d_F(v) \text{ is odd} \}$ a minimum weight T-join.

Conjecture : 4/3 is also true in these cases .

Tour d'horizon V : Cuts

Input : $G=(V,E)$, $c: E \to \mathbb{Z}$
Output: Partition $\{X, Y\}$ of V that minimizes $\sum_{x\in X,\ y\in Y,\ xy\in E} c(xy)$

minimum cut : c non-negative $\in \mathcal{P}$

maximum cut : c non-positive $\mathcal{NP}$ - complete

Randomized 2-approx : Flip a coin !
2-approx : Derandomize !

Tour d'horizon V cont'd: Cuts

MIN CUT $\in \mathscr{P}$

Ford Fulkerson: algorithm and
Max Flow Min Cut.
(Improvments, analysis: Dinits, Frank-Tardos …)

Menger's theorems.

Goldberg-Tarjan : preflow push

Karger : uniform distribution on edges. Choose an edge, contract. When |V|=2 stop. Simple, beautiful analysis

Nagamochi-Ibarraki derandomization

MAX CUT
$\mathscr{NP}$ - hard

NP-hard, see GJ.

In planar graphs = Chinese postman problem. $\in \mathscr{P}$

0.878-approx: Goemans-Williamson with Semidefinit Programming

Tour d'horizon VI : Submodular Functions

Def : $f : 2^S \rightarrow \mathbb{R}$ is *submodular on 2^S*, if

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$$

submodular $\Leftrightarrow \forall\ A \subseteq B\ ,\ x \in S$:

$$f(A \cup \{x\}) - f(A) \geq f(B \cup \{x\}) - f(B)$$

1.) occurs often   2.) useful   3.) can be played with

MIN $\in \mathscr{P}$

MAX $\mathscr{NP}$ - hard

versions: For machine learning, f(0)=0, mon, size k

Examples, special cases, connexions

Total « Information in » a subset of random variables

Probability of the product of a subset of events

Vector ranks in any vector space

Minus the number of components of a set of edges

Maximum size of a spanning tree

For k $\in \mathbb{N}$ and finite set S : min { k, the size of a subset }

Many essential is in matroids:

Def: M=(S,r) *matroid*: r $(\emptyset)$ =0,r monoton&submodular, r(s)=1,(s $\in \mathbb{N}$)

Approx for submod max mon, size k, f(0)=0

Algorithm (for sets of size k): Having X already,
WHILE |X|< k we choose x that maximizes
$f(X \cup \{x\}) - f(X)$

Lemma : $f(X \cup \{x\}) - f(X) \geq (f(OPT) – f(X)) /k$

Proof: Since mon. $f(OPT) \leq f(OPT \cup X) \leq$
$\leq f(X) + k (f(X \cup \{x\}) - f(X))$

Let X^i be what we find in step i. Then $f(X^k) - f(X^{k-1}) \geq$
$\geq f(OPT) / k - f(X^{k-1}) / k$, so

$$f(X^k) \geq f(OPT) / k + (1 – 1/k) f(X^{k-1})$$

$$f(X^k) \geq f(OPT) (1 - (1 – 1/k)^k) \geq (1-1/e) f(OPT)$$

Matroids

$M = (S, \mathcal{F})$ $\quad \mathcal{F} \subseteq \mathcal{P}(S)$

matroïde, si

(i) $\emptyset \in \mathcal{F}$ $\quad (\Leftrightarrow \mathcal{F} \neq \emptyset)$

(ii) $F \in \mathcal{F},\ F' \subseteq F \Rightarrow F' \in \mathcal{F}$

(iii) $F_1, F_2 \in \mathcal{F},\ |F_1| < |F_2|$

$\Rightarrow \exists\ x \in F_2 \setminus F_1 :$

$F_1 \cup \{x\} \in \mathcal{F}$

Prove the equivalence with the other def !

Exemples:

1.) $S \subseteq GF(q)^n$ fini

$\mathcal{F} := \{ F \subseteq S : \text{lin indép} \}$

M(G):=

2. G graphe $S = E(G)$ $\mathcal{F}$ = forêts

normal exemple?

