Journal of Combinatorial Theory, Series A 175 (2020) 105270

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

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Orientations and bijections for toroidal maps with prescribed face-degrees and essential girth ☆

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ARTICLE INFO

Article history:
Received 2 July 2018
Received in revised form 1 December 2019
Accepted 1 May 2020
Available online xxxx

Keywords:
Toroidal maps
d-angulations
Girth constraints
Orientations
Bijections
Generating functions

ABSTRACT

We present unified bijections for maps on the torus with control on the face-degrees and essential girth (girth of the periodic planar representation). A first step is to show that for $d \geq 3$ every toroidal d-angulation of essential girth d can be endowed with a certain 'canonical' orientation (formulated as a weight-assignment on the half-edges). Using an adaptation of a construction by Bernardi and Chapuy, we can then derive a bijection between face-rooted toroidal d-angulations of essential girth d (with the condition that, apart from the root-face contour, no other closed walk of length d encloses the root-face) and a family of decorated unicellular maps. The orientations and bijections can then be generalized, for any $d \geq 1$, to toroidal face-rooted maps of essential girth d with a root-face of degree d (and with the same root-face contour condition as for d-angulations), and they take a simpler form in the bipartite case, as a parity specialization. On the enumerative side we obtain explicit algebraic expressions for the generating functions of rooted essentially simple triangulations and bipartite quadrangulations on the torus. Our bijective constructions can be considered as toroidal counterparts of those obtained by Bernardi and the first author in the planar case, and they also build on ideas introduced by Despré, Gonçalves and the second author for

☆ This work was supported by the grant EGOS ANR-12-JS02-002-01 and GATO ANR-16-CE40-0009-01.
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https://doi.org/10.1016/j.jcta.2020.105270

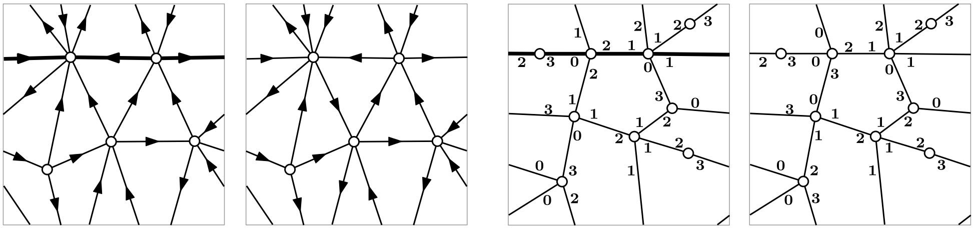
essentially simple triangulations, of imposing a balancedness condition on the orientations in genus 1.

1. Introduction

The enumerative study of (rooted) maps has been a very active research topic since Tutte's seminal results on the enumeration of planar maps [37,38], later extended to higher genus by Bender and Canfield [4]. Tutte's approach is based on so-called loop-equations for the associated generating functions with a catalytic variable for the root-face degree. Powerful methods have been developed to compute the solution of such equations (originally solved by guessing/checking), both in the planar case [28,13] and in higher genus [22].

The striking simplicity of counting formulas discovered by Tutte (e.g., the number of rooted planar simple triangulations with $n+3$ vertices is equal to $\frac{2}{n(n+1)}\binom{4n+1}{n-1}$) asked for bijective explanations. The first such constructions, bijections from maps to certain decorated trees, were introduced by Cori and Vauquelin [20] and Arquès [3] and later further developed by Schaeffer [35], who also introduced with Marcus the first bijection (for bipartite quadrangulations) that extends to higher genus [35, Chap. 6]. The bijection has been adapted in [19] to a form better suited for computing the generating functions, and has been recently extended to non-orientable surfaces [17,11].

In the planar case many natural families of maps considered in the literature are given by restrictions on the face-degrees and on the girth (length of a shortest cycle). For instance loopless triangulations are (planar) maps with all face-degrees equal to 3 and girth at least 2. The bijections developed over the years for such families (in particular, simple quadrangulations [35, Sect. 2.3.3], loopless triangulations [35, Sect. 2.3.4], simple triangulations [32], irreducible quadrangulations [25] and triangulations [24]) shared the feature that each map of the considered family can be endowed with a 'canonical' orientation that is usually specified by outdegree prescriptions (so-called α-orientations [23]), which is then exploited to associate to the map a decorated tree structure. For instance simple triangulations with a distinguished outer face can be endowed with an orientation where all outer vertices have outdegree 1 and all inner vertices have outdegree 3, such orientations being closely related to Schnyder woods [36]. In recent works [7,2] the methodology has been given a unified formalism, where each such bijective construction can be obtained as a specialization of a 'meta'-bijection between certain oriented maps and certain decorated trees, which itself is an adaptation of a bijection developed in [5] (and extended in [6] to higher genus) to count tree-rooted planar maps. A success of this strategy has been to solve for the first time [8] the problem of counting planar maps with control on the face-degrees and on the girth (this has been subsequently recovered



(a) (b)

Fig. 1. (a) Examples of 3-orientations for a toroidal essentially simple triangulation (the first example is not balanced as the bold cycle has outdegree 3 on the upper side and outdegree 1 on the lower side, the second example is balanced). (b) Examples of $\frac{5}{3}$-orientations for a toroidal pentagulation of essential girth 5 (the first example is not balanced as the bold cycle has total weight 4 on the upper side and total weight 2 on the lower side, the second example is balanced).

in [15] and extended to the so-called irreducible setting), and to adapt the bijections to hypermaps [9] and maps with boundaries [10].

Up to now this general strategy based on canonical orientations has been mostly applied in the planar case, while the only bijections known to extend to any genus $g \geq 0$ deal with maps (or bipartite maps) with control on the face-degrees but not on the girth: bijections to labeled mobiles [14,19,16] or to blossoming trees and unicellular maps [34,29]. It has however recently appeared [21] that in the case of genus 1, a bijection based on canonical orientations can be designed for essentially simple triangulations.[1] The canonical orientations used in this construction are 3-orientations (all vertices have outdegree 3) with an additional 'balancedness' property (every non-contractible cycle has the same number of outgoing edges to the left side as to the right side), see Fig. 1(a) for examples. The existence of such orientations builds on an earlier work on toroidal Schnyder woods [27] (see also [30]), and the bijection thus obtained can be considered as a toroidal counterpart of the one in [32]. This strategy has also been recently applied to essentially 4-connected triangulations [12], where the obtained bijection (based on certain 'balanced' transversal structures) is now a toroidal counterpart of the one in [24].

Main results and outline of the article.

In this article, we extend the strategy of [21] to toroidal maps of prescribed essential girth and face-degrees, thereby obtaining bijections with certain decorated unicellular maps. Our bijections can be seen as toroidal counterparts of those given in [7] for planar toroidal d-angulations of essential girth $d \geq 3$, and in [8] for planar maps with prescribed girth and face-degrees.

Our first results deal with toroidal d-angulations of essential girth d, for $d \geq 3$. In the planar case it is known [7] that d-angulations of girth d, with a marked face considered as the outer face, can be endowed with certain 'weighted biorientations' (given by assigning

[1] A map M on the torus is said to have 'essentially' property P if the periodic planar representation M^∞ of M has property P; thus M is essentially simple means that M^∞ is simple. Similarly the *essential girth* of M is defined as the girth of M^∞.

a weight in $\mathbb{N}$ to every half-edge) called $\frac{d}{d-2}$-orientations, such that for every inner edge (resp. inner vertex) the sum of the weights of the incident half-edges is $d-2$ (resp. d). Moreover, each d-angulation of girth d admits a 'canonical' such orientation, called the minimal one. The meta-bijection given in [7] can then be applied to the minimal $\frac{d}{d-2}$-orientations, giving a correspondence with well-characterized decorated trees.

We will prove that a parallel strategy can be applied in genus 1. Precisely, we show in Section 3 that every toroidal d-angulation of essential girth $d \geq 3$ admits a so-called *balanced* $\frac{d}{d-2}$*-orientation*, where again every half-edge is assigned a weight-value in $\mathbb{N}$ such that the total weight of each edge (resp. vertex) is $d-2$ (resp. d) and 'balanced' means that for every non-contractible cycle C, the total weight of half-edges incident to each side of C is the same, see Fig. 1 for examples ($d=3$ on the left side, $d=5$ on the right side). Similarly as in the planar case, when the d-angulation has a distinguished face, the map admits a 'canonical' such orientation, called the minimal one. An extension of the 'meta-bijection' to higher genus (described in Section 4.2 and obtained by adapting the construction of [6]) can then be applied to these orientations, yielding a bijection, stated in Section 4.5, between face-rooted toroidal d-angulations of essential girth d (with the extra condition that apart from the root-face contour, there is no other closed walk of length d that encloses the root-face) and a family of well-characterized decorated unicellular maps of genus 1.

Similarly as in the planar case [8], the strategy can then be extended to face-rooted toroidal maps of essential girth $d \geq 1$, with root-face degree d (with the same root-face contour conditions as for d-angulations). The canonical orientations in that case have similar weight conditions, now allowing for half-edges of negative weights, and the obtained bijections, stated in Section 4.6, keep track of the distribution of the face-degrees, and have a simpler form in the bipartite case (which can be seen as a parity specialization of the general bijection, as in the planar case [7,8]).

Regarding counting results, we show in Section 5 that in certain cases (essentially simple triangulations and essentially simple bipartite quadrangulations), the generating function of the corresponding mobiles can be computed by a similar approach as in [19], and the expressions simplify nicely. Unfortunately, for general d, even if the corresponding unicellular decorated trees are well-characterized, we have not succeeded in deriving an explicit simple expression of the generating function of rooted toroidal d-angulations of essential girth d, as was done in the planar case [7,8,15].

Higher genus extensions?

It is unclear to us if our results could be extended to higher genus. The nice property of the torus is that the Euler characteristic is zero, which is compatible with orientations having homogeneous outdegrees (e.g. for triangulations on the torus there are exactly 3 times more edges than vertices, and the orientations exploited to derive a bijection are those with outdegree 3 at each vertex).

In higher genus it has been shown in [1] that every simple triangulation has an orientation where every vertex-outdegree is a nonzero multiple of 3, hence all vertices have outdegree 3 except for $O(g)$ special vertices whose outdegree is a multiple of 3 larger than 3 (e.g. in genus 2 all vertices have outdegree 3 except for either two vertices of outdegree 6 or one vertex of outdegree 9), and the presence of these special vertices makes it more difficult to come up with a natural canonical orientation amenable to a bijection.

