Graphs and Combinatorics (2007) 23:291–301
Digital Object Identifier (DOI) 10.1007/s00373-007-0724-1

Graphs and Combinatorics

Precoloring Extension of Co-Meyniel Graphs

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Abstract. The pre-coloring extension problem consists, given a graph G and a set of nodes to which some colors are already assigned, in finding a coloring of G with the minimum number of colors which respects the pre-coloring assignment. This can be reduced to the usual coloring problem on a certain contracted graph. We prove that pre-coloring extension is polynomial for complements of Meyniel graphs. We answer a question of Hujter and Tuza by showing that "PrExt perfect" graphs are exactly the co-Meyniel graphs, which also generalizes results of Hujter and Tuza and of Hertz. Moreover we show that, given a co-Meyniel graph, the corresponding contracted graph belongs to a restricted class of perfect graphs ("co-Artemis" graphs, which are "co-perfectly contractile" graphs), whose perfectness is easier to establish than the strong perfect graph theorem. However, the polynomiality of our algorithm still depends on the ellipsoid method for coloring perfect graphs.

Key words. Precoloring, Meyniel graph, Artemis graph

1. Introduction

Often in applied optimization, one faces difficulty in the modelling process because of the need to express some constraints that are not extensively studied theoretically. One type of such constraints is that the organization of the system is partially fixed a priori, for technical, historical or social reasons. In terms of mathematical programming, this can be interpreted as fixing the value of some decision variables before the optimization process. Although these prerequirements cause the size of the problem to drop, they may alterate the structural properties of the problem in such a way that its complexity increases from polynomial to NP-hard. The pre-coloring extension problem, also called *PrExt*, is a good illustration of this phenomenon.

For an integer k, a *k-coloring* of the vertices of a graph G is the assignment of one element of $\{1, 2, \dots, k\}$ (a color) to each vertex of G so that any two adjacent vertices receive different colors. Since each color class induces a stable set of G, a coloring can also be seen as a partition of $V(G)$ into stable sets. The smallest k such that G admits a k-coloring is the *chromatic number* of G, denoted by $\chi(G)$. A *pre-coloring* of G is a coloring of the vertices of an induced subgraph of G, that is, a collection $\mathcal{Q} = \{C_1, \dots, C_m\}$ of pairwise disjoint stable sets of G. We say that a

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k-coloring $\{S_1, \ldots, S_k\}$ of G *extends* $\mathcal{Q}$ if for each $j = 1, \ldots, m$ we have $C_j \subseteq S_j$. The problem PrExt can be defined as follows:

> *Input:* A graph G, an integer k, and a pre-coloring $\mathcal{Q}$ of G using only colors from $\{1, \ldots, k\}$.
> *Question:* Is there a k-coloring of G that extends $\mathcal{Q}$?

PrExt is a generalization of coloring (which consists in taking $\mathcal{Q} = \emptyset$), and it is more difficult than coloring since PrExt is NP-complete even when restricted to bipartite graphs [5, 10], to interval graphs [4] or to permutation graphs [12]. On the other hand, polynomial cases of PrExt have been found, using several approaches, surveyed in [19], see also [1].

Given a pre-coloring $\mathcal{Q} = \{C_1, \ldots, C_m\}$ of G, we can define a graph $G/\mathcal{Q}$ as follows: for each $j = 1, \ldots, m$, remove C_j, add a new vertex c_j, add an edge between c_j and every vertex of $V(G) \setminus (C_1 \cup \cdots \cup C_m)$ that has at least one neighbour in C_j, and add an edge between any two c_j's. The operation of contraction is a special case of *homomorphism* [13]. A homomorphism between two graphs G, H is a mapping $f : V(G) \to V(H)$ such that if x and y are adjacent vertices in G then $f(x)$ and $f(y)$ are adjacent vertices in H. Contraction is a special case since we add an edge between any two c_j's. We use the term contraction here because we find it appropriate and suggestive, although it is a different operation from the classical edge contraction.

The following lemma, whose proof is obvious, shows how PrExt reduces to coloring via contraction:

Lemma 1. *For any graph G, and any pre-coloring $\mathcal{Q}$ of G, the set of pre-coloring extensions of $\mathcal{Q}$ is in one-to-one correspondence with the set of colorings of $G/\mathcal{Q}$. In particular, the minimum number of colors needed to extend $\mathcal{Q}$ is equal to $\chi(G/\mathcal{Q})$.*

□

Lemma 1 shows that if we are able to solve the coloring problem for $G/\mathcal{Q}$ then we are also able to solve PrExt for G. This paper attempts to give some insight on how this translation works in general and to show in particular how it applies to perfect graphs. The latter point is summarized in the following two theorems, which are the central results of this paper. Before presenting them we need some more definitions. A *cycle* of length p in a graph G is a sequence of p distinct vertices $v_1, \ldots, v_p$ of G such that $v_i v_{i+1}$ is an edge for each i modulo p. A *chord* of the cycle is an edge $v_i v_j$ such that $|j - i| \geq 2 \bmod p$. If the cycle has length at least four and is chordless, then it is also called a *hole*. If the cycle has length at least five and has only one chord $p_1 p_3$ (up to shifting indices), then it is called a *house*. An *antihole* is the complementary graph of a hole. A *Meyniel graph* [15, 16] is a graph in which every odd cycle has at least two chords. It is easy to see that a graph G is a Meyniel graph if and only if it contains no odd hole and no house.