3. $U_{n,r}$

$|S| = n$, $\mathcal{F} := \{ F \subseteq S : |F| \leq r \}$

3. $U_{n,r}$

$|S| = n, \quad \mathcal{F} := \{F \subseteq S : |F| \leq r\}$

4.) Fact: (some Nash-Williams)

$M_1 \bowtie M_2 \qquad (M_1 = (S, \mathcal{F}_1), \; M_2 = (S, \mathcal{F}_2))$

$\| $

$\{F_1 \cup F_2 : F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\}$

5.) [illegible] transversal:

$G = (A, B, E) \qquad S := A$

$\mathcal{F} := \{F \subseteq S : F \text{ couplable}\}$

Circuits

Def: $\mathcal{C}$ famille des dépendants minimaux — circuits

Fact: $C_1 \neq C_2 \in \mathcal{C}$, $x \in C_1 \cap C_2$ $\Rightarrow \exists C \in \mathcal{C} : C \subseteq C_1 \cup C_2 \setminus x$

Proof: $r(C_1) + r(C_2) \geq r(C_1 \cap C_2) + r(C_1 \cup C_2)$

$r(C_1) + r(C_2) = |C_1| + |C_2| - 2$, $\quad r(C_1 \cap C_2) = |C_1 \cap C_2|$, $\quad r(C_1 \cup C_2) \leq |C_1 \cup C_2| - 2$

$r(C_1 \cup C_2 \setminus x) \leq |C_1 \cup C_2 \setminus x|$

Prove the other direction !

Bases

B is a *base* if $B \in \mathscr{F}$, $|B| = r(S)$.

Set of bases : $\mathscr{B}$

Fact : $\forall\ B_1 , B_2 \in \mathscr{B}$, $\forall\ x \in B_1 \setminus B_2$
$\exists\ y \in B_2 \setminus B_1$: $(B_1 \setminus x) \cup \{y\} \in \mathscr{B}$

Proposition: $\mathscr{B} \neq \emptyset$ is the set of bases of a matr
$\Leftrightarrow$ the Fact holds.

So this def is also equivalent !

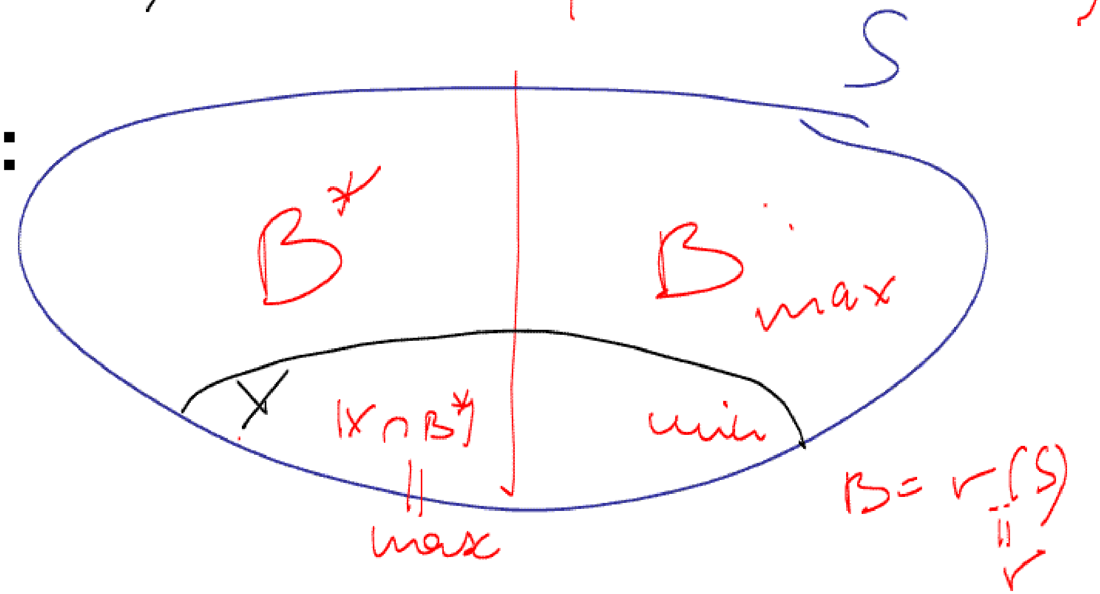
Matroid dual

Def: dual : $M^* = (S, \mathcal{B}^*)$ dual de $M = (S, \mathcal{B})$

$\mathcal{B}^* = \{ S \setminus B : B \in \mathcal{B} \}$

Fact: $r^*(X) = |X| - (r(S) - r(S \setminus X))$

Proof:



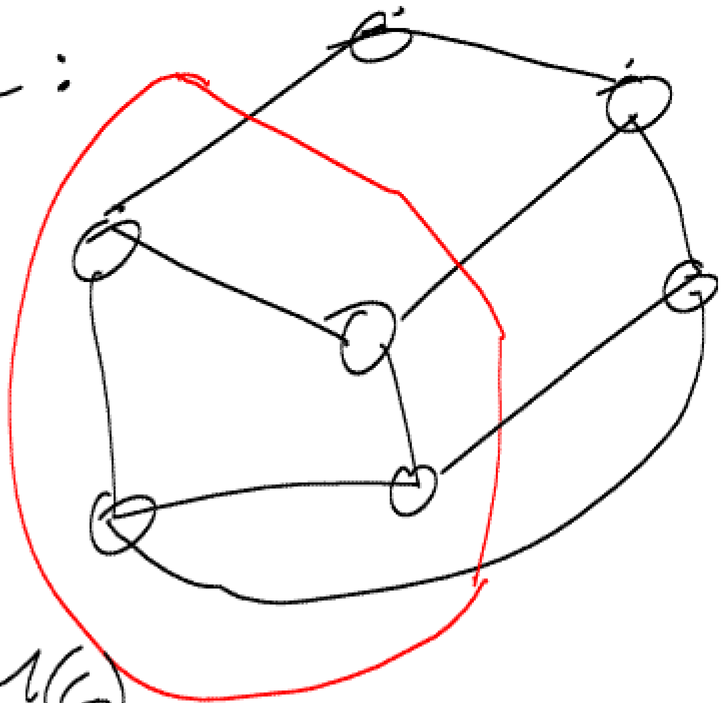
Def: cocircuit
coupe d'un matroïde : circ. du dual

Planarity and Duality

graphe planaire :

circuit de G =
circuit de M(G)

coupe min par
inclusion
= cocircuit de M(G)



Proposition : F is a spanning tree ⇔
E \ F is a spanning tree of the dual graph

Euler's formula : $n - 1 + f - 1 = m$

Greedy alg for max weight indep

Algorithme glouton (AG) : Si $x_1, \dots, x_i$ ont été choisis soit x_{i+1} t.q. $\{x_1, \dots, x_{i+1}\} \in \mathcal{H}$, $w(x_{i+1})$ max.

<u>Théorème</u> : $H = (S, \mathcal{H})$ héréditaire

$\forall w \, (\in \mathbb{N})$ AG trouve l'opt

$\iff$ H est un matroïde

$\Rightarrow$ 

Preuve : TROUVE : $c(x_1) \geq \dots \geq c(x_i) \geq \dots$

$\quad\quad\quad\quad\quad\quad\quad\quad \text{IV} \quad\quad\quad\quad\quad \wedge$

MAX : $c(x'_1) \geq \dots \geq c(x'_i)$

Par l'axiome des indep on aurait pu choisir en x_i : plus gros

Let us make it more complicated !

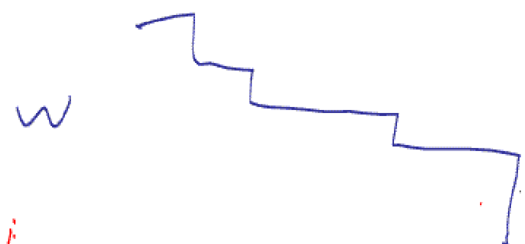
Thm (Edmonds) : $M = (S, \mathcal{F})$ mat.

$$\left\{ x : \begin{array}{l} x(A) \le r(A) \\ x \ge 0 \end{array} \right\} = \text{conv}(\chi_F : F \in \mathcal{F})$$

$$x(A) = \sum_{a \in A} x_a$$

<ins>Preuve</ins> : $w_1 \ge \dots \ge w_n$

$U_i = \{1, \dots, i\}$



Le F qu'on trouve : $|F \cap U_i| = r(U_i)$

$$\left.\begin{array}{l} w(F) = (w_1 - w_2)\,|F \cap U_1| + \\ + (w_2 - w_3)\,|F \cap U_2| + \dots \\ + \; w_n\,|F \cap U_n| \end{array}\right\} \text{id. dual}$$

The inverse of the duality theorem

Thm (Edmonds): $M = (S, \mathcal{F})$ matroïde

$\operatorname{conv}(\chi_F : F \in \mathcal{F}) =$

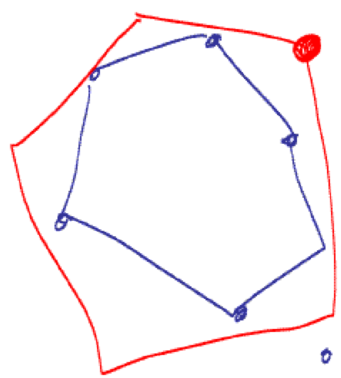
$= \{x \in \mathbb{R}^S : x(A) \leq r(A),\ x \geq 0\}$

$\subseteq$: clear !

