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Graph transformations preserving the stability number

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Abstract

We analyse the relations between several graph transformations that were introduced to be used in procedures determining the stability number of a graph. We show that all these transformations can be decomposed into a sequence of edge deletions and twin deletions. We also show how some of these transformations are related to the notion of even pair introduced to color some classes of perfect graphs. Then, some properties of edge deletion and twin deletion are given and a conjecture is formulated about the class of graphs for which these transformations can be used to determine the stability number.

Keywords: stability number, perfect graph, even pair

1 Introduction

A set of pairwise non-adjacent vertices in a graph G is called a *stable set*. The maximum size of a stable set in a graph G is called the *stability number* and is noted $\alpha(G)$. Given a vertex v of a graph, let $N(v)$ (resp. $N(v)$) denote the set of vertices that are adjacent (resp. non-adjacent) to v. The closed neighborhood $N[v]$ of a vertex v is $N(v) \cup \{v\}$. A clique in a graph is a complete subgraph. The *triangle* is the clique of size three. The maximum size of a clique in a graph G is called the *clique number* and is noted $\omega(G)$. The complement G of a graph G is the graph obtained by removing all the edges that were in G and adding all the edges that were not in G. Clearly, $\alpha(G) =$

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$\omega(\overline{G})$ for every graph G. The *chromatic number* of a graph G is the minimum number of colors that can be assigned to the vertices of a graph in such a way that two adjacent vertices receive two distinct colors. This number is noted $\chi(G)$. Clearly, $\omega(G) \leq \chi(G)$, for every graph G. The decision problems associated to determining α, ω and χ are known to be NP-complete [8].

A chordless path is a set of vertices $v_1, \ldots, v_k$, such that for $1 \leq i \leq k-1$, vertex v_i is adjacent to v_{i+1} and there is no other edge between these vertices. Such a path will sometimes be denoted v_1-$\cdots$-v_k. The length of a path is the number of its edges. A path is *odd* if it has odd length and *even* otherwise. A *hole* is a chordless cycle with at least five vertices. An *antihole* is the complement of a hole. A hole or an antihole is *odd* if it has an odd number of vertices and *even* otherwise.

2 Transformations preserving the stability number

Many graph transformations have been defined in order to determine the size of a maximum stable set in special classes of graphs.

A vertex v is called *simplicial* if $N(v)$ is a clique. If v is a simplicial vertex, then the deletion of $N(v)$ does not change the stability number. This transformation is called the *simplicial reduction*.

If a and b are two adjacent vertices such that $N(b) \subseteq N[a]$, then the deletion of a does not change the stability number. This transformation is called the *neighborhood reduction*. The neighborhood reduction generalizes the simplicial vertex reduction. If v is a simplicial vertex, then the deletion of $N(v)$ can be viewed as a sequence of neighborhood reductions for each vertex $a \in N(v)$, as $N(v) \subseteq N[a]$.

A *magnet* [10] consists of two adjacent vertices a, b such that $N(a) \backslash N(b)$ is completely linked to $N(b) \backslash N(a)$. If a and b form a magnet, then the deletion of a and of all the edges between b and $N(b) \backslash N(a)$ does not change the stability number. This transformation is called the *magnet reduction*. Clearly, the magnet reduction generalizes the neighborhood reduction. If a and b are two adjacent vertices such that $N(b) \subseteq N[a]$, then $N(b) \backslash N[a] = \emptyset$, so $N(a) \backslash N(b)$ is completely linked to $N(b) \backslash N(a)$. Deleting a corresponds to the magnet reduction as $N(b) \backslash N[a] = \emptyset$.

If a, b, c are three vertices such that a-b-c is a chordless path and $N(a) \subseteq N(b) \cup N(c)$, then the deletion of the edge bc does not change the stability number. This transformation is called the *edge deletion* [4].

As remarked in [1], the magnet reduction can be viewed as a sequence of edge deletions followed by one neighborhood reduction. If a, b are two adjacent vertices such that $N(a)\backslash N(b)$ is completely linked to $N(b)\backslash N(a)$, then for each c in $N(b)\backslash N[a]$, we have that a-b-c is a chordless path and $N(a) \subseteq N(b) \cup N(c)$, so edge bc can be deleted (edge deletion). When $N(b)\backslash N[a]$ is empty, then $N(b) \subseteq N[a]$ and a can be deleted (neighborhood reduction).

Two adjacent vertices a, b are *twins* if $N[a] = N[b]$. If a, b are twins, then the deletion of a does not change the stability number. We call this transformation the *twin deletion*. Clearly, twin deletion is a special case of neighborhood reduction, but the neighborhood reduction can be viewed as a sequence of edge deletions followed by one twin deletion. If a and b are two adjacent vertices such that $N(a) \subseteq N[b]$, then for each $c \in N(b)\backslash N[a]$ we have that a-b-c is a chordless path and $N(a) \subseteq N(b) \cup N(c)$, so edge bc can be deleted (edge deletion). When $N(b)\backslash N[a]$ is empty, then $N[a] = N[b]$ and a can be deleted (twin deletion).

