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Graph transformations preserving the stability number

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ABSTRACT

We analyze the relations between several graph transformations that were introduced to be used in procedures determining the stability number of a graph. We show that all these transformations can be decomposed into a sequence of edge deletions and twin deletions. We also show how some of these transformations are related to the notion of even pair introduced to color some classes of perfect graphs. Then, some properties of edge deletion and twin deletion are given and a conjecture is formulated about the class of graphs for which these transformations can be used to determine the stability number.

1. Introduction

The purpose of this note is to review a series of graph transformations which have been introduced for determining the stability number of graphs. We intend to derive some properties of these transformations and we will in particular exhibit some connections between transformations which were initially introduced in different contexts. Although most of the following remarks are based on elementary observations, we believe that our attempt to unify and clarify the presentation of these methods will be fruitful for the future developments of graph transformations for finding the stability number of graphs. This note is an expanded version of [24].

Graphs considered here are without loops or multiple edges. We say that a graph G contains H when H is isomorphic to an induced subgraph of G. A set of pairwise non-adjacent vertices in a graph G is called a *stable set*. The maximum size of a stable set in a graph G is called the *stability number of* G and is denoted $\alpha(G)$. Given a vertex v of a graph, let $N(v)$ denote the set of vertices that are adjacent to v. The closed neighborhood $N[v]$ of a vertex v is $N(v) \cup \{v\}$. In a specific graph H, we denote by $N_H(v)$ the neighborhood of v and $N_H[v]$ its closed neighborhood. A *clique* in a graph is a set of pairwise adjacent vertices. A *triangle* is a clique of size three. The maximum size of a clique in a graph G is called the *clique number of* G and is denoted $\omega(G)$. The complement $\overline{G}$ of a graph G is the graph obtained by removing all the edges that were in G and adding all the edges that were not in G. For a vertex v, the set $N_{\overline{G}}(v)$ will be simply denoted $\overline{N}(v)$. Clearly, $\alpha(G) = \omega(\overline{G})$ for every graph G. The *chromatic number* of a graph G is the minimum number of colors that can be assigned to the vertices of a graph in such a way that two adjacent vertices receive two distinct colors. This number is denoted $\chi(G)$. Clearly, $\omega(G) \leq \chi(G)$, for every graph G. The decision problems associated with determining α, ω and χ are known to be NP-complete [14].

A chordless path is a set of vertices $v_1, \ldots, v_k$, such that for $1 \leq i \leq k-1$, vertex v_i is adjacent to v_{i+1} and there is no other edge between these vertices. Such a path will be denoted $v_1 - \cdots - v_k$. The length of a path is the number of edges of the subgraph induced by its vertices. A path is *odd* if it has odd length and *even* otherwise. Let P_k denote the chordless path (of length $k-1$) on k vertices. A *hole* is a chordless cycle with at least five vertices. An *antihole* is the complement of a hole.

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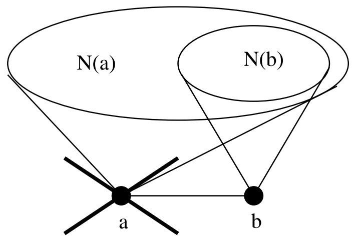

Fig. 1. Neighborhood reduction.

A hole or an antihole is *odd* if it has an odd number of vertices and *even* otherwise. A *square* is a chordless cycle of length four.

We will consider several graph transformations. When there is no ambiguity we often refer to G as the original graph and G' the graph obtained after applying the considered transformation.

We will introduce several graph transformations preserving α in Section 2 and ω in Section 3. We show how these transformations are linked to each other and how they can be decomposed into two basic transformations that are the edge deletion and twin deletion. In Section 4, we discuss how these simple transformations may be combined to compute α.