2. Preliminaries

2.1. Maps and essential girth for toroidal maps

A *map* M of genus g is an embedding of a connected graph (possibly with loops and multiple edges) on the orientable surface Σ of genus g, such that all components of $\Sigma\backslash M$ are homeomorphic to open disks; we will mostly consider maps of genus 1, which we call toroidal maps. A map is called *rooted* if it has a marked corner, and is called *face-rooted* if it has a marked face. The *dual* M^* of M is the map obtained by inserting a vertex in each face of M, every edge $e \in M$ yielding a dual edge e^* in M^* that connects the vertices dual to the faces on each side of e. A *walk* in M is a (possibly infinite) sequence of edges traversed in a given direction, such that the head of an edge in the sequence coincides with the tail of the next edge in the sequence (possibly two successive edges in the sequence are the same edge traversed in opposite directions). A *path* in M is a walk with no repeated vertices. A *closed walk* in M is a finite walk such that the head of the first edge in the sequence coincides with the tail of the last edge. We identify two closed walks if they differ by a cyclic shift of the sequence of edges. Hence a closed walk can be seen as a cyclic sequence of edges such that the head of each edge coincides with the tail of the next edge in the sequence. A closed walk is called *non-repetitive* if it does not pass twice by a same edge taken in the same direction. A *cycle* is a closed walk with no repeated vertices.

The *girth* of a map M is the length of a shortest cycle in M. The *essential girth* of a toroidal map M is the girth of the universal cover M^∞ (periodic planar representation). As we will see, the essential girth is at least the girth. A *contractible closed walk* of M (resp. of M^∞) is defined as a non-repetitive closed walk W having a contractible region on its right, which is called the *interior* of W.

Lemma 1. *Let M be a toroidal map. Then the essential girth of M coincides with the length of a shortest contractible closed walk in M.*

Proof. Let d be the essential girth of M and let d' be the length of a shortest contractible closed walk in M. We first make a few observations. Any contractible closed walk W of M^∞ yields a contractible closed walk w in M, called the *projection* of W. Any contractible closed walk of M^∞ that projects to w is called a *replication* of W (in the periodic planar representation, a replication of W is a translate of W by an integer linear combination of two vectors spanning an elementary cell). A closed walk of M^∞ is called *admissible* if its interior does not overlap with the interior of any of its other replications.

Clearly, for W a contractible closed walk of M^∞, the projection of W is a contractible closed walk of M iff W is admissible. This ensures that there is a contractible closed walk of length d' in M^∞, from which a cycle can be extracted. Hence $d' \geq d$. It remains to show that $d \geq d'$. For this, we just have to find an admissible cycle of length d in M^∞. Let C be a cycle of length d in M^∞, with the property that the interior of C does not contain the interior of another cycle of length d. We are going to show that C is admissible. Let $C' \neq C$ be a replication of C, and let R, R' be the respective interiors of C and C'. Assume by contradiction that $R \cap R' \neq \emptyset$. Note that we have

$$|\partial R| + |\partial R'| \geq |\partial(R \cap R')| + |\partial(R \cup R')|,$$

(indeed we have $\partial(R \cup R') \subseteq \partial R \cup \partial R'$, $\partial(R \cap R') \subseteq \partial R \cup \partial R'$, and $\partial(R \cup R') \cap \partial(R \cap R') \subseteq \partial R \cap \partial R'$, so that for every edge e of M, the contribution of e to $|\partial R| + |\partial R'|$ is at least its contribution to $|\partial(R \cap R')| + |\partial(R \cup R')|$). Hence

$$2d \geq |\partial(R \cap R')| + |\partial(R \cup R')|.$$

Since d is the minimal cycle-length in M^∞ we must have $|\partial(R \cap R')| = |\partial(R \cup R')| = d$. Hence the contour of $R \cap R'$ is a cycle of length d, contradicting the initial hypothesis on C. $\square$

The characterization given in Lemma 1 easily ensures that the girth of M is at most its essential girth (indeed, a cycle can be extracted from a shortest contractible closed walk). If M has essential girth d, a *d-angle* of M is a contractible closed walk of length d. It is called *maximal* if its interior is not contained in the interior of another d-angle. A toroidal map M is called *essentially simple* if it has essential girth at least 3 (it means that M^∞ is simple, i.e., has no loop nor multiple edges).

For $d \geq 3$, a map is called a d-angulation if all its faces have degree d. For $d = 3, 4, 5$, such maps are respectively called triangulations, quadrangulations, pentagulations. Note that a toroidal d-angulation has essential girth less than or equal to d (and it can be strictly less), since every face-contour is a d-angle. A toroidal d-angulation of essential girth d is called a *d-toroidal map*.[2] Note that 3-toroidal maps are exactly essentially simple toroidal triangulations. By Euler's formula, one can check that, in a toroidal map with all face-degrees even, a contractible closed walk must have even length. In particular, 4-toroidal maps are the same as essentially simple quadrangulations.

2.2. Constrained orientations and weighted biorientations of maps

For M a map with vertex-set V and edge-set E, and $\alpha : V \to \mathbb{N}$, an *$\alpha$-orientation* [23] of M is an orientation of M such that every vertex has outdegree $\alpha(v)$. A *biorientation* of

[2] The extra condition on the root-face contour mentioned in the abstract and introduction amounts to considering d-toroidal maps where the root-face contour is a maximal d-angle.

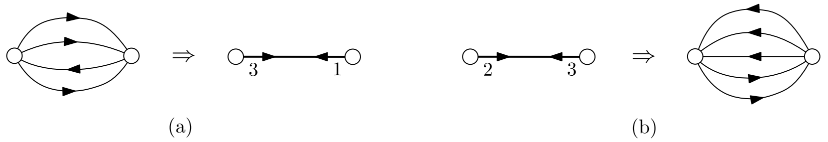


Fig. 2. (a) Rule to obtain an $\frac{\alpha}{\beta}$-orientation of M from an α-orientation of H (with H the β-expansion of M). (b) Rule to obtain an α-orientation of H from an $\frac{\alpha}{\beta}$-orientation of M.

M is the assignment of a direction to every half-edge (half-edges can be either outgoing or ingoing at their incident vertex). The *outdegree* of a vertex v is the total number of outgoing half-edges incident to v. An $\mathbb{N}$*-biorientation* of M is a biorientation of M where every half-edge is given a value in $\mathbb{N}$, which is in $\mathbb{Z}_{>0}$ if the half-edge is outgoing and equal to zero if the half-edge is ingoing. The *weight of a vertex* is the total weight of its incident half-edges. The *weight of an edge* is the total weight of its two half-edges. Note that an orientation can be identified with an $\mathbb{N}$-biorientation where every edge has weight 1. For $\alpha : V \to \mathbb{N}$ and $\beta : E \to \mathbb{N}$, an $\frac{\alpha}{\beta}$*-orientation* of M is an $\mathbb{N}$-biorientation of M such that every vertex v has weight $\alpha(v)$ and every edge e has weight $\beta(e)$. In all this paper, we assume that β takes only strictly positive values. By doing so we can define the β*-expansion* of M as the map H obtained from M after replacing every edge $e = \{u, v\}$ of M by a group of $\beta(e)$ parallel edges connecting u and v. Note that every α-orientation of H yields an $\frac{\alpha}{\beta}$-orientation of M, see Fig. 2.(a). Conversely every $\frac{\alpha}{\beta}$-orientation X of M yields an α-orientation of H, called the β-expansion of X, with the convention that the edge-directions in the group of parallel edges are chosen in the unique way consistent with the weights and such that there is no clockwise cycle within the group, as shown in Fig. 2(b).

Assume M is a face-rooted map, with f its marked face. An orientation of M is called *non-minimal* if there exists a non-empty set S of faces such that $f \notin S$ and every edge on the boundary of S has a face in S on its right (and a face not in S on its left). It is called *minimal* otherwise. An $\frac{\alpha}{\beta}$-orientation of M is called *minimal* if its β-expansion H is minimal (where the root-face of H is the one corresponding to f). Equivalently, an $\frac{\alpha}{\beta}$-orientation of M is non-minimal if there exists a non-empty set S of faces such that $f \notin S$ and every edge on the boundary of S either is simply directed with a face in S on its right or is bidirected.

Consider an orientation of M and a non-contractible cycle C^* of M^* given with a traversal direction (i.e., a cyclic ordering $(h_0, \dots, h_{2k-1})$ of the half-edges on the cycle such that any two successive half-edges h_{2i}, h_{2i+1} are opposite on the same edge, and any two successive half-edges $h_{2i+1}, h_{(2i+2) \bmod 2k}$ are at the same vertex). Let $\delta_R(C^*)$ (resp. $\delta_L(C^*)$) be the number of edges of M crossing C^* from left to right (resp. from right to left). Then the δ*-score* of C^* is defined as $\delta(C^*) = \delta_R(C^*) - \delta_L(C^*)$. Two α-orientations X, X' are called δ*-equivalent* if every non-contractible cycle of M^* has the same δ-score in X and in X'. The following statement is easily deduced from the results and observations in [33] (in particular the fact that the set of contours of non-root faces plus two non-homotopic non-contractible cycles form a basis of the cycle-space):

Theorem 2 *([33]). Let M be a face-rooted map on the orientable surface of genus g endowed with an α-orientation X. Then M has a unique α-orientation X_0 that is minimal*[3] *and δ-equivalent to X.*

Moreover, suppose that M is a toroidal map and X, X' are two α-orientations of M. If there exist two non-contractible non-homotopic[4] *cycles of M that have the same δ-score in X and in X', then X, X' are δ-equivalent.*

We now define the analogue of the function γ introduced in [21,26] for Schnyder woods (see also [30] for a detailed presentation).

If M is endowed with an orientation, and C is a non-contractible cycle of M given with a traversal direction, we denote by $\gamma_R(C)$ (resp. $\gamma_L(C)$) the total number of edges going out of a vertex on C on the right (resp. left) side of C, and define the *γ-score* of C as $\gamma(C) = \gamma_R(C) - \gamma_L(C)$. Two α-orientations X, X' of M are called *γ-equivalent* if every non-contractible cycle of M has the same γ-score in X as in X'. The following theorem is an analog (and a consequence) of Theorem 2; we only state it in genus 1, to keep the proof simpler and as it is the focus of the article.

Corollary 3. *Let M be a face-rooted toroidal map endowed with an α-orientation X. Then M has a unique α-orientation X_0 that is minimal and γ-equivalent to X.*

Moreover, for two α-orientations X, X' of M to be γ-equivalent, it is enough that two non-contractible non-homotopic cycles of M have the same γ-score in X and in X'.

Proof. The *completion-map* of M is the map $\hat{M}$ obtained by superimposing M and M^*. The vertices of $\hat{M}$ are of 3 types: primal vertices (those of M), dual vertices (those of M^*) and edge-vertices (those, of degree 4, at the intersection of an edge $e \in M$ with its dual edge $e^* \in M^*$). Let $\hat{\alpha}$ be the function from the vertex-set of $\hat{M}$ to $\mathbb{N}$ such that, if v is a primal vertex of $\hat{M}$ then $\hat{\alpha}(v) = \alpha(v)$, if v is a dual vertex of $\hat{M}$ then $\hat{\alpha}(v) = \deg(v)$, and if v is an edge-vertex of $\hat{M}$ then $\hat{\alpha}(v) = 1$. Note that any α-orientation Z of M yields an $\hat{\alpha}$-orientation $\hat{Z}$ of $\hat{M}$: each edge of $\hat{M}$ corresponding to a half-edge of an edge $e \in M$ is assigned the direction of e in Z, and each edge of $\hat{M}$ corresponding to a half-edge of an edge $e^* \in M^*$ is directed toward the incident edge-vertex. Clearly the mapping sending Z to $\hat{Z}$ is a bijection from the α-orientations of M to the $\hat{\alpha}$-orientations of $\hat{M}$, with the property that Z is minimal if and only if $\hat{Z}$ is minimal.