A graph G is *perfect* if, for every induced subgraph H of G, the chromatic number of H is equal to the maximum clique size in H. Perfect graphs were introduced in 1960 by Berge, see [3, 17], who also conjectured that a graph is perfect if and only if it does not contain an odd hole or an odd antihole of length at least five. This long-standing conjecture was proved by Chudnovsky et al. [6]. It was known since [15, 16] that Meyniel graphs are perfect.

A *prism* is a graph formed by three vertex-disjoint chordless paths $P_1 = u_0$-$\cdots$-u_r, $P_2 = v_0$-$\cdots$-v_s, $P_3 = w_0$-$\cdots$-w_t with $r, s, t \geq 1$, such that the sets $A = \{u_0, v_0, w_0\}$ and $B = \{u_r, v_s, w_t\}$ are cliques and there is no edge between the P_i's other than the edges in A and B. Note that a prism with $r = s = t = 1$ is an antihole on six vertices. A graph is an *Artemis* graph [7] if it contains no odd hole, no antihole on at least five vertices, and no prism. Artemis graphs are perfect by the strong perfect graph theorem [6] but also by a simpler result [14].

Given a graph G, a *co-coloring* of G is a partition of $V(G)$ into cliques of G. For our purpose it will be more convenient to talk about co-colorings than colorings. Obviously a co-coloring of G is a coloring of $\overline{G}$, and all the statements in this paper can be translated back and forth from co-colorings to colorings by taking complementary graphs and complementary classes of graphs.

Let $\mathcal{Q} = \{C_1, \ldots, C_m\}$ be a collection of pairwise disjoint cliques of G. Note that $\mathcal{Q}$ is a pre-coloring of $\overline{G}$; so $\mathcal{Q}$ will be called a *pre-co-coloring* of G. We denote by $G^{\mathcal{Q}}$ the graph obtained by the operation of *co-contraction* defined as follows. Each element C_j of $\mathcal{Q}$ is contracted into one vertex c_j. A vertex of $G \setminus (C_1 \cup \cdots \cup C_m)$ is adjacent to c_j in $G^{\mathcal{Q}}$ if and only if it is adjacent in G to every vertex of C_j; and there is no edge between any two c_j's. Clearly, $G^{\mathcal{Q}}$ is the complementary graph of $\overline{G}/\mathcal{Q}$.

Theorem 1. *The co-contraction $G^{\mathcal{Q}}$ of G is a perfect graph for every pre-co-coloring $\mathcal{Q}$ if and only if G is a Meyniel graph.*

Theorem 2. *If G is a Meyniel graph and $\mathcal{Q}$ is any pre-co-coloring of G, then the co-contracted graph $G^{\mathcal{Q}}$ is an Artemis graph.*

Theorem 1 has a nice algorithmic consequence:

Corollary 1. *PrExt is polynomial for co-Meyniel graphs.*

Proof. PrExt on co-Meyniel graphs is equivalent to co-PrExt on Meyniel graphs. Given a Meyniel graph G and a co-coloring $\mathcal{Q}$ of G, the co-contracted graph $G^{\mathcal{Q}}$ is perfect by Theorem 1. One can use a polynomial-time algorithm [8, 18] to find an optimum co-coloring for $G^{\mathcal{Q}}$. From this co-coloring, using Lemma 1, one deduces an optimal pre-co-coloring extension of $\mathcal{Q}$ for G. □

Corollary 1 contains and unifies several previously known cases of polynomiality of PrExt [10, 11, 19]: split graphs, cographs, P5-free bipartite graphs, complements of bipartite graphs, and the case of a co-Meyniel graph where every pre-color class has size 1 [9]. The proof of corollary 1 can also be used to derive a more general algorithmic consequence of Lemma 1. Given a class of graphs $\mathcal{G}$, let its "co-contraction closure" $\mathcal{G}^+$ be the class of all graphs obtained by the co-contraction of any pre-co-colored graph in $\mathcal{G}$.