2. Transformations preserving the stability number

Many graph transformations have been defined in order to determine the size of a maximum stable set in special classes of graphs. Here we shall concentrate on transformations which simplify the graph G by removing vertices and/or edges, while keeping the same value of $\alpha(G)$. Other operations have been introduced for computing $\alpha(G)$ by transforming the graph G into another graph G' with $\alpha(G') = \alpha(G) - p$ (where p is a fixed integer). The struction, derived initially with pseudo-Boolean arguments (see [11]), is such an example. We shall not discuss those kinds of transformations here, but refer the reader to [2].

2.1. Simplicial, neighborhood and magnet reductions

A vertex v is called *simplicial* if $N(v)$ is a clique. If v is a simplicial vertex, then the deletion of $N(v)$ is called the *simplicial reduction*. This transformation does not change the stability number. If a stable set S of G contains a vertex u in $N(v)$, then it contains no vertices of $N[v] \setminus \{u\}$, so $(S \setminus \{u\}) \cup \{v\}$ is a stable set of G' of the same size as S.

A graph is *chordal* if it contains no hole and no square as an induced subgraph. By a theorem of Dirac [10], every chordal graph contains a simplicial vertex. So, the simplicial vertex reduction can be used to determine the stability number of a chordal graph. This can be done in linear time by using algorithms Lexicographic Breadth First Search [29] or Maximum Cardinality Search [30].

If a and b are two adjacent vertices such that $N(b) \subseteq N[a]$, then the deletion of a is called the *neighborhood reduction* (see Fig. 1). This transformation does not change the stability number. If a stable set S of G contains a, then no vertices of $N(a)$ are in S, so $(S \setminus \{a\}) \cup \{b\}$ is a stable set of G' of the same size as S.

The neighborhood reduction generalizes the simplicial vertex reduction. If v is a simplicial vertex, then the deletion of $N(v)$ can be viewed as a sequence of neighborhood reductions for each vertex $a \in N(v)$, as $N(v) \subseteq N[a]$. In [15], the neighborhood reduction has been used to transform any circular arc graph into a canonical form for which the stability number could be easily determined.

It was shown in [11] that finding the stability number of a graph could be reduced to the problem of maximizing a pseudo-Boolean expression called a posiform (i.e., a polynomial of 0,1 variables x_i and their complements $\overline{x_i} = 1 - x_i$, where all coefficients of monomials are positive). Conversely, to any maximization problem of a pseudo-Boolean function, we can associate an equivalent problem of finding a maximum stable set in a graph. In some cases, algebraic manipulations in a posiform can be devised to simplify the posiform without affecting its maximum value. In this way, a purely graph theoretical transformation called the magnet reduction was discovered.

A *magnet* [17] (see also [18,19]) consists of two adjacent vertices a, b such that $N(a) \setminus N(b)$ is completely linked to $N(b) \setminus N(a)$. If a and b form a magnet, then the deletion of a and of all the edges between b and $N(b) \setminus N(a)$ is called the *magnet reduction* (see Fig. 2, where a dashed line means that there is no edge). This transformation does not change the stability number. If a stable set S of G contains a, then no vertices of $N(a)$ are in S, so $(S \setminus \{a\}) \cup \{b\}$ is a stable set of G' of the same size as S. Conversely, if a stable set S' of G', is not a stable set of G, then it contains b and at least one vertex c of $N(b) \setminus N(a)$. As c is completely linked to $N(a) \setminus N(b)$, there is no vertex of $N(a) \setminus N(b)$ in S. Also there is no vertex of $N(b)$ in S. So $(S' \setminus \{b\}) \cup \{a\}$ is a stable set of G of the same size as S'.

Clearly, the magnet reduction generalizes the neighborhood reduction. If a and b are two adjacent vertices such that $N(b) \subseteq N[a]$, then $N(b) \setminus N[a] = \emptyset$, so $N(a) \setminus N(b)$ is completely linked to $N(b) \setminus N(a)$. Deleting a corresponds to the magnet reduction as $N(b) \setminus N[a] = \emptyset$.