Let C be a non-contractible cycle of M given with a traversal direction. Let $(c_1, \ldots, c_k)$ be the cyclic sequence of corners of M that are encountered when walking "just to the right" of C. Since every corner of M corresponds to a face of $\hat{M}^*$, the cyclic sequence $(c_1, \ldots, c_k)$ identifies to a non-contractible cycle of $\hat{M}^*$, which we denote by C^*, see Fig. 3 (note that C^* is clearly homotopic to C). It is then easy to see that for every α-orientation Z of M, we have

[3] It is actually proved in [33] that the set of α-orientations that are δ-equivalent to X is a distributive lattice, of which X_0 is the minimum element.

[4] Two closed curves on a surface are called *homotopic* if one can be continuously deformed into the other.

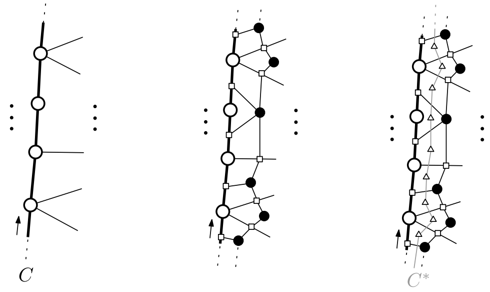


Fig. 3. Left: a non-contractible cycle C of M. Middle: the situation in $\hat{M}$ in the right neighborhood of C. Right: the corresponding non-contractible cycle C^* of $\hat{M}^*$ (which passes by vertices, represented as triangles, that are in the faces corresponding to the corners incident to C on its right side).

$$\gamma_R^Z(C) = \delta_R^{\hat{Z}}(C^*).$$

Hence, for two α-orientations X, X' of M, and for C a non-contractible cycle of M given with a traversal direction, we have $\gamma^X(C) = \gamma^{X'}(C)$ iff $\gamma_R^X(C) = \gamma_R^{X'}(C)$ iff $\delta_R^{\hat{X}}(C^*) = \delta_R^{\hat{X}'}(C^*)$ iff $\delta^{\hat{X}}(C^*) = \delta^{\hat{X}'}(C^*)$. Hence X, X' are γ-equivalent if and only if $\hat{X}, \hat{X}'$ are δ-equivalent, where we use the second statement in Theorem 2 to have the 'only if' direction.[5]

It is then easy to prove the theorem. For X an α-orientation of M, Theorem 2 ensures that there exists an $\hat{\alpha}$-orientation $\hat{X}_0$ of $\hat{M}$ that is minimal and δ-equivalent to $\hat{X}$. By what precedes, X_0 is γ-equivalent to X (and is minimal), hence we have the existence part. Moreover, if there was another α-orientation X_1 minimal and γ-equivalent to X, then $\hat{X}_1$ would be minimal, δ-equivalent to $\hat{X}$, and different from $\hat{X}_0$, yielding a contradiction. This gives the uniqueness part.

We now prove the second statement of the theorem. Let X, X' be two α-orientations of M that have the same γ-score for two non-contractible non-homotopic cycles C_1, C_2. By what precedes, C_1^* and C_2^* have the same δ-score in $\hat{X}$ and in $\hat{X}'$. Hence, by Theorem 2, $\hat{X}$ and $\hat{X}'$ are δ-equivalent, so that X and X' are γ-equivalent. $\square$

More generally if M is endowed with an $\mathbb{N}$-biorientation and C is a non-contractible cycle of M given with a traversal direction, we denote by $\gamma_R(C)$ (resp. $\gamma_L(C)$) the total weight of half-edges incident to a vertex on C on the right (resp. left) side of C, and define the *γ-score* of C as $\gamma(C) = \gamma_R(C) - \gamma_L(C)$.

Two $\frac{\alpha}{\beta}$-orientations, X, X' are called *γ-equivalent* if every non-contractible cycle of M has the same γ-score in X and in X'. The following theorem is a generalization (and a consequence) of Corollary 3 that will be useful for our purpose.

[5] While we do not need it here, we also mention that it is easy to prove by similar arguments that $\hat{X}, \hat{X}'$ are δ-equivalent iff X, X' are δ-equivalent. Hence the γ-equivalence classes on α-orientations are the same as the δ-equivalence classes on α-orientations (which are distributive lattices).

Corollary 4. *Let M be a face-rooted toroidal map endowed with an $\frac{\alpha}{\beta}$-orientation X. Then M has a unique $\frac{\alpha}{\beta}$-orientation X_0 that is minimal and γ-equivalent to X.*

Moreover, for two $\frac{\alpha}{\beta}$-orientations X, X' of M to be γ-equivalent, it is enough that two non-contractible non-homotopic cycles of M have the same γ-score in X and in X'.

Proof. Let H be the β-expansion of M. For Z an $\frac{\alpha}{\beta}$-orientation of M, let $\bar{Z}$ be the β-expansion of Z, i.e., the α-orientation of H obtained from Z by applying the rule of Fig. 2(b). For C a non-contractible cycle of M given with a traversal direction, let $\bar{C}$ be the non-contractible cycle of H that goes along C in the "rightmost" way, i.e. for each edge e of C, the cycle $\bar{C}$ passes by the rightmost edge in the group of $\beta(e)$ edges arising from e. Clearly

$$\gamma_R^Z(C) = \gamma_R^{\bar{Z}}(\bar{C}).$$

Hence, for two $\frac{\alpha}{\beta}$-orientations X, X' of M and for C a non-contractible cycle of M given with a traversal direction, we have $\gamma^X(C) = \gamma^{X'}(C)$ iff $\gamma_R^X(C) = \gamma_R^{X'}(C)$ iff $\gamma_R^{\bar{X}}(\bar{C}) = \gamma_R^{\bar{X}'}(\bar{C})$ iff $\gamma^{\bar{X}}(\bar{C}) = \gamma^{\bar{X}'}(\bar{C})$. Hence X, X' are γ-equivalent if and only if $\bar{X}, \bar{X}'$ are γ-equivalent (we use the second statement in Corollary 3 to have the 'only if' direction).

For X an $\frac{\alpha}{\beta}$-orientation of M, Corollary 3 ensures that H has an α-orientation $\bar{X}_0$ that is minimal and γ-equivalent to $\bar{X}$. Let X_0 be the $\frac{\alpha}{\beta}$-orientation obtained from $\bar{X}_0$ by applying the rule of Fig. 2(a). Since $\bar{X}_0$ is minimal, there is no clockwise cycle inside any group of $\beta(e)$ edges associated to an edge $e \in M$. Hence $\bar{X}_0$ is the β-expansion of X_0, so that (by definition) X_0 is minimal, and moreover it is γ-equivalent to X. This proves the existence part. If there was another $\frac{\alpha}{\beta}$-orientation X_1 minimal and γ-equivalent to X, then $\bar{X}_1$ would be minimal, γ-equivalent to $\bar{X}$, and different from $\bar{X}_0$, contradicting Corollary 3. This proves the uniqueness part.

Let us now prove the second statement of the theorem. Let X, X' be two α-orientations of M that have the same γ-score for two non-contractible non-homotopic cycles C_1, C_2. By what precedes, $\bar{C}_1$ and $\bar{C}_2$ have the same γ-score in $\bar{X}$ and in $\bar{X}'$. Hence, by Corollary 3, $\bar{X}$ and $\bar{X}'$ are γ-equivalent, so that X and X' are γ-equivalent. □

3. Balanced $\frac{d}{d-2}$-orientations on the torus

Let M be a toroidal map. We say that an $\mathbb{N}$-biorientation of M is *balanced* if the γ-score of any non-contractible cycle of M is 0. Note that Corollary 4 implies that if M is face-rooted and admits a balanced $\frac{\alpha}{\beta}$-orientation, then M admits a unique balanced $\frac{\alpha}{\beta}$-orientation that is minimal. For a toroidal d-angulation M, we define a $\frac{d}{d-2}$-*orientation* of M as an $\mathbb{N}$-biorientation of M such that every vertex has weight d and every edge has weight $d-2$ (our bijections for toroidal d-angulations of essential girth d will crucially rely on minimal balanced $\frac{d}{d-2}$-*orientation*). The purpose of this section is to show that a toroidal d-angulation admits a $\frac{d}{d-2}$-orientation iff it has essential girth d, and that in that case it admits a balanced $\frac{d}{d-2}$-orientation.

3.1. Necessary condition on the essential girth

The following lemma gives a necessary condition for a toroidal d-angulation to admit a $\frac{d}{d-2}$-orientation.

Lemma 5. *If a toroidal d-angulation admits a $\frac{d}{d-2}$-orientation then it has essential girth d (i.e. it is a d-toroidal map).*

To prove it, note that the essential girth is clearly at most d since faces have degree d. The fact that the essential girth is at least d is actually a direct consequence of the following statement (which will also be useful in proofs later):

Claim 1. *Let M be a toroidal d-angulation endowed with a $\frac{d}{d-2}$-orientation, and let W be a contractible closed walk of length k, with R the (contractible) enclosed region. Let ϵ be the sum of the weights of half-edges in R that are incident to a vertex on W. Then $\epsilon = k - d$.*

Proof. Let n', m', f' be respectively the numbers of vertices, edges and faces of M that are (strictly) inside R. Since all faces of M have degree d we have (i) $df' = 2m' + k$. Since the weight of every vertex (resp. edge) is d (resp. $d-2$), we have (ii) $dn' + \epsilon = (d-2)m'$. Finally, since R is contractible, the Euler relation ensures that (iii) $n' - m' + f' = 1$. Taking (i)+(ii) gives $d(n' - m' + f') = k - \epsilon$, which together with (iii) gives $d = k - \epsilon$. □

We will see in the Section 3.3 that, conversely, any d-toroidal map admits a $\frac{d}{d-2}$-orientation, and even more, it admits a balanced one.

3.2. Sufficient condition for balancedness

The next lemma shows that γ behaves well with respect to homotopy in $\frac{d}{d-2}$-orientations:

Lemma 6. *Let M be a d-toroidal map endowed with a $\frac{d}{d-2}$-orientation, let C be a non-contractible cycle of M given with a traversal direction, and let $\{B_1, B_2\}$ be a basis for the homotopy of M, such that B_1, B_2 are non-contractible cycles whose intersection is a single vertex or a common path. Let $k_1, k_2 \in \mathbb{Z}^2$, such that C is homotopic to $k_1 B_1 + k_2 B_2$. Then $\gamma(C) = k_1\, \gamma(B_1) + k_2\, \gamma(B_2)$.*

Proof. Let v and u be the two extremities of the path $B_1 \cap B_2$ (possibly $v = u$, if $B_1 \cap B_2$ is reduced to a single vertex). Consider a drawing of M^∞ obtained by replicating a flat representation of M to tile the plane. Let v_0 be a copy of v in M^∞. Consider the walk W

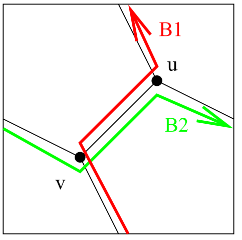


Fig. 4. Intersection of the basis.