Corollary 2. *Let $\mathcal{G}$ be a class of graph. If co-coloring is polynomial on graph class $\mathcal{G}^+$, then co-PrExt is polynomial on $\mathcal{G}$.*

One use of Corollary 2 is to reduce co-PrExt on a given class of graph $\mathcal{G}$ to asking "what is $\mathcal{G}^+$" and then solving co-coloring on $\mathcal{G}^+$. Unfortunately this strategy may fail in general, because even if we are able to describe $\mathcal{G}^+$, we do not necessarily know the complexity of co-coloring on $\mathcal{G}^+$. It may be more fruitful to try to translate results from co-coloring to co-PrExt. For instance, in this paper, we want to apply the ellipsoid method, which allows to (co)-color perfect graphs in polynomial time [8, 18]. So we can ask: "what is the class $\mathcal{G}$ such that $\mathcal{G}^+ =$ Perfect?". Unfortunately, such a class does not exist. To clarify this point, let us give another definition. Given a class $\mathcal{G}$, let $\mathcal{G}^-$ be the set of graphs G such that $G^{\mathcal{Q}}$ belongs to $\mathcal{G}$ for every $\mathcal{Q}$. It is easy to see that every class $\mathcal{G}$ of graphs satisfies $(\mathcal{G}^-)^+ \subseteq \mathcal{G} \subseteq (\mathcal{G}^+)^-$. Theorem 1 says that *Perfect*$^-$ = *Meyniel*, and Theorem 2 says that *Meyniel*$^+ \subseteq$ Artemis, which is a strict subclass of perfect graphs. It follows that (*Perfect*$^-$)$^+ \neq$ Perfect, and consequently that there is no class $\mathcal{G}$ of graphs such that $\mathcal{G}^+ =$ Perfect.

This discussion suggests a weaker but more directly usable version of corollary 2.

Corollary 3. *Let $\mathcal{G}$ be a class of graph. If co-coloring is polynomial on graph class $\mathcal{G}$, then co-PrExt is polynomial on $\mathcal{G}^-$.*

Let us also note that Theorem 1 answers the question of Hujter and Tuza [11] to characterize "PrExt-Perfect graphs" which, in our language, was precisely to characterize the class *Perfect*$^-$. Indeed, Hujter and Tuza's so-called "core condition" turns out to be equivalent to the clique condition $\chi(G/\mathcal{Q}) \geq \omega(G/\mathcal{Q})$. They called a graph G "PrExt-perfect" if both $G/\mathcal{Q}$ is perfect for every $\mathcal{Q}$ and the core condition is sufficient for extendibility. A consequence of our Lemma 1 is that this second condition is redundant, because perfection implies sufficiency of the clique condition in $G/\mathcal{Q}$. Hence, their PrExt-perfect graphs coincide with *Perfect*$^-$.

2. Proof of Theorems 1 and 2

Throughout this section, G is a graph and $\mathcal{Q}$ is a pre-co-coloring of G.

One way in Theorem 1 (namely, *Perfect*$^- \subseteq$ *Meyniel*) is easy:

Lemma 2. *If the co-contracted graph $G^{\mathcal{Q}}$ is perfect for all pre-co-coloring $\mathcal{Q}$ of G, then G is Meyniel.*

Proof. If G contains an odd hole, then the graph $G^{\emptyset}$ contains this hole. If G contains a house with the chord xy and G contains no odd hole, then the house has odd length and the co-contracted graph $G^{\{\{x\},\{y\}\}}$ contains an odd hole. So G must be a *Meyniel* graph for $G^{\mathcal{Q}}$ to be perfect for every $\mathcal{Q}$. □

The rest of this section is devoted to the study of *Meyniel*$^+$.

Lemma 3. *In a Meyniel graph G, let $P = p_0\text{-}\cdots\text{-}p_n$ be a chordless path and x be a vertex of $V(G) \setminus V(P)$ which sees p_0 and p_n. Then either x sees every vertex of P, or n is even and $N(x) \cap V(P) \subseteq \{p_{2i} \mid i = 0, \ldots, n/2\}$.*

Proof. Call segment any subpath of length at least 1 of P whose two endvertices see x and whose interior vertices do not. Since p_0 and p_n see x, path P is partitioned into its segments. Let p_h-$\cdots$-p_j be any segment with $j - h \geq 2$. Then $x, p_h, \dots, p_j$ induce a hole, so $j - h$ is even. Thus, every segment has length either even or equal to 1. Suppose that there is a segment of length 1 and a segment of even length. Then, there are consecutive such segments, that is, up to symmetry, there are integers $0 < h < j \leq n$ such that p_{h-1}-p_h is a segment of length 1 and p_h-$\cdots$-p_j is a segment of even length; but then $x, p_{h-1}, p_h, \dots, p_j$ induce a house, a contradiction. Thus either all segments have length 1 (i.e., x sees every vertex of P), or they all have even length, and the lemma holds. □