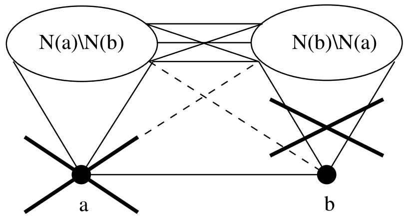

Fig. 2. Magnet reduction.

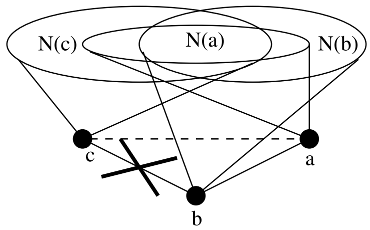

Fig. 3. Edge deletion.

2.2. Edge and twin deletions

If a, b, c are three vertices such that a–b–c is a chordless path and $N(a) \subseteq N(b) \cup N(c)$, then the deletion of the edge bc is called the *edge deletion* [5] (see Fig. 3). This transformation does not change the stability number. If a stable set S' of G', is not a stable set of G, then it contains b and c. As $N[a] \subseteq N(b) \cup N(c)$, the set S contains no vertices of $N[a]$, So $(S' \setminus \{b\}) \cup \{a\}$ is a stable set of G of the same size as S'.

As remarked in [2], the magnet reduction can be viewed as a sequence of edge deletions followed by one neighborhood reduction. If a, b are two adjacent vertices such that $N(a) \setminus N(b)$ is completely linked to $N(b) \setminus N(a)$, then for each c in $N(b) \setminus N[a]$, we have that a–b–c is a chordless path and $N(a) \subseteq N(b) \cup N(c)$, so edge bc can be deleted (edge deletion). When $N(b) \setminus N[a]$ is empty, then $N(b) \subseteq N[a]$ and a can be deleted (neighborhood reduction).

Two adjacent vertices a, b are *twins* if $N[a] = N[b]$. If a, b are twins, then the deletion of a is called the *twin deletion*. This transformation does not change the stability number. If a stable set S of G contains a, then no vertices of $N(a)$ are in S, so $(S \setminus \{a\}) \cup \{b\}$ is a stable set of G' of the same size as S.

The twin deletion has been used in [9] to determine the stability number of cographs. Clearly, twin deletion is a special case of neighborhood reduction, but the neighborhood reduction can be viewed as a sequence of edge deletions followed by one twin deletion. If a and b are two adjacent vertices such that $N(a) \subseteq N[b]$, then for each $c \in N(b) \setminus N[a]$ we have that a–b–c is a chordless path and $N(a) \subseteq N(b) \cup N(c)$, so edge bc can be deleted (edge deletion). When $N(b) \setminus N[a]$ is empty, then $N[a] = N[b]$ and a can be deleted (twin deletion).

Now we can decompose the magnet reduction into the two simple transformations that are edge and twin deletions:

Proposition 1. *The magnet reduction is a sequence of edge deletions followed by one twin deletion.*

Proof. By the previous remarks, the magnet reduction is a sequence of edge deletions followed by one neighborhood reduction, and the neighborhood reduction is a sequence of edge deletions followed by one twin deletion. So, the magnet reduction is a sequence of edge deletions followed by one twin deletion. □

3. Transformations preserving the clique number

Some other graph transformations have been defined to determine the chromatic number of some subclasses of perfect graphs. The class of *perfect graphs* [3] has been defined as the class of graphs G such that for every induced subgraph H of G, we have $\chi(H) = \omega(H)$. The strong perfect graph theorem [6] asserts that a graph is perfect if and only if it contains no odd hole and no odd antihole. The class of perfect graphs is a general class of graphs in which the problem of determining the chromatic number can be solved in polynomial time [16] with the ellipsoid method [21]. But the problem of finding a purely combinatorial algorithm for determining χ is still open. We will give below graph transformations that might be useful to solve this problem.