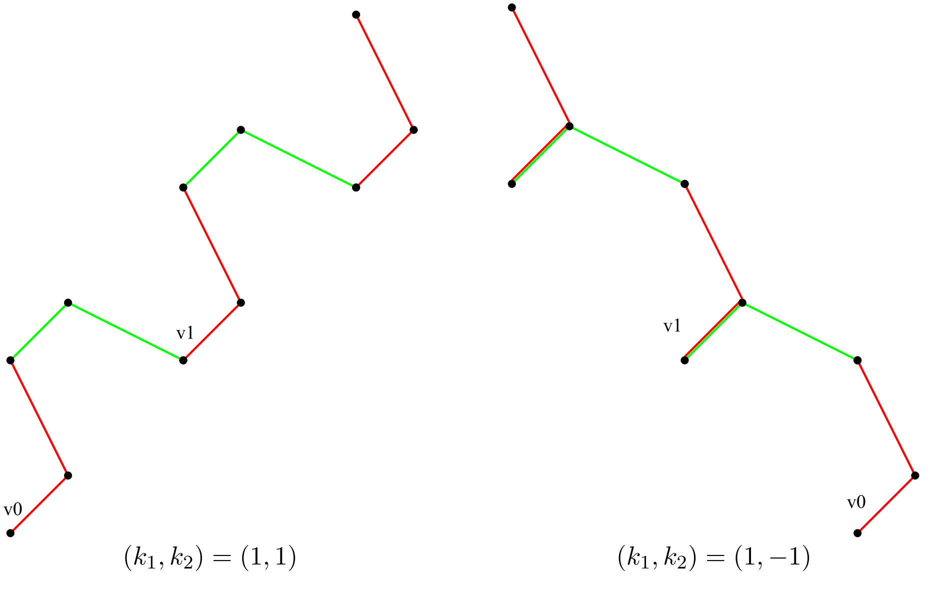


Fig. 5. Replicating "$k_1 B_1$" and "$k_2 B_2$" in the universal cover.

starting from v_0 and following k_1 times the edges corresponding to B_1 and then k_2 times the edges corresponding to B_2 (we are going backward if k_i is negative). This walk ends at a copy v_1 of v. Since C is non-contractible we have k_1 or k_2 not equal to 0 and thus v_1 is distinct from v_0. Let W^∞ be the infinite walk obtained by replicating W (forward and backward) from v_0. Note that their might be some repetition of vertices in W^∞ if the intersection of B_1, B_2 is a path. But in that case, by the choice of B_1, B_2, the walk W^∞ is almost a path, except maybe at all the transitions from "$k_1 B_1$" to "$k_2 B_2$", or (exclusive or) at all the transitions from "$k_2 B_2$" to "$k_1 B_1$", where it can go back and forth a path corresponding to the intersection of B_1 and B_2. The existence or not of such "back and forth" parts depends on the signs of k_1, k_2 and the way B_1, B_2 are going through their common path. Fig. 5 gives an example of this construction with $(k_1, k_2) = (1, 1)$ and $(k_1, k_2) = (1, -1)$ when B_1, B_2 intersect on a path and are oriented the same way along this path as in Fig. 4.

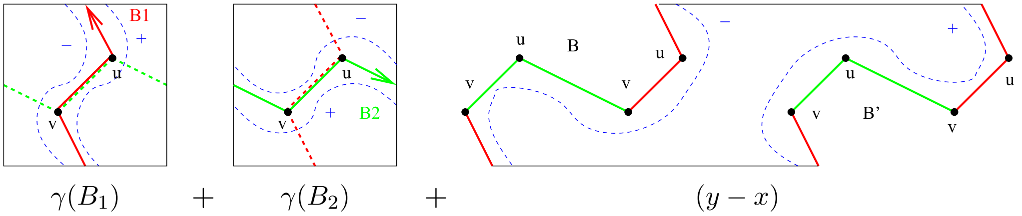


Fig. 6. Case $(k_1, k_2) = (1, 1)$.

We "simplify" W^∞ by removing all the parts that consist of going back and forth along a path (if any) and call B^∞ the obtained walk that is now without repetition of vertices. By the choice of v, the walk B^∞ goes through copies of v. If v_0, v_1 are no more a vertex along B^∞, because of a simplification at the transition from "k_2B_2" to "k_1B_1", then we replace v_0 and v_1 by the next copies of v along W^∞, i.e., at the transition from "k_1B_1" to "k_2B_2".

Since C is homotopic to $k_1B_1+k_2B_2$, we can find an infinite path C^∞, that corresponds to copies of C replicated, that does not intersect B^∞ and situated on the right side of B^∞. Now we can find a copy B'^∞ of B^∞, such that C^∞ lies between B^∞ and B'^∞ without intersecting them. We choose two copies v'_0, v'_1 of v_0, v_1 on B'^∞ such that the vectors v_0v_1 and $v'_0v'_1$ are equal.

Let R_0 be the region bounded by B^∞ and B'^∞. Let R_1 (resp. R_2) be the subregion of R_0 delimited by B^∞ and C^∞ (resp. by C^∞ and B'^∞). We consider R_0, R_1, R_2 as cylinders, where the lines $(v_0, v'_0), (v_1, v'_1)$ (or part of them) are identified. Let B, B', C' be the cycles of R_0 corresponding to $B^\infty, B'^\infty, C^\infty$ respectively.

Let x be the sum of the weights of the half-edges of M incident to B and in the strict interior of R_1. Let y be the sum of the weights of the half-edges of M incident to B' and in the strict interior of R_2. Let x' (resp. y') be the sum of the weights of the half-edges of M incident to C' and in the strict interior of R_2 (resp. R_1). Note that C' corresponds to exactly one copy of C, so $\gamma(C) = x' - y'$. Similarly, B (and B' as well) "almost" corresponds to k_1 copies of B_1 followed by k_2 copies of B_2, except for the fact that we may have removed a back and forth part (if any). In any case we have the following:

Claim. $k_1\,\gamma(B_1) + k_2\,\gamma(B_2) = x - y$.

Proof of the claim. We prove the case where the common intersection of B_1, B_2 is a path (if the intersection is a single vertex, the proof is very similar and even simpler). We assume, by possibly reversing one of B_1 or B_2, that B_1, B_2 are oriented the same way along their intersection, so we are in the situation of Fig. 4.

Fig. 6 shows how to compute $k_1\,\gamma(B_1) + k_2\,\gamma(B_2) + y - x$ when $(k_1, k_2) = (1, 1)$. Then, one can check that the weight of each half-edge of M is counted exactly the same number of times positively and negatively. So everything compensates and we obtain $k_1\,\gamma(B_1) + k_2\,\gamma(B_2) + y - x = 0$.

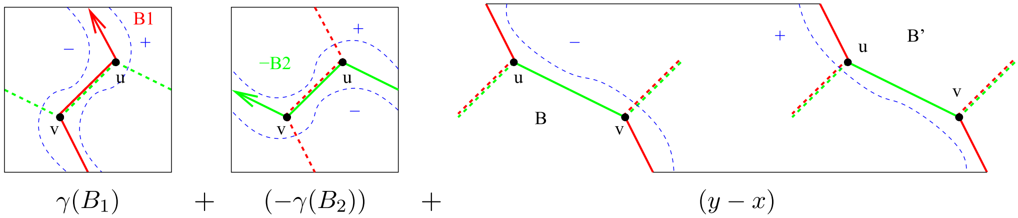


Fig. 7. Case $(k_1, k_2) = (1, -1)$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Fig. 7 shows how to compute $k_1\,\gamma(B_1) + k_2\,\gamma(B_2) + y - x$ when $(k_1, k_2) = (1, -1)$. As above, most of the things compensate but, in the end, we obtain $k_1\,\gamma(B_1)+k_2\,\gamma(B_2)+y-x$ equals the sum of the weights of the half-edges incident to u minus the sum of the weights of the half-edges incident to v. Since the sum of the weights of the half-edges at each vertex is equal to d, we again conclude that $k_1\,\gamma(B_1) + k_2\,\gamma(B_2) + y - x = 0$.

One can easily be convinced that when $|k_1| \geq 1$ and $|k_2| \geq 1$ then the same arguments apply. The only difference is that the red or green part of the figures in the universal cover would be longer (with repetitions of B_1 and B_2). These parts being "smooth", they do not affect the way we compute the equality. Finally, if one of k_1 or k_2 is equal to zero, the analysis is much simpler and the conclusion holds. $\diamond$

For $i \in \{0, 1, 2\}$, let G_i be the cylinder map made of all the vertices and edges of M^∞ that are in the cylinder region R_i. Let k (resp. k') be the length of B (resp. C'). Let n_1, m_1, f_1 be respectively the number of vertices, edges and faces of G_1. Since G_1 is a d-angulation we have $2m_1 = df_1 + (k + k')$. The total weight of the edges of G_1 is $(d-2)m_1 = dn_1 - (x' + y)$. Combining these equalities with Euler's formula $n_1 - m_1 + f_1 = 0$, one obtains $k + k' = x' + y$. Similarly, by considering G_2, one obtains $k + k' = x + y'$. Thus $x' + y = x + y'$, which gives $\gamma(C) = k_1\,\gamma(B_1) + k_2\,\gamma(B_2)$ using the claim. $\square$

Lemma 6 implies the following:

Lemma 7. *Let M be a d-toroidal map endowed with a $\frac{d}{d-2}$-orientation. If the γ-score of two non-contractible non-homotopic cycles of M is* 0*, then the orientation is balanced.*

Proof. Consider two non-contractible non-homotopic cycles C, C' of M, each with a chosen traversal direction, such that $\gamma(C) = \gamma(C') = 0$. Consider a homotopy basis $\{B_1, B_2\}$ of M, such that B_1, B_2 are non-contractible cycles whose intersection is a single vertex or a path. Note that one can easily obtain such a basis by considering a spanning tree T of M, and a spanning tree T^* of M^* that contains no edges dual to T. By Euler's formula, there are exactly 2 edges in M that are not in T nor dual to edges of T^*. Each of these edges forms a unique cycle with T. These two cycles, given with any traversal direction, form the wanted basis.