Lemma 4. *In a Meyniel graph G, let H be an even hole and x be a vertex of $V(G)\setminus V(H)$ that sees two consecutive vertices of H. Then x sees either all vertices of H or exactly three consecutive vertices of H.*

Proof. Let $x_1, \dots, x_n$ be the vertices of H ordered cyclically. Suppose that x sees x_1 and x_2 but not all vertices of H, and let x_i be a vertex of H that is not seen by x. Suppose that x sees a vertex x_j with $j \notin \{1, 2, 3, n\}$. By symmetry we can assume that $i < j$. Then either x_1-$\cdots$-x_j or x_2-$\cdots$-x_j is an odd chordless path, and in either case x sees the two endvertices and not all vertices of that path, a contradiction to Lemma 3. So $N(x) \cap V(H) \subseteq \{x_1, x_2, x_3, x_n\}$. If x sees none of x_3, x_n, then $V(H) \cup \{x\}$ induces a house, a contradiction. If x sees both x_3, x_n, then $x, x_3, \dots, x_n$ induce an odd hole, a contradiction. So x sees exactly one of x_3, x_n, and the lemma holds. □

Lemma 5. *In a Meyniel graph G, let Q be a clique, $P = p_0$-$\cdots$-p_n be a chordless path in $G \setminus Q$, and z be a vertex not in $Q \cup V(P)$. Suppose that z and p_0 see all vertices of Q, that z does not see any of p_0, p_1, and that some vertex $q \in Q$ sees p_n and not p_1. Then p_n sees all vertices of Q.*

Proof. We prove this lemma by induction on n. Suppose that some vertex $q' \in Q$ does not see p_n. By Lemma 3 applied to P and q, since q sees p_0, p_n and not p_1, path P has even length and $N(q) \cap V(P) \subseteq \{p_{2i} \mid i = 1, \dots, n/2\}$. Let j be the largest integer such that q sees p_j and $j < n$. Since j is even, $p_j, \dots, p_n, q$ induce an even hole C. Suppose that $j = 0$. Then q' sees q, p_0 on C and not p_n, so Lemma 4 implies that q' sees p_1 and none of $p_2, \dots, p_n$. Call C' the even hole induced by $(V(C) \setminus p_0) \cup \{q'\}$. Vertex z sees q, q' of C' and not p_1, so Lemma 4 implies that z sees p_n and none of $p_1, \dots, p_{n-1}$. But then $V(C) \cup \{z\}$ induces a house, a contradiction. So $j \geq 2$. By the induction hypothesis, p_j sees all vertices of Q. Then q' sees q, p_j on C and not p_n, so Lemma 4 implies that q' sees p_{j+1} and none of $p_{j+2}, \dots, p_n$. But then $p_0, q, q', p_{j+1}, \dots, p_n$ induce a house, a contradiction. □

Lemma 6. *In a Meyniel graph G, let Q be a clique, X be a connected set of vertices of $G \setminus Q$, and z be a vertex not in $Q \cup X$. Suppose that z sees all the vertices of Q and none of X, and that each vertex of X has a non-neighbour in Q. Then some vertex of Q has no neighbour in X.*

Proof. We prove the lemma by induction on the size of X. If $|X| = 1$ there is nothing to prove, so assume $|X| \geq 2$. Let x, x' be two distinct vertices of X such that $X \setminus \{x\}$ and $X \setminus \{x'\}$ are connected (for example let x, x' be two leaves of a spanning tree of X). By the induction hypothesis, there are vertices q, q' of Q such that q has no neighbour in $X \setminus \{x\}$ and q' has no neighbour in $X \setminus \{x'\}$. If either q does not see x or q' does not see x', then the lemma holds, so suppose that q sees x and q' sees x'. Let P be a shortest path from x to x' in X. Then either $V(P) \cup \{q, q'\}$ induces an odd hole or $V(P) \cup \{q, q', z\}$ induces a house, a contradiction. □

Lemma 7. *Let G be a Meyniel graph and $\mathcal{Q}$ be a precocoloring of G. Then the cocontracted graph $G^{\mathcal{Q}}$ contains no antihole of size at least* 6.