Montrer que $\forall w$

$$\max_{x \in \text{gauche}} w^T x = \max_{x \in \text{droite}} w^T x$$

SUFFIT :



cas contraire

$\exists\ x_0 \in$ droite $\setminus$ gauche

$\{x : c^T x = b\}$ $\Leftarrow$ hyperplan séparateur

qui sépare x_0 de gauche

Farkas' Lemma

$c^T x_0 > b$

$c^T x \leq b \quad \forall x \in$ gauche

Intersection des matroïdes (S,r_1) et (S,r_2)
Edmonds (1979)

$\mathrm{conv}(\chi_F : F \in \mathcal{F}_1 \cap \mathcal{F}_2)$

$\overset{?}{=} \{x : x(A) \le r_1(A),\ x(A) \le r_2(A)\ \forall A \subseteq S\}$

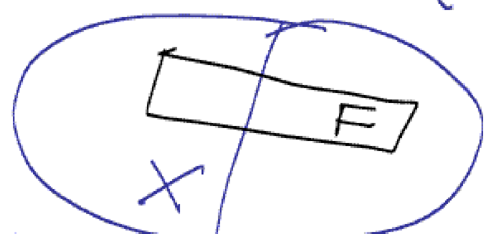
$\subseteq$ triv

pour 1 ? :

Théorème de $\cap$

$$\max_{F \in \mathcal{F}_1 \cap \mathcal{F}_2} |F| = \min_{X \subseteq S} r_1(X) + r_2(S \setminus X)$$

Proof : $\le$



$F \in \mathcal{F}_1 \cap \mathcal{F}_2$: $|F| = |F \cap X| + |F \setminus X| \le$

$\le r_1(X) + r_2(S \setminus X)$

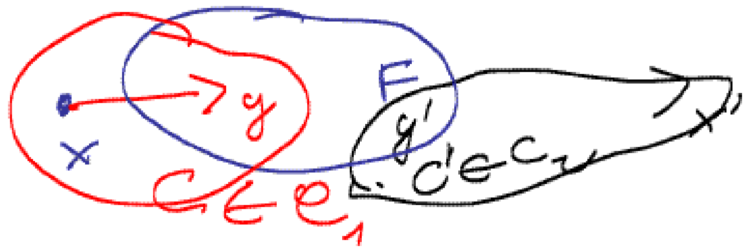
Exo : 2 arbres disj bases $\quad M_1 = (S, \mathcal{B}_1) \quad M_2 = M_1^*$

Algorithme d'intersection

Preuve : (algorithme)

0.) Soit $F \in \mathcal{F}_1 \cap \mathcal{F}_2$ max par inclusion

1.) cycles uniques



2.) Chercher un chemin S, T

S ~~sources~~ $:= \{x \in S \setminus F : F \cup x \in \mathcal{F}_2\}$, T puits

si S ou T = $\emptyset$?

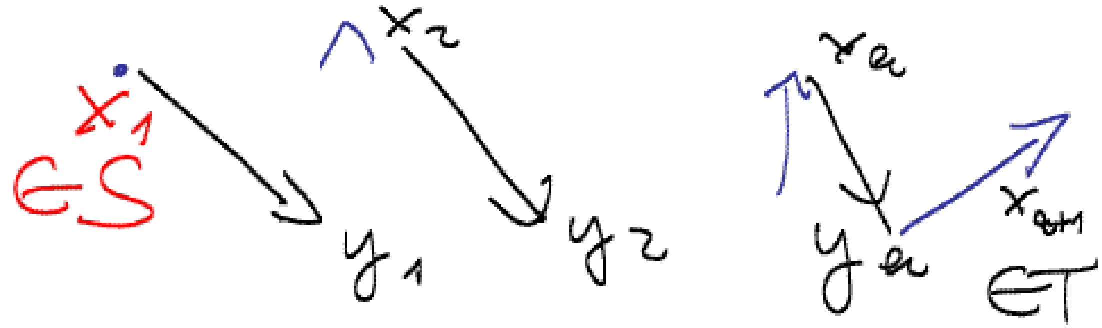
3.) - S'il n'y en a pas X : accessible de S

si $x \in X \setminus F$: $F \cup x \notin \mathcal{F}_1$ (car pas puits) $\Rightarrow r(X) = |F|$