3.1. Even pair contraction

If a, b are two non-adjacent vertices in a graph G, the *contraction* of a and b consists in deleting a and b and adding a new vertex called ab that is adjacent to every vertex of $N(a) \cup N(b)$. When contracting any two nonadjacent vertices, the chromatic number can only increase. An *even pair* [27] is a pair of non-adjacent vertices such that there is no odd chordless

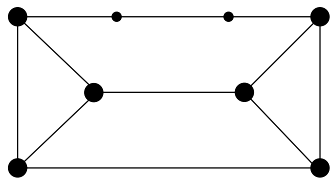

Fig. 4. The odd prism on eight vertices.

path between them. It has been shown that the even pair contraction preserves the chromatic number and the clique number in any graph (not necessarily perfect) [13]. The notion of even pair plays a major role in the class of perfect graphs. It has been used to find a substantial shortcut [7] in the proof of the strong perfect graph theorem [6]. Moreover it can be used to determine χ with combinatorial tools in many subclasses of perfect graphs [12].

A graph is called *contractile* [4] if it can be reduced to a clique by a sequence of even pair contractions. As the even pair contraction preserves χ and ω, one can determine the chromatic number and clique number of a contractile graph as soon as one has a sequence of even pair contractions that transforms the graph into a clique. A graph is called *perfectly contractile* [4] if all its induced subgraphs are contractile.

A *prism* is a graph that consists of two vertex-disjoint triangles and three vertex-disjoint paths between them, with no other edge than those in the two triangles and in the three paths. When odd holes are forbidden the length of the three paths of a prism must have the same parity. A prism is *odd*, if the length of the three paths is odd, and *even* if the length of the three paths is even (see Fig. 4).

The following conjecture tries to characterize the class of perfectly contractile graphs by forbidden induced subgraphs:

Conjecture 1 (*[28]*)**.** *A graph is perfectly contractile if and only if it contains no odd hole, no antihole and no odd prism.*

A weaker form of this conjecture has been proved:

Theorem 1 (*[26]*)**.** *A graph that contains no odd hole, no antihole and no prism is perfectly contractile.*

An $\mathcal{O}(n^2m)$ algorithm [25] has been deduced from the proof of Theorem 1 to compute the chromatic and clique number of graphs with no odd hole, no antihole and no prism.

A P_4*-free pair* is a pair of non-adjacent vertices such that there is no chordless path of length three between them. This generalization of even pair has been introduced by the first author in [22]. To prove that the even pair contraction preserves ω, as done in [13], there is no need to exclude odd chordless paths of length ≥ 5 and a corollary of this remark is that contracting a P_4-free pair preserves the clique number. Consider a P_4-free pair u, v that is contracted into w. If Q is a clique of G that is not a clique of G', then it contains at least one vertex in $\{u, v\}$. The clique Q cannot contain both u and v, so $(Q \setminus \{u, v\}) \cup \{w\}$ is a clique of G' of the same size of Q. If Q' is a clique of G' that is not a clique of G, then it contains w. If none of $(Q \setminus \{w\}) \cup \{u\}$ and $(Q \setminus \{w\}) \cup \{u\}$ is a clique of G, this means that u (resp. v) has a non-neighbor x (resp. y) in Q. Then u–y–x–v is a P_4, a contradiction.

The P_4-free pair contraction does not preserve the chromatic number as one can remark by contracting the end vertices of a P_6. But we can nevertheless define classes of contractile graphs as in [4]. A graph is called P_4*-free-contractile* [22] if it can be reduced to a clique by a sequence of P_4-free pair contractions. One can determine the chromatic and clique numbers of a P_4-free-contractile graph G as soon as one has a sequence of P_4-free pair contractions that transforms the graph into a clique. Suppose G is transformed into a clique Q of size k by a sequence of P_4-free pair contractions. As for any graph, we have $\chi(G) \geq \omega(G)$. The P_4-free pair contraction preserves ω, so we have $\omega(G) = \omega(Q)$. For the clique Q of size k, we have $\omega(Q) = k = \chi(Q)$. And as observed before, the contraction of non-adjacent vertices can only increase the chromatic number, thus $\chi(Q) \geq \chi(G)$. By combining all these (in)equalities together we get $\chi(G) \geq \omega(G) = \omega(Q) = k = \chi(Q) \geq \chi(G)$ and so all these values are equal.