Proof. Suppose that $G^{\mathcal{Q}}$ contains an antihole A of size at least 6. Since the cliques of $\mathcal{Q}$ are cocontracted into a stable set, there are at most two vertices in A that result from the cocontraction of a clique and if there are two such vertices they are consecutive in the cyclic ordering of $\overline{A}$. If there are five consecutive vertices of A that do not result from the cocontraction of a clique, then these five vertices form a house of G, a contradiction. So there are no such five vertices, which implies that A is of size six and has exactly two cocontracted vertices. Let $x_1, \ldots, x_6$ be the vertices of A ordered cyclically, such that x_1, x_2 are the cocontracted vertices. Let C_1 be the clique whose cocontraction results in x_1. Since x_1 and x_6 are not adjacent, x_6 does not see all the vertices of C_1, so there is a vertex q_1 of C_1 that does not see x_6 in G. Then q_1, x_3, x_4, x_5, x_6 induce a house in G, a contradiction. □

Lemma 8. *Let G be a Meyniel graph and $\mathcal{Q}$ be a precocoloring of G. Then the cocontracted graph $G^{\mathcal{Q}}$ contains no odd hole.*

Proof. We prove the lemma by induction on $m = |\mathcal{Q}|$. If $m = 0$, then $G^{\mathcal{Q}} = G$ and the lemma holds. So assume $m > 0$ and let $\mathcal{Q} = \{C_1, \ldots, C_m\}$. Suppose that $G^{\mathcal{Q}}$ contains an odd hole $\mathcal{H}$. Let $x_1, \ldots, x_n$ be the vertices of $\mathcal{H}$ ordered cyclically. For each $j = 1, \ldots, m$, we may assume that the vertex that results from the cocontraction of C_j lies in $\mathcal{H}$, for otherwise $\mathcal{H}$ is an odd hole in $G^{\mathcal{Q}\setminus\{C_j\}}$, which contradicts the induction hypothesis. So let us call x_{i_j} the vertex of $\mathcal{H}$ that results from the cocontraction of C_j, and assume without loss of generality that $1 \leq i_1 < i_2 < \cdots < i_m \leq n$.

Suppose that $m = 1$. We may assume that $i_1 = 1$. Since x_3 and x_1 are not adjacent in $G^{\mathcal{Q}}$, x_3 does not see all vertices of C_1 in G, so there is a vertex q_1 of C_1 that does not see x_3 in G. Then the path $P = x_2 \text{-} \cdots \text{-} x_n$ is chordless and odd, and q_1 sees both endvertices of P and misses vertex x_3 of P, a contradiction to Lemma 3. Therefore $m \geq 2$.

The cocontracted vertices $x_{i_1}, \ldots, x_{i_m}$ form a stable set in $\mathcal{H}$. So for all j, we have $i_{j+1} - i_j \geq 2$. Since n is odd and $m \geq 2$, there exists j such that $i_{j+1} - i_j$ is odd (and so $i_{j+1} - i_j \geq 3$). We can assume without loss of generality that $i_2 - i_1$ is odd and $i_1 = 1$ (so $i_2 \geq 4$). Let R be the odd path $x_2 \text{-} \cdots \text{-} x_{i_2 - 1}$.

Since x_3 and x_1 are not adjacent in $G^{\mathcal{Q}}$, there is a vertex q_1 of C_1 that does not see x_3 in G. Likewise, there is a vertex q_2 of C_2 that does not see $x_{i_2 - 2}$ in G. Moreover, if $m \geq 3$, we can apply Lemma 6 to the clique C_j, the connected set R and x_{i_j-1}, which implies that:

$$\text{For } j = 3, \ldots, m, \text{ there is a vertex } q_j \in C_j \text{ that sees no vertex of } R. \quad (1)$$

Now we select vertices $y_1, \ldots, y_n$ of G as follows. For $k = 1, \ldots, n$, if there exists j such that $i_j = k$, let $y_k = q_j$; else let $y_k = x_k$. The selected vertices $y_1, \ldots, y_n$ form an odd cycle H of G. Note that, in H, vertex y_2 is adjacent only to y_1, y_3 and possibly to $y_{i_2} = q_2$, by (1).

$$\text{Every neighbour of } q_1 \text{ in } V(H)\setminus\{y_2, y_n\} \text{ is in } \{q_2, \ldots, q_m\}. \quad (2)$$

For let u be a neighbour of q_1 in $V(H) \setminus \{y_2, y_n\}$. Since x_2 and x_{i_2} are not adjacent in $G^{\mathcal{Q}}$, there is a vertex q_2' of C_2 that does not see y_2 in G. The subgraph of G induced by $V(H) \cup \{q_2'\} \setminus \{y_1, y_n, q_2\}$ is connected, so it contains a shortest path U from y_2 to u. Since y_3 is the only neighbour of y_2 in that subgraph, y_3 lies on U as the neighbour of y_2. Now Lemma 5 can be applied to clique C_1, path U and vertex y_n, which implies that u sees all of C_1. Then u must be in $\{q_2, \ldots, q_m\}$ for otherwise q_1 and u would be adjacent in $G^{\mathcal{Q}}$. So (2) holds. Likewise:

$$\text{Every neighbour of } q_2 \text{ in } V(H)\setminus\{y_{i_2-1}, y_{i_2+1}\} \text{ is in } \{q_1, q_3, \ldots, q_m\}. \quad (3)$$

Now each of $y_2, \ldots, y_{i_2-1}$ has degree 2 in H.