- S'il y en a, P : $F' = (F \setminus P) \cup (P \setminus F)$

What remains: If $P=\{x_1, y_1, x_2, \dots x_k, y_k, x_{k+1}\}$ is a chordless path, then $F \,\Delta\, P \in \mathscr{F}_1 \cap \mathscr{F}_2$

Lemma: $M = (S, \mathscr{F})$



$F \in \mathscr{F}$

$x_i \notin F$, $y_i \in F$ $(i = 1, \dots, a)$

y_i est dans le cycle unique de $F \cup x_i$

mais y_j $j > i$ n'y est pas.

Alors $F \setminus \{y_1, \dots, y_a\} \cup \{x_1, \dots, x_a\} \in \mathscr{F}$

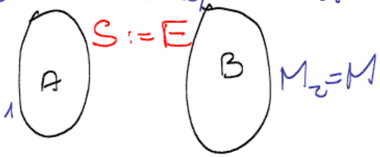
Preuve : Echanger x_a, y_a et rec. Appliquer à $F \cup \{x_{a+1}\}$

Corollaries

Corollaire (Rado) 1 : $G=(A,B,E)$ avec un matroïde $M=(B,\mathcal{F})$.

$$\max_{\substack{X\subseteq A\\ X \text{ couplable à un indépendant de } B}} |X| = \min_{X\subseteq A} r(\Gamma(X)) + |A\setminus X|$$

Preuve : $M_1 : U_{\delta(v),1}$

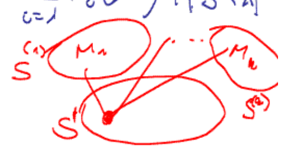


Cor 2 : $\exists$ matching dans A à un indép de M $\iff r(\Gamma(X)) \geq |X| \quad \forall X\subseteq A$.

3 Fonction de rang de $M_1 \vee \dots \vee M_k$:

$S'\subseteq S$: $r_\vee(S') = \min_{X\subseteq S'} \sum_{i=1}^{k} r_i(X) + |S'\setminus X|$

Preuve : avec Rado



Pour $r_i = r\ \forall i$, $r_\vee = kr$

4 Corollaire (Nash-Williams) : Dans un matroïde $\exists$ k bases disjointes

$\iff k\,(r(S) - r(X)) \leq |S\setminus X| \quad \forall X\subseteq S$

Remarque : suffit pour fermés.

5 Corollaire (Nash-Williams) : G graphe

G possède k arbres couvrants disjoints

$\forall$ partition $\mathcal{P}$ de sommets $\exists \geq (k-1)|\mathcal{P}|$ arêtes entre les classes de $\mathcal{P}$.

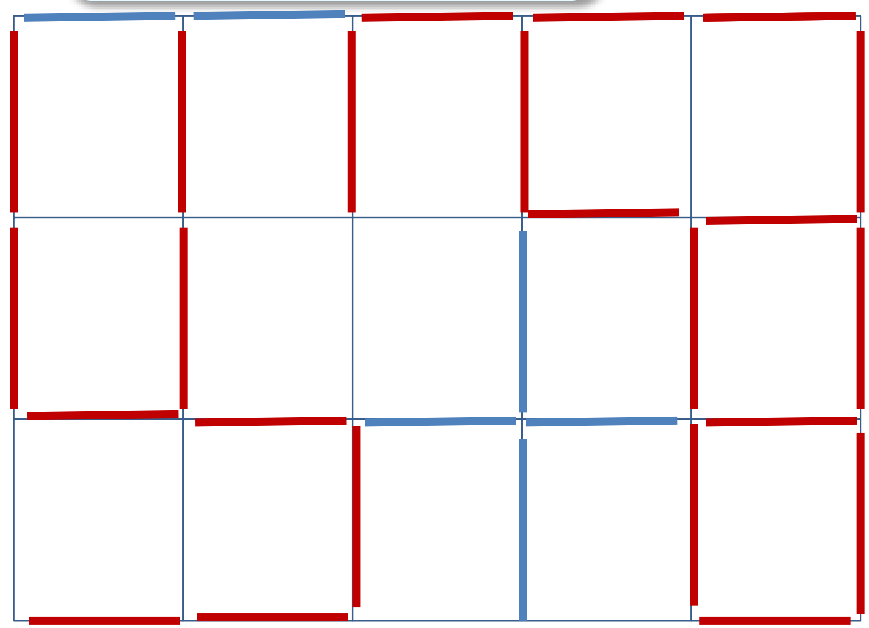
[illegible]