Let $k_1, k_2, k'_1, k'_2 \in \mathbb{Z}^4$, such that C (resp. C') is homotopic to $k_1B_1 + k_2B_2$ (resp. $k'_1B_1 + k'_2B_2$). Since C is non-contractible we have $(k_1, k_2) \neq (0, 0)$. By possibly exchanging B_1, B_2, we can assume, without loss of generality that $k_1 \neq 0$. By Lemma 6, we have $k_1\gamma(B_1) + k_2\gamma(B_2) = \gamma(C) = 0 = \gamma(C') = k'_1\gamma(B_1) + k'_2\gamma(B_2)$. So $\gamma(B_1) = (-k_2/k_1)\gamma(B_2)$ and thus $(-k_2k'_1/k_1 + k'_2)\gamma(B_2) = 0$. So $k'_2 = k_2k'_1/k_1$ or $\gamma(B_2) = 0$. Suppose by contradiction, that $\gamma(B_2) \neq 0$. Then $(k'_1, k'_2) = \frac{k'_1}{k_1}(k_1, k_2)$, and C' is homotopic to $\frac{k'_1}{k_1}C$. Since C and C' are both non-contractible cycles, it is not possible that one is homotopic to a multiple of the other, with a multiple different from $-1, 1$. So C, C' are homotopic, a contradiction. So $\gamma(B_2) = 0$ and thus $\gamma(B_1) = 0$. Then by Lemma 6 we have $\gamma(C) = 0$ for any non-contractible cycle C of M, and thus the orientation is balanced. □

3.3. *Existence of balanced toroidal $\frac{d}{d-2}$-orientations*

The main goal of this section is to prove the following existence result:

Proposition 8. *Any toroidal d-angulation with essential girth d admits a balanced $\frac{d}{d-2}$-orientation.*

In the case of toroidal triangulations, essentially toroidal 3-connected maps, or essentially 4-connected toroidal triangulations, the proof of existence of analogous "balanced orientations" can be done by doing edge-contractions until reaching a map with few vertices (see [30,12]). We do not know if such a strategy could be applied for $d \geq 5$ (indeed the contraction of an edge in a d-toroidal map results in some faces of size strictly less than d). So we use a different technique in the current paper.

The method consists in defining orientations that are "totally unbalanced" —which we call biased orientations— then taking linear combinations of these biased orientations to obtain a balanced orientation but with rational weights, and finally proving that the orientation that is minimal and γ-equivalent to it is a balanced orientation with integer weights.

3.3.1. *Biased orientations*

Consider a d-toroidal map M, and let C be a non-contractible cycle of M of length k given with a traversal direction. A *biased orientation w.r.t. C* is a $\frac{d}{d-2}$-orientation of M such that $\gamma(C) = 2k$. Note that in a $\frac{d}{d-2}$-orientation of M, the sum of the weights of the half-edges incident to vertices of C is dk and the sum of the half-edges that are on C is $(d-2)k$. So we have $\gamma_L(C) + \gamma_R(C) = dk - (d-2)k = 2k$. Thus a $\frac{d}{d-2}$-orientation of M is a biased orientation w.r.t. C if and only if all the half-edges incident to the left side of C have weight 0.

The goal of this section is to prove the following lemma:

Lemma 9. *Let M be a d-toroidal map and C a non-contractible cycle of M that is shortest in its homotopy class and is given with a traversal direction. Then M admits a biased orientation w.r.t. C.*

To prove Lemma 9 we need to introduce some more general terminology concerning α-orientations.

If S is a subset of vertices of a graph M, then $E[S]$ denotes the set of edges of M with both ends in S. We need the following lemma[6] from [7]:

Lemma 10 *([7]). A graph G admits an $\frac{\alpha}{\beta}$-orientation if and only if $\sum_{e\in E(G)}\beta(e) = \sum_{v\in V(G)}\alpha(v)$, and, for every subset of vertices S of G, we have $\sum_{e\in E[S]}\beta(e) \leq \sum_{v\in S}\alpha(v)$.*

Consider a non-contractible cycle C of M that is a shortest cycle in its class of homotopy and given with a traversal direction. Consider the annular map A obtained from M by cutting M along C and open it as a planar map where vertices of C are duplicated to form the outer face and a *special* inner face of A. Without loss of generality, we assume that A is represented such that the special inner face is on the left side of C. Let $\alpha : V(A) \to \mathbb{N}$ be such that $\alpha(v) = 0$ if v is an outer-vertex of A and $\alpha(v) = d$ otherwise. Let $\beta : E(A) \to \mathbb{N}$ be such that $\beta(e) = 0$ if e is an outer-edge of A and $\beta(e) = (d-2)$ otherwise. Then one can transform any $\frac{\alpha}{\beta}$-orientation of A to a biased orientation of M by gluing back the two copies of C and giving to the half-edges of C the weight they have on the special face of A. Indeed, it is clear by the definition of A and the choice of α, β, that in the obtained $\frac{d}{d-2}$-orientation of M all the weights on half-edges incident to the left side of C are equal to 0, and thus the orientation is biased w.r.t. C by the above discussion. So the existence of a biased orientation (Lemma 9), is reduced to the existence of an $\frac{\alpha}{\beta}$-orientation of A. It is proved in Theorem 24 of [8] that A admits an $\frac{\alpha}{\beta}$-orientation, where the proof is done first in the bipartite case (case of even d) using Lemma 10, and then the general case is derived from the bipartite case using a subdivision argument. We reproduce here in the general case the arguments given in [8] for the bipartite case, for the sake of completeness and since this is one of the key ingredients to obtain a balanced orientation of M.

Lemma 11 *(Theorem 24 in [8]). The annular map A admits an $\frac{\alpha}{\beta}$-orientation.*

Proof. It is not difficult to check that by Euler formula that the first condition of Lemma 10 is satisfied. Let us now prove that the second condition of the lemma is also satisfied.

[6] This lemma can be seen as an application of Hall's theorem regarding the existence of a perfect matching in the bipartite graph obtain from G by copying $\beta(e)$ times each edge e, then subdividing once each edge of the resulting graph, and finally copying $\alpha(v)$ times each initial vertex of G.

Let S be any subset of vertices of A. Suppose first that $A[S]$, the subgraph of A induced by S, is connected. We consider two cases whether S contains some outer vertices of A or not.

- *S contains at least one outer vertex of A:*
  Let S' be the set of vertices obtained by adding to S all the outer vertices of A. Since α equals to 0 for outer vertices, we have $\sum_{v\in S}\alpha(v)=\sum_{v\in S'}\alpha(v)$. Moreover, $E[S]$ is a subset of $E[S']$, so $\sum_{e\in E[S]}\beta(e)\le\sum_{e\in E[S']}\beta(e)$.
  Let n',m',f' be the number of vertices, edges and faces of $A'=A[S']$. Euler's formula says that $n'-m'+f'=2$. The outer face of A' has size k. Since C is a shortest cycle in its class of homotopy, the inner face of A' containing the special face of A has size at least k. Moreover M is a d-angulation, so all the other inner faces of A have size at least d. So finally $2m'\ge d\,(f'-2)+2k$. By combining the two (in)equalities, we obtain $d\,n'-(d-2)\,m'-2k\ge 0$. So $\sum_{v\in S}\alpha(v)-\sum_{e\in E[S]}\beta(e)\ge\sum_{v\in S'}\alpha(v)-\sum_{e\in E[S']}\beta(e)=d(n'-k)-(d-2)(m'-k)\ge 0$.
- *S does not contain any outer vertices of A:*
  Let n',m',f' be the number of vertices, edges and faces of $A'=A[S]$. Then Euler's formula says that $n'-m'+f'=2$. The planar map A' has at most two faces that can be of size strictly less than d: its outer face, and the face of A' containing the special face of A. Note that these two faces are not necessarily distinct and can also be of size more than d. In any case we have $2m'>d\,(f'-2)$. By combining the two (in)equalities, we obtain $d\,n'-(d-2)\,m'>0$. So $\sum_{v\in S}\alpha(v)-\sum_{e\in E[S]}\beta(e)=dn'-(d-2)m'>0$.

In both cases, the second condition of Lemma 10 is satisfied when $A[S]$ is connected. If $A[S]$ is not connected, then we can sum over the different connected components to obtain the result. □

By the above remarks, Lemma 11 implies Lemma 9.

3.3.2. Linear combinations of biased orientations

Consider a d-toroidal map M and B_1,B_2 two non-contractible non-homotopic cycles of M that are both shortest cycles in their respective class of homotopy. Suppose that B_1,B_2 are given with a traversal direction. Let k_1 (resp. k_2) be the length of B_1 (resp. B_2).

Consider D_1,D_2,D_3,D_4 the four $\frac{d}{d-2}$-orientations of M that are biased with respect to $B_1,-B_1,B_2,-B_2$ respectively. The γ-score of B_1,B_2 in these four orientations are given in Table 1 where a,b are integers in $\{-2k_2,\dots,2k_2\}$ and c,d are integers in $\{-2k_1,\dots,2k_1\}$.

For $1\le i\le 4$, let w_i be the *weight function* of D_i. i.e., the function defined on the half-edges of M such that the weight of a half-edge h is $w_i(h)$ in the $\frac{d}{d-2}$-orientation D_i.

Table 1
γ-score of the orientations D_1, D_2, D_3, D_4.

	D_1	D_2	D_3	D_4
$\gamma(B_1)$	$2k_1$	$-2k_1$	c	d
$\gamma(B_2)$	a	b	$2k_2$	$-2k_2$

Let $k = 2k_1k_2$. Let w be the weight function defined on the set of half-edges of M by the following:

$$w = \begin{cases} (2k+bc)k_2 \times w_1 + (2k-ac)k_2 \times w_2 - (a+b)k \times w_3 & \text{if } a+b<0 \\ w_1 + w_2 & \text{if } a+b=0 \\ (2k-bd)k_2 \times w_1 + (2k+ad)k_2 \times w_2 + (a+b)k \times w_4 & \text{if } a+b>0 \end{cases}$$

Note that in all cases, with weight function w, the γ-score of both B_1, B_2 is zero. Indeed, we have:

$$\begin{bmatrix} (2k+bc)k_2 & (2k-ac)k_2 & -(a+b)k & 0 \\ 1 & 1 & 0 & 0 \\ (2k-bd)k_2 & (2k+ad)k_2 & 0 & (a+b)k \end{bmatrix} \times \begin{bmatrix} 2k_1 & a \\ -2k_1 & b \\ c & 2k_2 \\ d & -2k_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & a+b \\ 0 & 0 \end{bmatrix}.$$

Note also that in all cases, for $1 \leq i \leq 4$ the coefficient of w_i is in $\mathbb{N}$, hence $w(h) \in \mathbb{N}$ for every half-edge h of M. We denote by σ the sum of the coefficients, i.e.,

$$\sigma = \begin{cases} (2k+bc)k_2 + (2k-ac)k_2 - (a+b)k & \text{if } a+b<0, \\ 2 & \text{if } a+b=0, \\ (2k-bd)k_2 + (2k+ad)k_2 + (a+b)k & \text{if } a+b>0. \end{cases}$$

Note that $\sigma \geq 1$ in all cases.

Then the total w-weight at any vertex (resp. edge) of M equals σd (resp. $\sigma(d-2)$). Hence w is the weight function of a $\frac{\sigma d}{\sigma(d-2)}$-orientation D^σ of M. In a sense D^σ/σ, obtained from D^σ by dividing all the weights by σ, is a $\frac{d}{d-2}$-orientation of M but with rational weights instead of integers. Note that the proof of Lemma 7 is not using the fact that the weights are integers thus the conclusion holds with rational weights as well.

We have defined the linear combination of biased orientations in such a way that we precisely have $\gamma(B_1) = \gamma(B_2) = 0$ for the orientation D^σ. A variant of Lemma 7 with rational weights implies that D^σ is a balanced $\frac{\sigma d}{\sigma(d-2)}$-orientations and D^σ/σ can be viewed as a balanced $\frac{d}{d-2}$-orientation of M but with rational weights. So we almost have what we are looking for, except for the rational weights that we would like to be integers.