Let us color blue some vertices of the path $\mathcal{H} \setminus x_1 = x_2\text{-}\cdots\text{-}x_n$ of $G^{\mathcal{Q}}$ as follows. Vertices x_2 and x_n are colored blue. For $j = 2, \ldots, m$, vertex x_{i_j} is colored blue if and only if all vertices of the corresponding clique C_j see q_1. All other vertices of $\mathcal{H}$ are uncolored. Call blue segment any subpath of length at least 1 of $\mathcal{H} \setminus x_1$ whose two endvertices are blue and whose interior vertices are uncolored. Since $\mathcal{H} \setminus x_1$ has odd length and its endvertices are blue, it has an odd blue segment. Let $x_h\text{-}\cdots\text{-}x_i$ be any odd blue segment, with $2 \leq h < i \leq n$. Suppose that $i - h \geq 3$. Then (2) implies that $q_1, x_h, x_{h+1}, \ldots, x_{i-1}, x_i$ induce an odd hole in $G^{\mathcal{Q}} \setminus \{C_1\}$, which contradicts the induction hypothesis on $|\mathcal{Q}|$. So we must have $i - h = 1$. Since $i_{j+1} - i_j \geq 2$ for all j and $i_2 \geq 4$, this is possible only if $h = n - 1$. This implies that $x_{n-1}\text{-}x_n$ is the only odd blue segment, and that every blue vertex x_k different from x_n has even k.

Likewise, we color red some vertices of the path $\mathcal{H} \setminus x_{i_2}$ of $G^{\mathcal{Q}}$ as follows. Vertices x_{i_2-1} and x_{i_2+1} are colored red. For $j = 1, 2, \ldots, m$ and $j \neq 2$, vertex x_{i_j} is colored red if and only if all vertices of the corresponding clique C_j see q_2. Call red segment any subpath of length at least 1 of $\mathcal{H} \setminus x_{i_2}$ whose two endvertices are red and whose interior vertices are not red. Just like in the preceding paragraph, we obtain that $x_{i_2+1}\text{-}x_{i_2+2}$ is the only odd red segment, and that every red vertex x_l different from x_{i_2-1} and x_{i_2+1} has either even l or $l = 1$.

If $i_2 = n - 1$, then $m = 2$ and $V(R) \cup \{q_1, q_2, x_n\}$ induces an odd hole (if q_1, q_2 are not adjacent) or a house (if q_1, q_2 are adjacent) in G, a contradiction. So suppose $i_2 \leq n - 3$. Since x_{i_2+2} is red and x_{n-1} is blue, there is a subpath $x_k\text{-}\cdots\text{-}x_l$ of $x_{i_2+2}\text{-}\cdots\text{-}x_{n-1}$ such that x_k is red, x_l is blue, and no interior vertex of $x_k\text{-}\cdots\text{-}x_l$ is colored. By the preceding paragraphs, both k, l are even. If $k = l$, then (2) implies that there is a clique C_j such that $k = i_j$, and then $V(R) \cup \{q_1, q_2, q_j\}$ induces an odd hole (if q_1, q_2 are not adjacent) or a house (if q_1, q_2 are adjacent) in G, a contradiction. So

$k \neq l$. If q_1, q_2 are adjacent, then (2) and (3) imply that $\{q_1, x_k, \dots, x_l, q_2\}$ induces an odd hole in $G^{\mathcal{Q}} \setminus \{C_1, C_2\}$, a contradiction to the induction hypothesis on $|\mathcal{Q}|$. If q_1, q_2 are not adjacent then $V(R) \cup \{q_1, x_k, \dots, x_l, q_2\}$ induces an odd hole in $G^{\mathcal{Q}} \setminus \{C_1, C_2\}$, again a contradiction. This completes the proof of the lemma. □

Lemma 9. *Let G be a Meyniel graph and $\mathcal{Q}$ be a precocoloring of G. Then the cocontracted graph $G^{\mathcal{Q}}$ contains no prism.*

Proof. Suppose that $G^{\mathcal{Q}}$ contains a prism K formed by paths $P_1 = u_0\text{-}\cdots\text{-}u_r$, $P_2 = v_0\text{-}\cdots\text{-}v_s$, $P_3 = w_0\text{-}\cdots\text{-}w_t$, with $r, s, t \geq 1$, and with triangles $A = \{u_0, v_0, w_0\}$ and $B = \{u_r, v_s, w_t\}$. By Lemma 7, K is not an antihole on 6 vertices, so we can assume that one of r, s, t is not equal to 1. By Lemma 8, $G^{\mathcal{Q}}$ contains no odd hole, thus r, s, t have the same parity. Let $\mathcal{Q} = \{C_1, \dots, C_m\}$. We have $m \geq 1$ since a Meyniel graph contains no prism, because a prism contains a house. For each $j = 1, \dots, m$, call c_j the vertex of $G^{\mathcal{Q}}$ that results from the cocontraction of C_j, and let $C = \{c_1, \dots, c_m\}$. Note that $V(K) \setminus C \subset V(G)$, and, since C is a stable set, $N(c_j) \subset V(G)$ for each $j = 1, \dots, m$. We claim that:

$$|A \cap C| = 1 \quad \text{and} \quad |B \cap C| = 1. \tag{4}$$

Note that $|A \cap C| \leq 1$ and $|B \cap C| \leq 1$ since A, B are cliques and C is a stable set of $G^{\mathcal{Q}}$. Now, suppose up to symmetry that $A \cap C = \emptyset$. For each $j = 1, \dots, m$, we can apply Lemma 6 in G to the clique C_j, the connected set $A \setminus N(c_j)$, and some neighbour z of c_j in K (more precisely: if $c_j = u_i$ with $i < r$ then take $z = c_{i+1}$; if $c_j = u_r$ and either $s \geq 2$ or $t \geq 2$, take $z = v_s$ or $z = w_t$ respectively; if $c_j = u_r$ and $s = t = 1$, then $r \geq 3$ and take $z = u_{r-1}$; a similar such z exists if $c_j \in V(P_2) \cup V(P_3)$). Lemma 6 implies that there is a vertex $q_j \in C_j$ that sees no vertex of $\mathcal{A} \setminus N(c_j)$. Let P be the subgraph of G induced by $(V(K) \setminus C) \cup \{q_1, \dots, q_m\}$. Let u_1' be the neighbour of u_0 in $P \setminus \{v_0, w_0\}$ (so u_1' is either u_1 or some q_j), and similarly let v_1' be the neighbour of v_0 in $P \setminus \{u_0, w_0\}$. Let R be a shortest path from u_1' to v_1' in $P \setminus \{u_0, v_0, w_0\}$. Then $V(R) \cup \{u_0, v_0, w_0\}$ induces a house in G, a contradiction. So (4) holds.

By (4) and up to symmetry we may assume that $u_0 = c_1$, and so v_0, w_0 are vertices of G. As above, by Lemma 6, for $j = 2, \dots, m$, we can select a vertex q_j in C_j that misses all of $\{v_0, w_0\} \setminus N(c_j)$. We claim that:

$$\text{Vertices } v_1 \text{ and } w_1 \text{ of } G^{\mathcal{Q}} \text{ are in } C. \tag{5}$$

For suppose, up to symmetry, that v_1 is not in C. Then we can select a vertex $q_1' \in C_1$ that misses v_1. Let P be the subgraph of G induced by $(V(K) \setminus C) \cup \{q_1', q_2, \dots, q_m\}$. Let R be a shortest path from q_1' to v_1 in $P \setminus \{v_0, w_0\}$. Then R has length at least 2 and $V(R) \cup \{v_0, w_0\}$ induces a house in G, a contradiction. So (5) holds.

By (5), we may assume that $v_1 = c_2$ and $w_1 = c_3$. Recall that q_2 is a vertex of C_2 that misses w_0, and q_3 is a vertex of C_3 that misses v_0. Since v_1 and w_1 are not adjacent, the lengths s, t of P_2, P_3 cannot both be equal to 1; thus let us assume up to symmetry that $s \geq 2$. We claim that:

$$\text{Vertex } q_2 \text{ is adjacent to all of } C_1. \tag{6}$$

For suppose that q_2 is not adjacent to some vertex $q_1'' \in C_1$. Let P be the subgraph of G induced by $(V(K) \setminus C) \cup \{q_1'', q_2, \dots, q_m\}$. Let R be a shortest path from q_1'' to q_2 in $P \setminus \{v_0, w_0\}$. Then R has length at least 2 and $V(R) \cup \{v_0, w_0\}$ induces a house in G, a contradiction. So (6) holds.

Now let q_1 be any vertex of C_1, and let P be the subgraph of G induced by $(V(K) \setminus C) \cup \{q_1, \dots, q_m\}$. We claim that:

$$\text{Every neighbour of } q_2 \text{ in } V(P)\setminus\{v_0, v_2\} \text{ is in } \{q_1, \dots, q_m\}. \qquad (7)$$

For let x be a neighbour of q_2 in $V(P) \setminus \{v_0, v_2\}$. We can suppose that $x \neq q_1$. The subgraph $P \setminus \{q_1, q_2, v_2\}$ is connected, so it contains a shortest path X from v_0 to x. Since v_0 has no neighbour in $V(P) \setminus \{q_1, q_2, w_0\}$, the neighbour of v_0 in X is w_0. Now Lemma 5 can be applied to clique C_2, path X and vertex v_2, which implies that x sees all of C_2. This means that x is in $\{q_1, \dots, q_m\}$ for otherwise v_1 and x would be adjacent in $G^{\mathcal{Q}}$. So (7) holds.