Tour d'horizon VI : submodular (examples)

minus deficit, d(X) , total information of a set of events, ...
minus number of components of a set of edges, vector rank, ...

MIN CUT

MAX CUT



Input : G=(V,E), w: E → [0,1]

Task : maximize

$c(V,L) + w(L) : L \subseteq E$

Supermodular

3/4 1/4

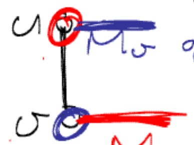
Particular submodular function minimization solved efficiently by Anglès d'Auriac, Iglói, Preissmann, S. (2001)

Tutte's theorem

5. Tutte's theorem, Edmonds' algorithm

Exo 1 :

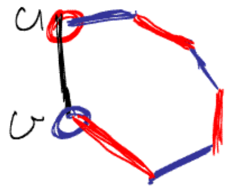
Exo 2 :



$M_u \cup M_v$ contient un (u,v) path !

Kőnig : $\tau(G) \le \tau(G-u)+1 \overset{récurr.}{=} \nu(G-u)+1 = \nu(G)$

Exo 3 :



min dans G/P

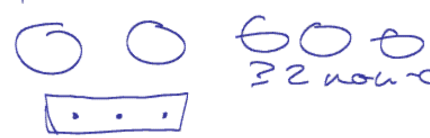
≤ min dans G

5...

Exo 4 : min sommets non-couverts

= max $q(X)-|X|$

≥ trivial :



= – si $\nu(G-u) < \nu(G)$:

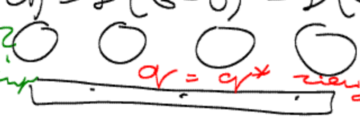
le nombre de sommets non-couv a augmenté (de 1) en supprimant u :

X_u Tutte-set de G−u

$X_u \cup \{u\}$ – " – de G

– si $\nu(G-u) = \nu(G-v) = \nu(G)$

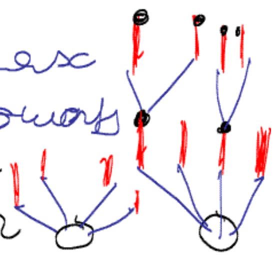
Où est C* ? dans comp imp



q = q* rien ne change !

Edmonds' algorithm

Algorithme



1. Construire un arbre max à partir de sommets non-couverts

pair ○ • : non-couv et à dist pair

impair :

2. Si deux sommets pairs sont adjacents

a.) entre 2 comp différentes ✓

b.) même composante Exo 3

CONTRACT, GOTO 1.

3. S'il n'y a pas d'arêtes entre sommets pairs FINI, et soit X := ensemble de sommets impairs

Thm: X est un ensemble de Tutte, le couplage est maximum

(tout ce qui est dans les sommets pairs forme les composantes impaires)

Method of variables: bipartite matching

6/1 méthode des variables
Classe RP

Un autre algorithme :

$G = (A, B, E)$ biparti, $|A| = |B|$

$M = (x_{ij}$ si $ij \in E$, 0 sinon$)$

$M = \begin{pmatrix} x_{ij} & 0 \\ 0 & \end{pmatrix}$: $\det(M)$: polynôme de $n!$ termes

Ecrire tous les termes : trop
Substituer et calculer ça

$\deg(\det(M)) \leq n-1$

- Peut-on déduire qqchose si la substitution est $\neq 0$?
- combien de substitution faut-il pour être sûr que $\neq 0$? (pour un polynôme de 1 variable) $(d+1)^n$