A graph is called *perfectly* P_4*-free-contractile* [22] if all its induced subgraphs are P_4-free-contractile. It is conjectured in [22] that a graph is perfectly P_4-free-contractile if and only if it contains no odd hole and no antihole.

At this stage, one may wonder whether there are some connections between the various transformations preserving α described in Section 2 and the contractions mentioned above which were devised independently using entirely different arguments. It turns out that we have the following:

Proposition 2. *A* P_4*-free pair contraction in a graph G is precisely a magnet reduction in the complement* $\overline{G}$*.*

Proof. The vertices a and b form a P_4-free pair in a graph G, if and only if, in $\overline{G}$, they are adjacent and they are not the middle vertices of a P_4, so if and only if they are adjacent and $\overline{N}(a) \setminus \overline{N}(b)$ is completely linked to $\overline{N}(b) \setminus \overline{N}(a)$. In $\overline{G}$, the deletion of a and of all the edges between b and $\overline{N}(b) \setminus \overline{N}(a)$ coincides with the contraction of a, b in G. □

The simple observation leading to Proposition 2 was indeed given in a preliminary version of this note [23] and it has been exploited in [20] to simplify the presentation of some algorithms. In fact in [20] one characterizes in terms of forbidden subgraphs a class of graphs G for which by repeated applications of magnet reductions one ends up with a stable set S (with

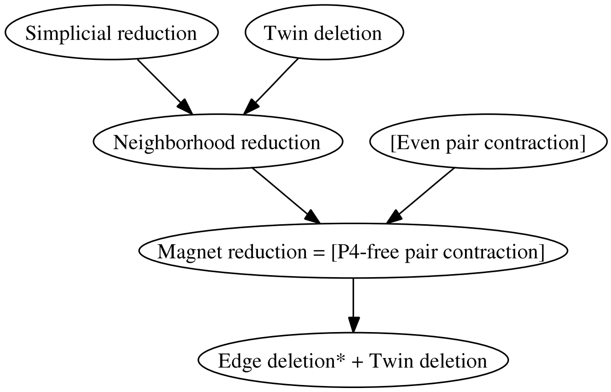

Fig. 5. Relations between considered graph transformations.

$|S| = \alpha(G)$). If one requires that any magnet found in the graph can be used for the reduction, then the class is characterized by the absence of eleven induced subgraphs and of chordless cycles of length at least 5.

Fig. 5 summarizes the relations between the graph transformations that we have considered. An edge $A \to B$ means that the transformation A is a special case of transformation B. A transformation $[A]$ is the transformation A considered in the complement, and $A^* + B$ means a sequence of As followed by one B.

3.2. Odd pair insertion

Asserting that an even pair contraction does not change the chromatic number amounts to saying that there exists an optimum coloring where the two contracted vertices receive the same color. In a symmetric way, one may ask when it is legitimate to say that there exists an optimum coloring where two non-adjacent vertices a, b have different colors. This would mean that we can introduce an edge ab without changing the chromatic number. Two non-adjacent vertices a, b form an *odd pair* [12] if there is no even chordless path between them. If $\omega \geq 2$, adding the edge ab preserves the chromatic number and the clique number [12]. This transformation is called the *odd pair insertion*.

As for even pairs, one may remark that to prove that the odd pair insertion preserves ω, there is no need to exclude even chordless paths of length ≥ 4. A *P_3-free pair* is a pair of non-adjacent vertices a, b that have no common neighbors. Adding the edge ab is called the *P_3-free pair insertion*. If $\omega \geq 2$, this transformation preserves ω (but not χ as one may remark by adding an edge between the end vertices of a P_5). If Q' is a clique of G' that is not a clique of G, then it contains both u and v. As u and v have no common neighbor, Q' has size 2.