In $G^{\mathcal{Q}}$, let us mark some vertices of $K \setminus v_1$ as follows. Vertices v_0 and v_2 are marked. For $j = 1, \dots, m$ and $j \neq 2$, vertex c_j is marked if and only if in G vertex q_2 sees all vertices of the corresponding clique C_j in G. All other vertices of K are unmarked. Call segment any subpath of length at least 1 of $K \setminus v_1$ whose two end-vertices are marked and whose interior vertices are unmarked. Suppose there exists an odd segment X of length ≥ 3. Then $V(X) \cup \{q_2\}$ induces an odd hole in $G^{\mathcal{Q}\setminus\{C_2\}}$, which contradicts Lemma 8. So every segment has length even or equal to 1. Note that $V(P_1) \cup V(P_2) \setminus \{v_0, v_1\}$ induces a chordless path, of odd length (because r, s have the same parity), and its two extremities are marked; so this path contains an odd segment, which as noted above has length 1. Call y the neighbour of v_2 on that path. Note that we have either $s \geq 3$ and $y = v_3$ or $s = 2$ and $y = u_r$. By (7) and the fact that vertices of C are pairwise non-adjacent, the only possible segment of length one is v_2-y, so y is marked, and (7) implies $y \in C$. Suppose that $s \geq 3$. Then $V(P_3) \cup V(P_2) \setminus \{v_1, v_2\}$ induces a chordless odd path, whose two extremities are marked, so it contains a segment of length 1. The only possible such segment is v_0-w_0, so w_0 is marked, and (7) implies $w_0 \in C$, which contradicts (4). So $s = 2$ and $y = u_r$. Now, since B contains u_r it cannot contain another vertex of C, so w_t is not in C, which by (5) implies $t \geq 2$. Now symmetry between s and t is restored, and as above we can prove that $t = 2$ and q_3 is adjacent to u_r. But then q_2, v_0, w_0, q_3, u_r induce an odd hole or a house in G, a contradiction. □

Now Lemmas 7, 8 and 9 imply that $G^{\mathcal{Q}}$ is an Artemis graph, which proves Theorem 2. Theorem 2 and Lemma 2 imply Theorem 1.

3. Concluding Remarks

This is still not the end of the story. The general method is as follows. Assume that we want to apply a co-coloring algorithm A whose validity is proved on a class $\mathcal{G}$. Then we can use A for the problem co-PrExt on $\mathcal{G}^-$. Since we know that $(\mathcal{G}^-)^+ \subseteq \mathcal{G}$ only, we can wonder what is the class $(\mathcal{G}^-)^+$, because there might exist algorithms that are better than A to co-color graphs in $(\mathcal{G}^-)^+$ (or for solving co-PrExt on $\mathcal{G}^-$). Here we proved that *Perfect*$^-$ = *Meyniel* and that (*Perfect*$^-$)$^+$ = *Meyniel*$^+$ $\subseteq$ Artemis $\subsetneq$ Perfect. Improving from Perfect to Artemis (in the last strict inclusion) has two interesting aspects: First, perfection of Artemis graphs [14] is easier to establish than perfection of Berge graphs. Second, since co-PrExt is polynomial on *Meyniel* graphs with the ellipsoid method, the question arises of finding a combinatorial algorithm for this question. However, we do not have an answer for this and we leave it as an open problem.

The scope of applications of Lemma 1 might not be completely exploited yet: for instance the computational complexity equivalence may work in any computational class $\mathcal{C}$ (APX, NP, ...), provided that the reduction in the proof of Lemma 1 preserves the properties of $\mathcal{C}$. For instance, it is an AP-reduction (see [2] for the background concerning approximability, both in general and concerning coloring problems); so approximability results can be transposed from coloring on $\mathcal{G}$ to PrExt on $\mathcal{G}^-$. An extension could be the converse of Corollary 2; this would allow for a translation of inaproximability results from coloring to precoloring extension. The difficulty here is, given a graph class $\mathcal{G}$ and a graph G (not necessarily in $\mathcal{G}^+$), to find a graph $H \in \mathcal{G}$ and a precoloring Q of H such that $H/Q = \mathcal{G}$ or to certify that there is no such pair H, Q. The complexity of this problem is open, even if we restrict $\mathcal{G}$ to be *Meyniel* (note here that *Meyniel*$^+$ is not even well characterized yet).

Acknowledgements. This research was supported by the ADONET network of the European Community, which is a Marie Curie Training Network.

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Received: September 1, 2005
Final version received: January, 2007