By the previous remarks, the magnet reduction is a sequence of edge deletions followed by one neighborhood reduction, and the neighborhood reduction is a sequence of edge deletions followed by one twin deletion. So we have the following:

Proposition 2.1 *The magnet reduction is a sequence of edge deletions followed by one twin deletion.*

3 Transformations preserving the clique number

Some other graph transformations have been defined to determine the chromatic number of some subclasses of perfect graphs. The class of perfect graphs [2] has been defined as the class of graphs G such that for every induced subgraph H of G, we have $\chi(H) = \omega(H)$. The strong perfect graph theorem [5] asserts that a graph is perfect if and only if it contains no odd hole and no odd antihole. The class of perfect graphs appears to be a general class of graphs in which the problem of determining the chromatic number can be solved in polynomial time [9]. But the problem of finding a purely combinatorial algorithm for determining χ is still open.

If a, b are two non adjacent vertices in a graph G, the *contraction* of a and b consists in deleting a and b and adding a new vertex called ab that is adjacent to every vertex of $N(a) \cup N(b)$. An *even pair* [12] is a pair of non adjacent vertices such that there is no odd chordless path between them. It has been shown that the even pair contraction preserves the chromatic number and the clique number in any graph (not necessarily perfect) [7].

A graph is called *contractile* [3] if it can be reduced to a clique by a sequence of even pair contractions. As the even pair contraction preserves χ and ω, one can determine the chromatic number and clique number of a contractile graph as soon as one has a sequence of even pair contractions that transforms the graph into a clique. A graph is called *perfectly contractile* [3] if all its induced subgraphs are contractile.

A *prism* is a graph that consists of two vertex-disjoint triangles and three vertex-disjoint paths between them, with no other edge than those in the two triangles and in the three paths. When odd holes are forbidden the length of the three paths of a prism must have the same parity. A prism is *odd*, if the length of the three paths is odd, and *even* if the length of the three paths is even.

The following conjecture tries to characterize the class of perfectly contractile graphs by forbidden induced subgraphs:

Conjecture 3.1 ([13]) *A graph is perfectly contractile if and only if it contains no odd hole, no antihole and no odd prism.*

A weaker form of this conjecture has been proved:

Theorem 3.2 ([11]) *A graph that contains no odd hole, no antihole and no prism is perfectly contractile.*

If two vertices a and b form an even pair in a graph G, then in G, they are adjacent and $N(a)\backslash N(b)$ is completely linked to $N(b)\backslash N(a)$. In G, the deletion of a and of all the edges between b and $N(b)\backslash N(a)$ coincide with the contraction of a, b in G. So we have the following:

Proposition 3.3 *An even pair contraction in a graph G is a special case of a magnet reduction in the complement G.*

So, by Proposition 2.1 an even pair contraction is a special case of a sequence of edge and twin deletions in the complement. (It should be noted that even pair contraction preserves the chromatic number whereas the magnet reduction and edge-deletion reduction in the complement does not.)

4 Edge and twin deletions

If one can transform a graph into a stable set by applying a sequence of edge and twin deletions, then one can determine the stability number of the original graph. It is exactly the number of vertices of the stable set that is obtained.

The following proposition is easy to prove:

Proposition 4.1 *If a graph G can be transformed into a stable set by applying a sequence of edge and twin deletions, then G can be transformed into a stable set by applying first a sequence of edge deletions, then a sequence of twin deletions.*

A graphs can be reduced to a stable set by applying a sequence of twin deletions if and only if it consists of a set of disjoint cliques. So if one wants to determine the stability number of a graph by using edge and twin deletions, one can apply only edge deletion and stop when the graph is a disjoint set of cliques. Then, the stability number of the original graph is exactly the number of disjoint cliques when the process stops. A graph is called *edge-deletable* if it can be reduced to a set of disjoint cliques by a sequence of edge deletions. A graph is called *perfectly edge-deletable* if all its induced subgraphs are edge-deletable. As a corollary of Theorem 3.2, we can state:

Corollary 4.2 *A graph that contains no hole, no odd antihole, and no complement of a prism is perfectly edge-deletable.*

Holes and odd antiholes are not edge-deletable but complements of even prisms and complements of odd prisms different from C_6 are edge-deletable. So we propose the following conjecture:

Conjecture 4.3 *A graph is perfectly edge-deletable if and only if it contains no hole and no odd antihole.*

Proving such a conjecture will be a substantial step in the process of finding a purely combinatorial algorithm for determining the chromatic number of perfect graphs.

A result of [6] provides the first step of a proof of conjecture 4.3. A chordless path v_1-$\cdots$-v_k is *simplicial* if it cannot be extended to a chordless path v_0-v_1-$\cdots$-v_k-v_{k+1}. It is easy to see that, given a graph G and an integer k, if every non empty induced subgraph of G contains a simplicial path on at most k vertices, then G contains no hole of length $\geq k+3$. The converse is also true:

Theorem 4.4 ([6]) *For all positive integers k, a graph contains no hole of size $\geq k+3$ if and only if all its non empty induced subgraphs contain a simplicial path on at most k vertices.*

Applied for $k = 2$, we get the following corollary.

Corollary 4.5 *Given a graph with no hole, either it is a set of pairwise disjoint cliques or edge deletion can be applied.*

This corollary shows that it is always possible to start the edge deletion method in a graph with no hole and no odd antihole. But it cannot be guaranteed that this first step will not create a hole or an odd antihole.

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