Here again there is a connection with transformations preserving α.

Proposition 3. *Let a, b be a P_3-free pair in a graph G. Either the P_3-free pair insertion of the edge ab is an edge deletion in $\overline{G}$ or a and b are both isolated vertices of G (and thus a and b are twins in $\overline{G}$).*

Proof. Supose a, b is a P_3-free pair in G and that one of a and b is not isolated. By symmetry we can assume that a is not isolated and that there exists $c \in N(a)$. As a and b have no common neighbor, all the vertices of G are not adjacent to at least one of a or b. Thus vertex c is not in $N(b)$ and all the vertices of G are in $\overline{N}(a) \cup \overline{N}(b)$. Thus in $\overline{G}$, a–b–c is a P_3 and $\overline{N}(c) \subseteq \overline{N}(a) \cup \overline{N}(b)$, so the deletion of ab is an edge deletion. □

4. Edge and twin deletions

As shown in Fig. 5, many graph transformations that were introduced to compute α, ω or χ are simply combinations of edge and twin deletions.

4.1. Edge-deletable graphs

If one can transform a graph into a stable set by applying a sequence of edge and twin deletions, then one can determine the stability number of the original graph. It is exactly the number of vertices of the stable set that is obtained.

Proposition 4. *If a graph G can be transformed into a stable set by applying a sequence of edge and twin deletions, then G can be transformed into a stable set by applying first a sequence of edge deletions, then a sequence of twin deletions.*

Proof. We prove that a twin deletion followed by an edge deletion can be replaced by one or two edge deletions followed by a twin deletion. By repeatedly applying this to a sequence of edge and twin deletions, one can place all the twin deletions at the end of the sequence and get the result.

Suppose that x and y are twins in a graph G. Let H be the graph obtained after deleting x. Suppose there exist three vertices a, b, c of H such that a–b–c is a chordless path and $N_H(a) \subseteq N_H(b) \cup N_H(c)$ (y is possibly one of a, b, c, but x is not). Let H' be the graph obtained from H by deleting edge bc.

In G, the three vertices a, b, c form a chordless path a–b–c. Suppose $N(a) \not\subseteq N(b) \cup N(c)$, then $x \in N(a) \setminus (N(b) \cup N(c))$. If $y = a$, then $b \in N[y] = N[x]$, a contradiction. If $y \neq a$, then y is also in $N(a) \setminus (N(b) \cup N(c))$ and bc cannot be deleted in H, a contradiction. So $N(a) \subseteq N(b) \cup N(c)$ and we can delete the edge bc to obtain the graph G'.

If x, y are still twins in G', then we can delete x to obtain H'. Suppose now that x, y are not twins in G'. As x is distinct from b, c we have that y is one of b, c. If $y = b$, then the three vertices a, x, c form a chordless path a–x–c of G' with $N_{G'}(a) \subseteq N_{G'}(x) \cup N_{G'}(c)$ so we can delete the edge xc, then x and y become twins and we can delete x to obtain H'. If $y = c$, then the three vertices a, b, x form a chordless path a–b–x of G' with $N_{G'}(a) \subseteq N_{G'}(b) \cup N_{G'}(x)$ so we can delete the edge bx, then x and y become twins and we can delete x to obtain H'. □

Proposition 5. *A graph can be reduced to a stable set by applying a sequence of twin deletions if and only if it consists of a set of disjoint cliques.*

Proof. If a graph is a set of disjoint cliques that is not a stable set, then there exists a clique containing at least two vertices that are twins. We can remove one of them to get a set of disjoint cliques with strictly fewer vertices. This operation can be repeated until we get a stable set.

If a graph is not a set of disjoint cliques, then it contains a P_3. The graph will still contain a P_3 after any twin deletion and so it is not possible to get a stable set. □

By Propositions 4 and 5, if one wants to determine the stability number of a graph by using edge and twin deletions, one can apply only edge deletion and stop when the graph is a disjoint set of cliques. Then, the stability number of the original graph is exactly the number of disjoint cliques when the process stops.