$\det(M) \not\equiv 0 \iff \exists$ perfect matching

The probabilities precisely

Lemme (Schwartz, Zippel)
q polynôme multivariable, le cas $q \neq 0$
$S \subseteq \mathbb{N}$ (valeurs à substituer)
$X_1, \ldots$ variables aléatoires pair
deux indé uniformes dans S.
Alors $Pr(q(X_1, \ldots X_a) = 0) \leq \frac{\deg q}{|S|}$

Pour des polynômes d'un seul variable si $|S| \leq$ degré
$>$ degré

Preuve : Par exemple X_n a une puissance dont le coeff (un polynôme) est $\neq 0$.
Soit μ le plus grand exposant de X_n pour lequel c'est vrai :
Soit $Q(X_1, \ldots, X_{n-1})$ le coeff.
Les choix de $X_1, \ldots, X_{n-1}$ et de X_n sont indép.

$$Pr(q(X_1, \ldots, X_a) = 0) \leq Pr(Q(X_1, \ldots X_{a-1}) = 0)$$
$$+ Pr(q(X_1, \ldots, X_a) = 0 \mid Q(X_1, \ldots, X_{a-1}) \neq 0)$$
$$\leq \frac{d - \mu}{|S|} + \frac{\mu}{|S|} = \frac{d}{|S|}$$

Method of variables: nonbipartite

If G not bipartite

$$\begin{pmatrix} & x_{ij} & & ij \in E \\ -x_{ij} & 0 & & 0 \text{ sinon} \end{pmatrix} =: M(G)$$

skew symmetric

Théorème (Tutte) G has a perfect m.

$\Longleftrightarrow \det(M(G)) \neq 0$

Pourquoi c'est moins trivial que dans le cas biparti ?

$$\begin{pmatrix} 0 & M \\ -M & 0 \end{pmatrix} \qquad \det(M(G)) = -\det(M)^2$$

Preuve : $\det(M(G)) =$

$= Pf(M(G))^2$ où

$$Pf(A) = \sum_{M=\{i_1 j_1, \ldots, i_{\frac{n}{2}} j_{\frac{n}{2}}\}} sg(M)\, a_{i_1 j_1} \cdots a_{i_{\frac{n}{2}} j_{\frac{n}{2}}}$$

Muir 1882, 1906

Dress, Wenzel 1995

Lovász 4.24.

ALGORITHME : $G=(V,E)$

1.) Soit $S=\{1,\dots,2n\}$

2.) Soit x_{ij} choix aléatoires indépendants dans S.

3.) Calculer le déterminant

si $\neq 0$ $\exists$ perfect matching

si $=0$ $\nexists$ — " —

error $\leq \frac{1}{2}$

Pour diminuer l'erreur, que faire?

choisir plus grand $|S|$?

Répéter plusieurs fois ?

Combien de fois si on la veut $\leq \varepsilon$

$|S| \geq \frac{n}{\varepsilon}$

mieux : $|S| = 2n$ répéter $\log_2 \frac{1}{\varepsilon}$ fois!

Classe RP

Σ alphabet $L \subseteq \Sigma^*$

Def : $L \in NP \iff \exists R_L : \Sigma^* \times \Sigma^* \to \{0,1\}$ calculable en temps polynomial (est polynôme p)