3.3.3. Integrality by minimality

We use the same terminology as in the previous subsection.

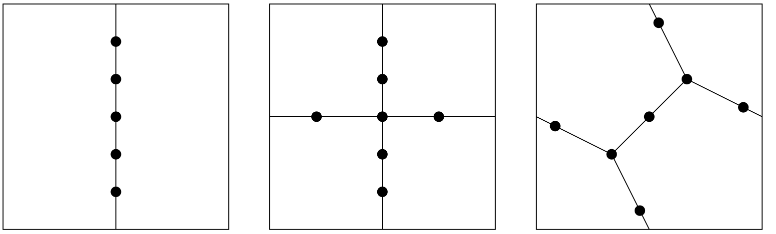


Fig. 8. The three possible cases for M_Q.

Let M be a d-toroidal map, with a distinguished face f_0. By Corollary 4, the map M has a unique minimal $\frac{\sigma d}{\sigma(d-2)}$-orientation $D^\sigma_{\min}$ that is γ-equivalent to D^σ, i.e. that is balanced. In the next lemma, we now prove that the weights of $D^\sigma_{\min}$ are multiple of σ. So $D^\sigma_{\min}/\sigma$, obtained from $D^\sigma_{\min}$ by dividing all the weights by σ, is a balanced $\frac{d}{d-2}$-orientation of M with integer weights and thus this proves Proposition 8.

Lemma 12. *All the weights of $D^\sigma_{\min}$ are multiples of σ.*

Proof. Since the total weight of an edge is a multiple of σ, for each edge $e \in M$ either its two half-edges are not multiple of σ or they are both multiple of σ. We denote by Q the set of edges with weights (on both half-edges) not multiple of σ, and let M_Q be the embedded graph induced by edges in Q and their incident vertices. Note that M_Q is embedded on the torus but is not necessarily a map as some of its faces may not be homeomorphic to an open disk. Since the total weight at any vertex is a multiple of σ, a vertex of M can not be incident to a single edge in Q, hence all the vertices of M_Q have degree at least 2.

Suppose by contradiction, that M_Q has at least two faces (the embedded subgraph M_Q is not necessarily a map, a 'face' refers here to a connected component of the torus cut by M_Q). Let f be a face of M_Q not containing f_0. Let F be the set of edges on the border of f. The weights of the half-edges of F are not multiple of σ. So none of their weights is equal to 0. So in the underlying biorientation of M, all edges of F are bioriented. Thus, in the $\sigma(d-2)$-expansion H of M, the set S of faces of H within f is such that every edge on the boundary of S has a face in S on its right, contradicting the minimality of $D^\sigma_{\min}$. So M_Q has a unique face.

Since the vertices of M_Q have degree at least 2 and M_Q has a unique face, the embedded toroidal graph M_Q has to be one of the graphs depicted in Fig. 8, i.e. it is either a non-contractible cycle, or the union of two non-contractible cycles that are edge-disjoint and intersect at a unique vertex, or it is the union of three edge-disjoint paths such that the union of any two of these paths forms a non-contractible cycle.

In any case, there exists a non-contractible cycle C of M such that on each side of C there is a single incident half-edge in Q. This implies that the sum of the weights of incident half-edges on the left (resp. right) side of C is not a multiple of σ.

On the other hand, since $D^{\sigma}_{\min}$ is balanced we have $\gamma(C) = 0$. Let ℓ be the length of C, so that the sum of the weights of the half-edges of M incident to each side of C is equal to $\frac{1}{2}(\sigma d\ell - \sigma(d-2)\ell) = \sigma\ell$. This is a multiple of σ, giving a contradiction. $\square$

3.4. Bipartite case

For the particular case where d is even and the map is bipartite, we can prove the existence of balanced orientations with even weights, as discussed below.

Consider a d-toroidal map M where d is even, i.e. $d = 2b$ with $b \geq 2$. Note that $d/2 = b$ and $(d-2)/2 = b-1$. So if the weights of a $\frac{d}{d-2}$-orientation of M are even, they can be divided by two to obtain a $\frac{b}{b-1}$-*orientation* of M, i.e., an $\mathbb{N}$-biorientation where every vertex has weight b and every edge has weight $b-1$. Then one can ask the question of existence of balanced $\frac{b}{b-1}$-orientations in that case. The answer to this question is given as follows.

Proposition 13. *When $d = 2b$ with $b \geq 2$, a toroidal d-angulation M with essential girth d admits a balanced $\frac{b}{b-1}$-orientation if and only if M is bipartite. In this case, for any choice of a distinguished face f_0 of M, the unique balanced $\frac{d}{d-2}$-orientation that is minimal has all its weights that are even (hence is a balanced $\frac{b}{b-1}$-orientation upon dividing the weights by 2).*

Proof. If M admits a balanced $\frac{b}{b-1}$-orientation we want to show that M is bipartite. Since the face-degrees of M are even it is enough to check that every non-contractible cycle C of M has even length. Recall that $\gamma_R(C)$ (resp. $\gamma_L(C)$) is the sum of the weights of the half-edges incident to the right (resp. left) side of C. Since the orientation is balanced, we have $\gamma_R(C) = \gamma_L(C)$. Denoting by k the length of C, the sum of the weights of all the half-edges of C is equal to $(b-1)k$. The sum of the weights of all the half-edges incident to vertices of C is bk. Hence $bk = (b-1)k + \gamma_R(C) + \gamma_L(C) = (b-1)k + 2\gamma_R(C)$. So $k = 2\gamma_R(C)$ and thus k is even.

Now suppose that M is bipartite, and consider an arbitrary face f_0 of M. By Proposition 8, M admits a balanced $\frac{d}{d-2}$-orientation. By Corollary 4 we can consider the unique minimal $\frac{d}{d-2}$-orientations D that is balanced. We have the following:

Claim. *The weights of D are even.*

Proof of the claim. The proof follows the same arguments as the proof of Lemma 12. Since each edge has even total weight, either its two half-edges have both even weights, or they have both odd weights. We let Q be the set of edges with odd weights, and assume for contradiction that Q is not empty. Let M_Q be the embedded graph induced by the edges in Q and their incident vertices. Since every vertex has even total weight, it can not be incident to a single edge in Q, hence all vertices of M_Q have degree at least 2.

Suppose by contradiction that M_Q has at least two faces. Let f be a face of M_Q not containing f_0. Let F be the set of edges on the border of f. The weights of the half-edges of F are odd, hence non-zero. Hence, in the underlying biorientation of M, all edges of F are bioriented. Thus, in the $(d-2)$-expansion H of M, the set of faces S of H corresponding to f is such that every edge on the boundary of S has a face in S on its right, contradicting the minimality of D. So M_Q has a unique face.

Since the vertices of M_Q have degree at least 2 and M_Q has a unique face, we again have the property that M_Q is in one of the configurations shown in Fig. 8. In any case, there exists a non-contractible cycle C of M that has a single edge in Q on each side. Hence $\gamma_L(C)$ is odd. On the other hand, since each edge has weight $d-2$ and each vertex has weight d, we have $\gamma_L(C) + \gamma_R(C) = 2\ell$, with ℓ the length of C. Since the orientation is balanced, we have $\gamma_L(C) = \ell$; and since the map is bipartite ℓ is even, contradicting the fact that $\gamma_L(C)$ is odd. $\diamond$

The claim ensures that all the weights of D are even. Thus, dividing all the weights of D by 2, one obtains a balanced $\frac{b}{(b-1)}$-orientation of M. $\square$

4. Bijective results

In this section we state our main bijective results. Similarly as in the planar case [7,8], our starting point is a 'meta-bijection' Φ_+ in any genus g between a family of oriented maps and a family of decorated unicellular maps. The families are defined in Section 4.1 and Φ_+ is presented in Section 4.2 in the oriented setting, and then extended in Section 4.3 to the weighted bioriented setting. In Section 4.5 we then specialize Φ_+ to the balanced $\frac{d}{d-2}$-orientations studied in Section 3, and obtain a bijection for toroidal d-angulations of essential girth d (Theorem 19) which admits a parity specialization in the bipartite case (Corollary 20). Each of these two bijections can be further extended to a bijection for toroidal maps of fixed essential girth with a certain root-face condition (Theorem 21, and Theorem 22 in the bipartite case, both stated in Section 4.6 without proofs, which are delayed to Section 6.3).

4.1. Terminology for oriented maps and mobiles

Consider a face-rooted map M of genus $g \geq 0$. Suppose that M is given with an orientation of its edges such that every vertex has at least one outgoing edge. For an edge $e \in M$, the *rightmost walk* starting from e, is the (necessarily unique and eventually looping) walk starting from e by following the orientation of e, then taking at each step the rightmost outgoing edge, i.e., for any pair e', e'' of consecutive edges along the walk, all edges between e' and e'' in counterclockwise order around their common vertex are ingoing.

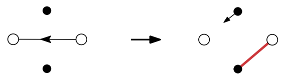


Fig. 9. The local rule applied by the bijection Φ_+ to each edge.

An orientation of M is called a *right orientation* if the following conditions are satisfied:

- every vertex has at least one outgoing edge,
- for every edge e of M, the rightmost walk starting from e eventually loops on the contour of the root-face f_0 with f_0 on its right side.

For $d \geq 1$ and $g \geq 0$, we denote by $\mathcal{O}_d^g$ the family of right orientations of face-rooted maps of genus g whose root-face has degree d.

Let us now define the unicellular maps to be set in bijective correspondence with $\mathcal{O}_d^g$. A *mobile of genus* g is defined as a unicellular map of genus g that is bipartite (it has black vertices and white vertices and every edge connects a black vertex to a white vertex) such that each corner at a black vertex is allowed to carry additional dangling half-edges called *buds*, represented as outgoing arrows. The *excess* of a mobile T is the number of edges minus the number of buds in T, and the family of mobiles of genus g and excess d is denoted by $\mathcal{T}_d^g$.

4.2. Bijection Φ_+ between $\mathcal{O}_d^g$ and $\mathcal{T}_d^g$

Let $d \geq 1$. Similarly as in the planar case developed in [7] we adapt the bijection from [6] into a bijection[7] between $\mathcal{O}_d^g$ and $\mathcal{T}_d^g$ (see Section 6.1 for proof details). For $O \in \mathcal{O}_d^g$ we denote by $\Phi_+(O)$ the embedded graph obtained by inserting a black vertex in each face of O, then applying the local rule of Fig. 9 to every edge of O (thereby creating an edge and a bud), and finally erasing the isolated black vertex in the root-face of O (since the root-face contour is directed clockwise, this black vertex is incident to d buds and no edge). See Fig. 10 for an example.

Theorem 14 *(Oriented case). For $d \geq 1$ and $g \geq 0$, the mapping Φ_+, with the local rule of Fig. 9, is a bijection between the family $\mathcal{O}_d^g$ of oriented maps and the family $\mathcal{T}_d^g$ of mobiles.*

The proof of Theorem 14 is delayed to Section 6.1.