A graph is called *edge-deletable* if it can be reduced to a set of disjoint cliques by a sequence of edge deletions. A graph is called *perfectly edge-deletable* if all its induced subgraphs are edge-deletable.

4.2. Forbidden induced subgraphs

Proposition 6. *A hole is not edge-deletable.*

Proof. Consider a hole x_1–$\cdots$–x_k–x_1, with $k \geq 5$. Three vertices that form a P_3 must be three consecutive vertices, for example x_1–x_2–x_3. We have $x_k \in N(x_1) \setminus (N(x_2) \cup N(x_3))$ and $x_4 \in N(x_3) \setminus (N(x_2) \cup N(x_1))$, so no edge deletion can be applied. □

Proposition 7. *An odd antihole is not edge-deletable.*

Proof. Suppose there exists an odd antihole A that is edge-deletable. The stability number of A is 2, so one can delete some edges of A to obtain a set of two disjoint cliques. But the vertices of odd antiholes cannot be partitioned into two cliques, a contradiction. □

A consequence is the following:

Corollary 1. *A perfectly edge-deletable graph contains no hole and no odd antihole.*

As a corollary of Theorem 1, we can state:

Corollary 2. *A graph that contains no hole, no odd antihole, and no complement of a prism is perfectly edge-deletable.*

Moreover we have:

Proposition 8. *Complements of even prisms and complements of odd prisms different from C_6 are edge-deletable.*

Proof. Even prisms are contractile, so complements of even prisms are edge-deletable.

Consider a graph G that is the complement of an odd prism different from C_6. Let H_1, H_2, H_3 be the three vertex-disjoint paths partitioning the vertices of $\overline{G}$. Since $\overline{G}$ is not C_6, at least one of the path has ≥ 4 vertices. We can assume w.l.o.g. that $|H_1| \geq 4$. One can transform H_1 (resp. H_2 and H_3) to a P_4 (resp. to an edge) by a sequence of even pair contractions. The graph that is obtained is the odd prism on eight vertices Π_8 (see Fig. 4). So G can be transformed into $\overline{\Pi_8}$ by a sequence of edge deletions. The graph $\overline{\Pi_8}$ is edge-deletable as shown in Fig. 6. So G is edge-deletable. □

The following conjecture tries to characterize the class of perfectly edge-deletable graphs by forbidden induced subgraphs:

Conjecture 2. *A graph is perfectly edge-deletable if and only if it contains no hole and no odd antihole.*

Proving such a conjecture would be a substantial step in the process of finding a purely combinatorial algorithm for determining the chromatic number of perfect graphs.