$x \notin L$: $R(x,y)=0$ $\forall y \in \Sigma^*$

$x \in L$: $R(x,y) = 1$ pour ≥ 1 $y \in \Sigma^*$, $|y| \leq p(|x|)$

RP, max matchings

RP

Σ alphabet

$L \subseteq \Sigma^*$

$L \in NP \iff \exists R_L : \Sigma^* \times \Sigma^* \to \{0,1\}$

$x \notin L : \quad R(x,y) = 0 \quad \forall y \in \Sigma^*$

$x \in L : \exists y \in \Sigma^* : R(x,y) = 1$

graphe certificat

$L \in RP \iff$ — " — $x \to Y_x$

— " — —, " —

$x \in L : \quad \{ y \in \Sigma^* : R(x,y) = 1 \} \geq \frac{|Y_x|}{2}$

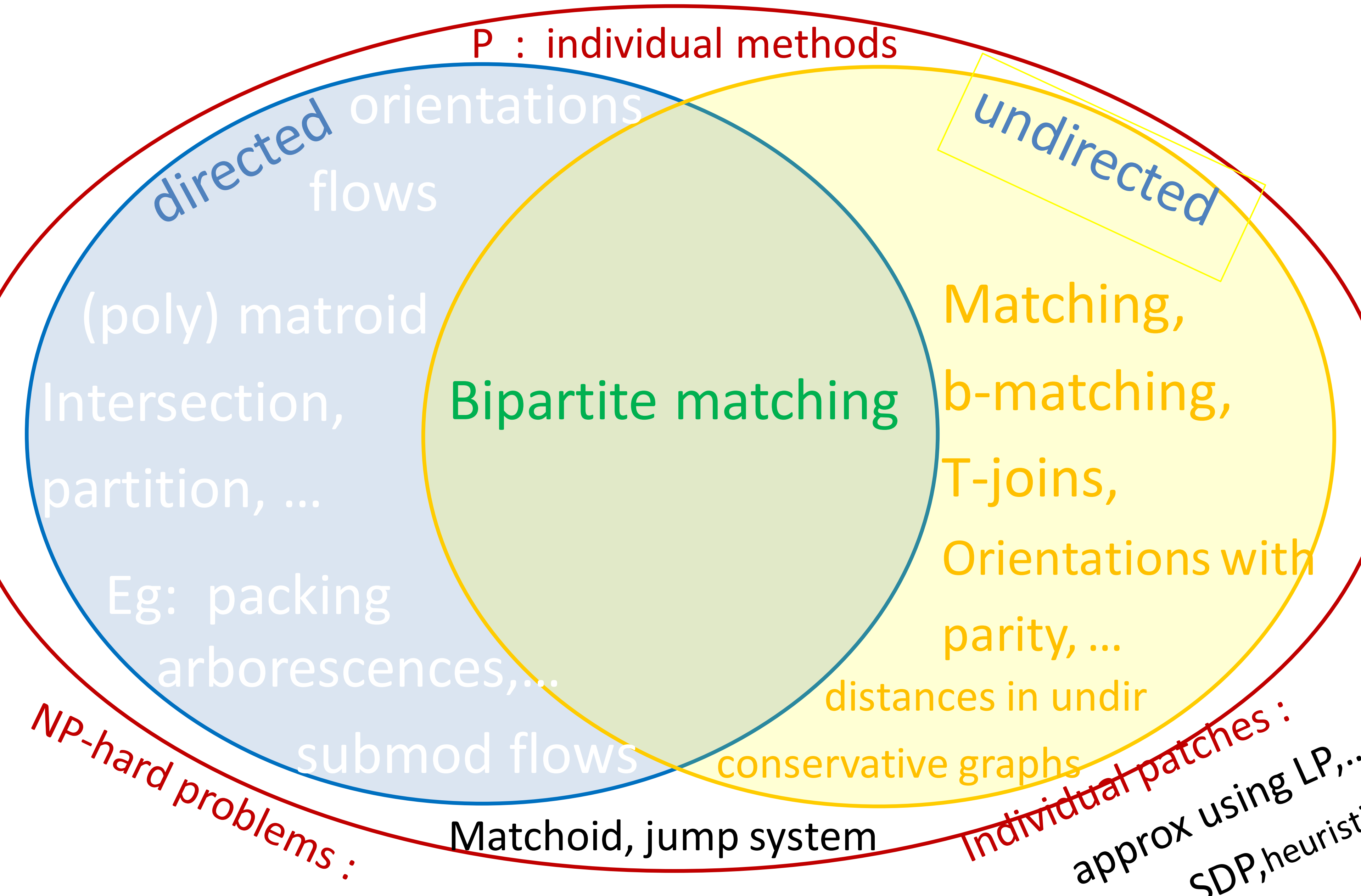
beaucoup de certificats !

$RP \approx P$

couplage biparti (méthode hongroise)

couplage nonbiparti (Edmonds)

Directions from bird's eyes …



We have seen :

Directed optimization : bipartite matching, max spanning tree or matroid independent, matroid intersection

Undirected optimization: non-bipartite matchings, undirected distances, Chinese Postman Problem

Approximation Algorithms : 2-approx for bin packing, 2-Approx of max cut, 3/2 approx of TSP, $1 - 1/e$ approximation of submod max with bounded size

Generic methods : method of variables, derandomization, improving paths (« test-sets ») , complexity analysis, polyhedral method, separation-optimization, ellipsoid method

Polyhedra : (bipartite) matching polytope, matroid independent set polytope

Complexity : P, NP, RP, good characterization, NP-hard, NP-complete