The inverse mapping Ψ_+ is done as follows. Starting from a mobile $T \in \mathcal{T}_d^g$, we insert an *ingoing bud* in every corner of a black vertex u that is just after an edge (not a bud)

[7] It should also be possible, for any $d \leq 0$, to adapt the bijection from [6] into a bijection between the family of genus g mobiles of excess d and a well-characterized family of genus g oriented map, but we will not need it here to get our bijections for toroidal maps with prescribed essential girth.

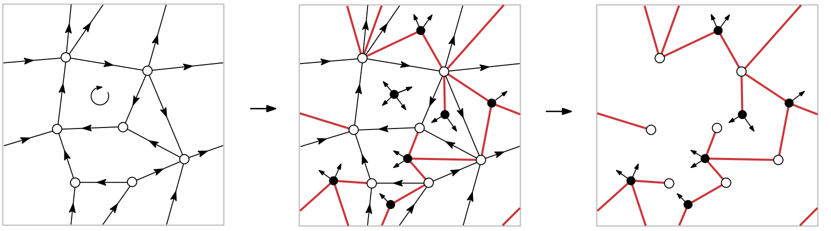


Fig. 10. The bijection Φ_+ from a toroidal orientation in $\mathcal{O}_4^1$ to a toroidal mobile of excess 4 (the root-face is indicated by the small clockwise circular arrow).

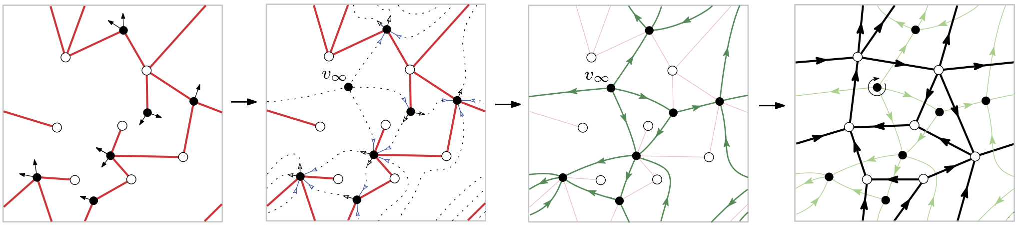


Fig. 11. The inverse mapping Ψ_+: from a mobile in $\mathcal{T}_4^1$ to an orientation in $\mathcal{O}_4^1$.

in counterclockwise order around u. Since T has excess d, there are d more ingoing buds than outgoing buds. We then match the outgoing and ingoing buds according to a walk (with the face on our right) around the unique face of T, considering outgoing buds as opening parentheses and ingoing buds as closing parentheses. Every matched pair yields a directed edge, and we are left with d unmatched ingoing buds (all in the same face of the obtained figure), which we call the *exposed buds* of T. For each such bud, the consecutive half-edge in clockwise order around the incident black vertex is called an *exposed half-edge* of T.

We then join the exposed buds to a newly created vertex v_∞, see Fig. 11 for an example. Let X be the oriented map obtained after erasing the edges of T and the white vertices; and let O be the dual map endowed with the face-rooted dual orientation (that is, for every edge $e \in O$, with $e^* \in X$ the dual edge, we orient e from the left side of e^* to the right side of e^*), where the root-face is taken to be the face dual to v_∞. Then Ψ_+ is the mapping that maps T to O (it is quite easy to check that $\Phi_+(\Psi_+(T)) = T$ when superimposing O, X and T).

4.3. Extension of Φ_+ to the weighted bioriented setting

Similarly as in [7] we may now extend this bijection to the context of biorientations, and then to weighted biorientations. Recall from Section 2, that in a bioriented map M, every half-edge receives a direction (ingoing or outgoing). For $i \in \{0, 1, 2\}$ an edge is said to be *i-way* if it has i outgoing half-edges among its two incident half-edges. For O a bioriented map, the *induced* oriented map $O' = \mu(O)$ is obtained by replacing each 2-way edge by a double edge (enclosing a face of degree 2) directed counterclockwise, and

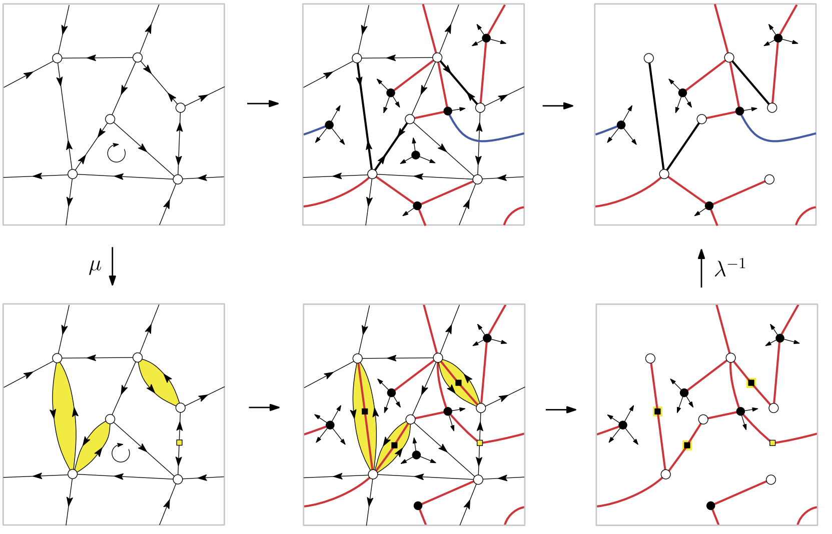


Fig. 12. Top-row: the bijection Φ_+ from a biorientation in $\mathcal{O}_3^1$ to a bimobile in $\mathcal{T}_3^1$, applying the local rules of Fig. 14. The bottom-row shows that the construction amounts to applying the bijection Φ_+ in the oriented setting, upon blowing each 2-way edge into a counterclockwise 2-cycle and inserting a sink of degree 2 in the middle of every 0-way edge.

inserting a vertex of (out)degree 2 in the middle of each 0-way edge, see the left column of Fig. 12 for an example. For $d \geq 1$ and $g \geq 0$ we can now extend the definition of the families $\mathcal{O}_d^g$ to the bioriented setting: a face-rooted bioriented map is said to belong to $\mathcal{O}_d^g$ if the induced oriented face-rooted map is in $\mathcal{O}_d^g$.

Let us now formulate rightmost walks directly on the biorientation to be a bit more explicit on the properties that a biorientation needs to satisfy to be in $\mathcal{O}_d^g$. Consider a face-rooted map M of genus $g \geq 0$. Suppose that M is given with a biorientation such that every vertex has at least one outgoing half-edge. For an outgoing half-edge h of M, we define the *rightmost walk* from h as the (necessarily unique and eventually looping) sequence of half-edges starting from h, and at each step taking the opposite half-edge and then the rightmost outgoing half-edge at the current vertex.

A biorientation of M is called a *right biorientation* if the following conditions are satisfied:

- every vertex has at least one outgoing half-edge,
- for every outgoing half-edge h, the rightmost walk starting from h loops on the contour of the root-face f_0 with f_0 on its right side.

Thus with this definition, a face-rooted bioriented map belongs to $\mathcal{O}_d^g$ if and only if it is a right biorientation, it has genus g and the degree of the root-face is d.

As illustrated in Fig. 12 (forgetting for now the second and third drawing of the top-row), $\Phi_+ \circ \mu$ induces a bijection between bioriented maps in $\mathcal{O}_d^g$ and mobiles in $\mathcal{T}_d^g$ where

Fig. 13. The rules to obtain a mobile $\lambda(T) \in \mathcal{T}_d^g$ (where the new added vertices, of degree 2, are distinguished as square) from a bimobile $T \in \mathcal{T}_d^g$.

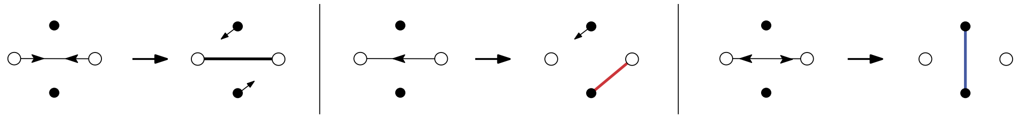


Fig. 14. The local rules applied to each edge by the bijection Φ_+ in the bioriented setting.

some vertices of degree 2 are marked as square vertices (square black vertices correspond to the 2-way edges, square white vertices correspond to the 0-way edges).

We call *bimobile* of genus g a unicellular map of genus g with two kinds of vertices, white or black (this time, black-black edges and white-white edges are allowed), and such that each corner at a black vertex might carry additional dangling half-edges called buds. (Note that a mobile is a special case of bimobile, where all the edges are black-white.) The *excess* of a bimobile is the number of black-white edges plus twice the number of white-white edges, minus the number of buds. We now extend the definition of the family $\mathcal{T}_d^g$ to bimobiles: a bimobile of genus g is said to belong to $\mathcal{T}_d^g$ if its excess is d. For T a bimobile, the *induced mobile* $\lambda(T)$ is obtained by inserting in each white-white edge a square black vertex of degree 2, and inserting in each black-black edge a square white vertex of degree 2 together with two buds at the incident edges, as shown in Fig. 13. Clearly $\lambda(T)$ has the same excess as T. As shown in Fig. 12 the mapping $\lambda^{-1} \circ \Phi_+ \circ \mu$ thus yields a bijection from bioriented maps in $\mathcal{O}_d^g$ to bimobiles in $\mathcal{T}_d^g$ (it just amounts to marking some counterclockwise faces of degree 2 and some sinks of degree 2 in the bijection of Theorem 14). By a slight abuse of notation we refer to $\lambda^{-1} \circ \Phi_+ \circ \mu$ as Φ_+ (adapted to the bioriented setting). It is easy to see that the effect of λ^{-1}, of μ, and of the local rules of Fig. 9 can be shortcut as the local rules shown in Fig. 14 applied to the three types of edges (0-way, 1-way, or 2-way), so that, given a biorientation O in $\mathcal{O}_d^g$, $\Phi_+(O)$ is obtained after applying these rules to every edge of O, and then deleting the isolated black vertex in the root-face.

We obtain:

Corollary 15 *(Extension to the bioriented setting). For $d \geq 1$ and $g \geq 0$, the mapping Φ_+, with the local rules of Fig. 14, is a bijection between the family $\mathcal{O}_d^g$ of bioriented maps and the family $\mathcal{T}_d^g$ of bimobiles.*

Finally, similarly as in the planar case [7], the bijection is directly extended to the weighted setting. A $\mathbb{Z}$-biorientation of a map is a biorientation where every half-edge is given a value in $\mathbb{Z}$, which is in $\mathbb{Z}_{>0}$ (strictly positive) if the half-edge is outgoing and in $\mathbb{Z}_{\leq 0}$ (negative or zero) if the half-edge is ingoing. A $\mathbb{Z}$-bimobile is a bimobile where

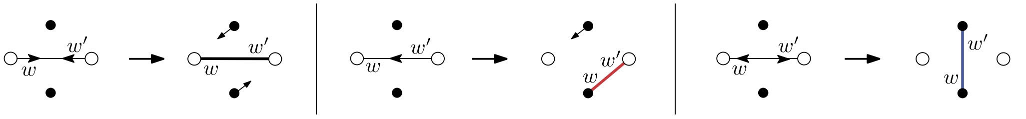


Fig. 15. The local rules applied to each edge by the bijection Φ_+ in the weighted bioriented setting.