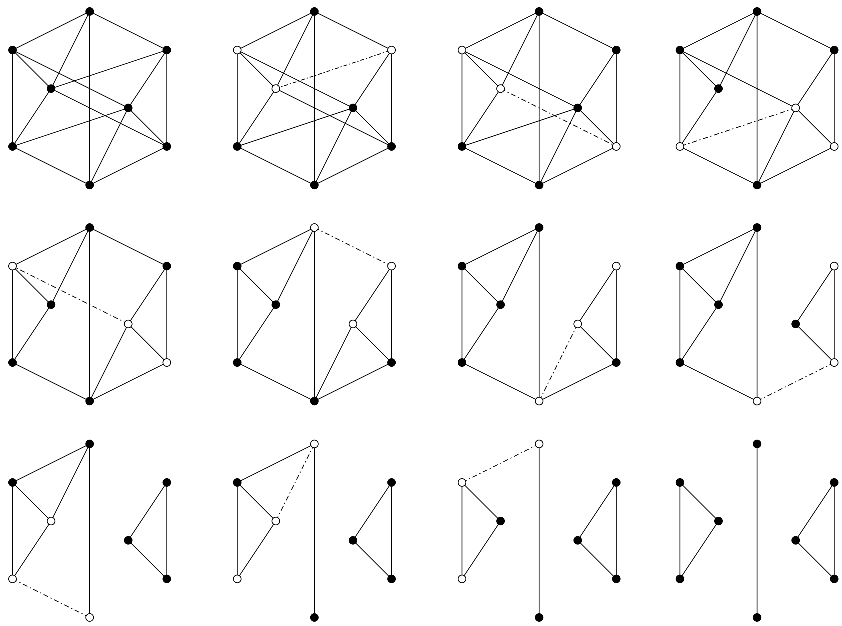

Fig. 6. The complement of the odd prism on eight vertices is edge-deletable. At each step, the three white vertices form the $P_3 a$–b–c as in the definition of edge deletion and the dashed edge is the deleted edge bc.

A result of [8] provides a first step of a proof of Conjecture 2. A chordless path $v_1 - \cdots - v_k$ is *simplicial* if it cannot be extended to a chordless path $v_0 - v_1 - \cdots - v_k - v_{k+1}$. Given a graph G and an integer k, if every non-empty induced subgraph of G contains a simplicial path on at most k vertices, then G contains no hole of length $\geq k+3$ (in a hole of length $\geq k+3$, any path on at most k vertices is not simplicial). The converse is also true:

Theorem 2 ([8]). *For all positive integers k, a graph contains no hole of size $\geq k+3$ if and only if all its non-empty induced subgraphs contain a simplicial path on at most k vertices.*

Applied for $k = 2$, we get the following corollary.

Corollary 3. *Given a graph with no hole, either it is a set of pairwise disjoint cliques or edge deletion can be applied.*

Proof. Let G be a graph with no hole that is not a set of pairwise disjoint cliques. Let H be a connected component of G that contains a P_3. Let H' be a maximal (inclusionwise) induced subgraph of H that contains no twins. Then H' is connected and still contains a P_3. By Theorem 2, H' contains a simplicial path P on at most two vertices.

If $|P| = 1$, let $P = \{a\}$. Vertex a is a simplicial vertex. Let $b \in N(a)$ (it exists as H' is connected and contains at least three vertices). Vertices a, b are not twins and $N(a) \subseteq N[b]$, so there exists $c \in N(b) \setminus N[a]$. If $|P| = 2$, let $P = \{a, b\}$. Vertices a, b are not twins, so we can assume that there exists $c \in N(b) \setminus N[a]$. As P is simplicial, $N(a) \setminus N[b] = \emptyset$. In both cases, vertices a, b, c are such that a–b–c is a chordless path and $N(a) \subseteq N(b) \cup N(c)$, so edge deletion can be applied. □

This corollary shows that it is always possible to start the edge deletion method in a graph with no hole and no odd antihole. But it cannot be guaranteed that this first step will not create a hole or an odd antihole. This is for example the case for the complement of the odd prism on eight vertices (see Fig. 6): whatever edge deletion is performed, a hole is created.

5. Comments

We have shown that many graph transformations preserving the stability number (or equivalently the clique number in the complement) can be expressed as a sequence of edge deletions followed by one twin deletion. This is not the case for all transformations that preserve α. For example, the *BAT*, defined in [18], cannot be decomposed into edge and twin deletions. We have to allow a third basic transformation called *edge insertion* [5] to be able to decompose the BAT. The problem of allowing edge insertion is that the reduction of a graph to a stable set by a sequence of edge deletions, edge insertions and twin deletions is not necessarily polynomial anymore. One may delete and insert the same edge many times.

It would be interesting to explore further a combination of the transformations discussed here with the other techniques like the struction to compute α. Some preliminary experiments are reported in [2].

The goal of the work presented here is to decompose some classical transformations preserving α into simpler ones. We hope that simplifying these transformations will provide a way of computing α in some classes of graphs such as graphs that contain no hole and no odd antihole. But one may consider the opposite direction. In [1], some graph transformations called *exchange plans* were devised in order to give a general framework in which many transformations preserving α can be expressed.

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