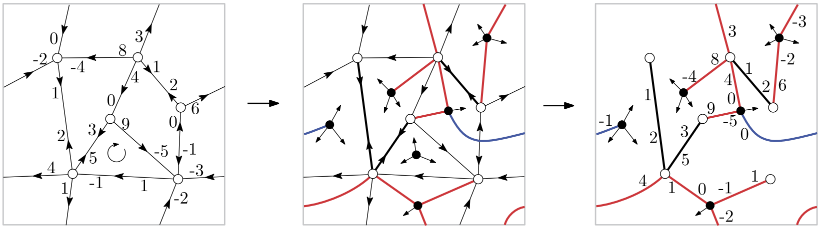


Fig. 16. Example of the bijection Φ_+ from a $\mathbb{Z}$-biorientation in $\mathcal{O}_3^1$ to a $\mathbb{Z}$-bimobile in $\mathcal{T}_3^1$.

every non-bud half-edge is given a value in $\mathbb{Z}$, which is in $\mathbb{Z}_{>0}$ if the half-edge is incident to a white vertex and in $\mathbb{Z}_{\leq 0}$ if the (non-bud) half-edge is incident to a black vertex.

A $\mathbb{Z}$-bioriented face-rooted map is said to belong to $\mathcal{O}_d^g$ if the underlying unweighted face-rooted bioriented map belongs to $\mathcal{O}_d^g$; and a $\mathbb{Z}$-bimobile T is said to belong to $\mathcal{T}_d^g$ if the underlying unweighted bimobile is in $\mathcal{T}_d^g$.

For a $\mathbb{Z}$-bioriented map, the *weight* of a vertex v is the sum of the weights of the outgoing half-edges at v, and the *weight* of a face f is the sum of the weights of the ingoing half-edges that have f on their left (traversing the half-edge toward its incident vertex); and the *weight* of an edge e is the sum of the weights of its two half-edges. For a $\mathbb{Z}$-bimobile, the *weight* of a vertex v is the sum of the weights of the incident half-edges, and the *weight* of an edge e is the sum of the weights of its two half-edges. We extend the bijection Φ_+ to the weighted bioriented setting by the rules of Fig. 15.

Then we obtain the following:

Corollary 16 *(Extension to the weighted bioriented setting). For $d \geq 1$ and $g \geq 0$, the mapping Φ_+, with the local rules shown in Fig. 15, is a bijection between the family $\mathcal{O}_d^g$ of $\mathbb{Z}$-bioriented maps and the family $\mathcal{T}_d^g$ of $\mathbb{Z}$-bimobiles.*

An example is given in Fig. 16 (the weights are omitted in the middle drawing).

As in the planar case [7], for O a $\mathbb{Z}$-bioriented map in $\mathcal{O}_d^g$ and $T = \Phi_+(O)$ the corresponding $\mathbb{Z}$-bimobile, several parameters can be traced:

- each vertex v of O corresponds to a white vertex w of T: the outdegree of v corresponds to the degree of w and the weight of v is the same as the weight of w,
- each non-root face f of O corresponds to a black vertex b of T of the same degree and same weight,
- each edge e of O corresponds to an edge of T of the same weight.

When all the weights of a $\mathbb{Z}$-biorientation are in $\mathbb{Z}_{\geq 0}$ then we have an $\mathbb{N}$-biorientation, as defined in Section 2. Note that an $\mathbb{N}$-biorientation is a $\mathbb{Z}$-biorientation where all the ingoing half-edges have weight 0. The corresponding $\mathbb{Z}$-bimobiles are called $\mathbb{N}$*-bimobiles* (these are the $\mathbb{Z}$-bimobiles where the half-edges at black vertices have weight 0).

We will use specializations of the weighted formulation of Φ_+ (Corollary 16) in order to obtain bijections for d-toroidal maps (relying on $\frac{d}{d-2}$-orientations, so we are in the $\mathbb{N}$-bioriented setting), and more generally for toroidal maps of essential girth d with a root-face of degree d (relying on a generalization of $\frac{d}{d-2}$-orientations in the $\mathbb{Z}$-bioriented setting).

4.4. Necessary condition for $\frac{\alpha}{\beta}$-orientations to be right biorientations

We prove here that minimality is a necessary condition for an $\frac{\alpha}{\beta}$-orientation to be a right biorientation:

Lemma 17. *If a face-rooted $\frac{\alpha}{\beta}$-oriented map belongs to $\mathcal{O}_d^g$, then it is minimal.*

Proof. Suppose by contradiction that a face-rooted map M has an $\frac{\alpha}{\beta}$-orientation X in $\mathcal{O}_d^g$ that is non-minimal. Let f_0 be the root face of M. By definition of non-minimality, there exists a non-empty set S of faces of M, not containing f_0, such that every edge on the boundary of S is either simply directed with a face in S on its right or is bidirected. Hence, while walking clockwise on the contour of S, each half-edge that is encountered just after a vertex is outgoing. Consider such a half-edge h and let W be the rightmost walk starting from h. Then W necessarily stays in S union its contour (it can not escape), and moreover if it loops on the contour of S, then it does so with S on its right side. Since S does not contain f_0, W can not eventually loop on the contour of f_0 with f_0 on the right side, a contradiction. □

4.5. Bijection for toroidal d-angulations of essential girth d

Let $d \geq 3$. We define a toroidal $\frac{d}{d-2}$-mobile as an $\mathbb{N}$-bimobile of genus 1, where every white vertex has weight d, every edge has weight $d-2$ and every black vertex has degree d. We denote by $\mathcal{U}_d$ the family of these $\mathbb{N}$-bimobiles. (Note that there is no black-black edges in an element of $\mathcal{U}_d$.) A simple counting argument gives:

Lemma 18. *Every $\mathbb{N}$-bimobile in $\mathcal{U}_d$ has excess d.*

Proof. For $T \in \mathcal{U}_d$, let $n_{\bullet\circ}$ be the number of black-white edges, $n_{\circ\circ}$ the number of white-white edges, $e = n_{\bullet\circ} + n_{\circ\circ}$ the total number of edges, $n_\bullet$ the number of black vertices, $n_\circ$ the number of white vertices, and k the number of buds. By definition the excess of T is $n_{\bullet\circ} + 2n_{\circ\circ} - k$, so we want to prove that this quantity equals d. Since T is unicellular, Euler's formula gives $e = n_\bullet + n_\circ + 1$. Since every white vertex has weight d, every black

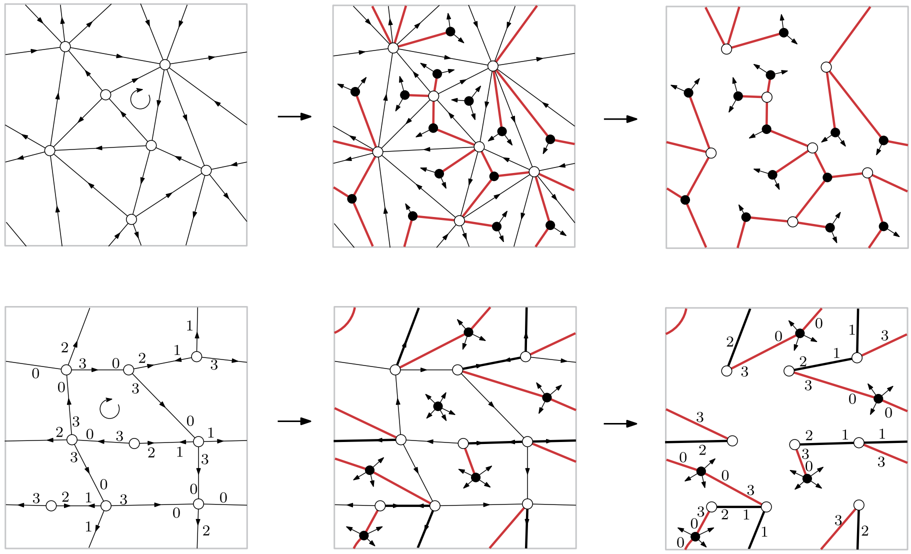


Fig. 17. Left: a toroidal d-angulation in $\mathcal{F}_d$ endowed with its unique balanced $\frac{d}{d-2}$-orientation in $\mathcal{O}_d^1$ ($d = 3$ for the top example, $d = 5$ for the bottom example). Right: the associated $\mathbb{N}$-bimobile.

vertex has weight 0, and every edge has weight $d-2$ we have $dn_\circ = (d-2)e$. Since every black vertex has degree d we have $dn_\bullet = n_{\bullet\circ} + k$. Hence we have at the same time $d(n_\circ + n_\bullet) = (d-2)e + n_{\bullet\circ} + k$ and $d(n_\circ + n_\bullet) = de - d$, so that $2e = k + n_{\bullet\circ} + d$, and thus $n_{\bullet\circ} + 2n_{\circ\circ} - k = d$. $\square$

Clearly the bijection Φ_+ specializes as a bijection between face-rooted toroidal d-angulations endowed with a $\frac{d}{d-2}$-orientation in $\mathcal{O}_d^1$, and the family $\mathcal{U}_d$.

Consider a toroidal $\frac{d}{d-2}$-mobile T and a cycle C of T with a traversal direction. Let $w_L(C)$ (resp. $w_R(C)$) be the total weight of half-edges incident to a white vertex of C on the left (resp. right) side of C; and let $s_L(C)$ (resp. $s_R(C)$) be the number of half-edges, including buds, incident to a black vertex of C on the left (resp. right) side of C. We define $\gamma_L(C) = w_L(C) + s_L(C)$, $\gamma_R(C) = w_R(C) + s_R(C)$, and define the γ-score of C as $\gamma(C) = \gamma_R(C) - \gamma_L(C)$. Then T is called *balanced* if the γ-score of any non-contractible cycle of T is 0. We denote by $\mathcal{U}_d^{Bal}$ the subset of elements of $\mathcal{U}_d$ that are balanced. We will show (Lemma 29 in Section 6.2.1) that Φ_+ specializes into a "balanced version" of the bijection, i.e., a bijection between face-rooted toroidal d-angulations endowed with a balanced $\frac{d}{d-2}$-orientation in $\mathcal{O}_d^1$, and the family $\mathcal{U}_d^{Bal}$.

We denote by $\mathcal{F}_d$ the family of face-rooted d-toroidal maps such that the only d-angle enclosing the root-face is its contour. We will show (Lemma 31 in Section 6.2.1) that a face-rooted toroidal d-angulation M has a balanced $\frac{d}{d-2}$-orientation in $\mathcal{O}_d^1$ if and only if $M \in \mathcal{F}_d$, and in that case M has a unique balanced $\frac{d}{d-2}$-orientation in $\mathcal{O}_d^1$, which is the minimal one (by Lemma 17). Thus we obtain the following bijection:

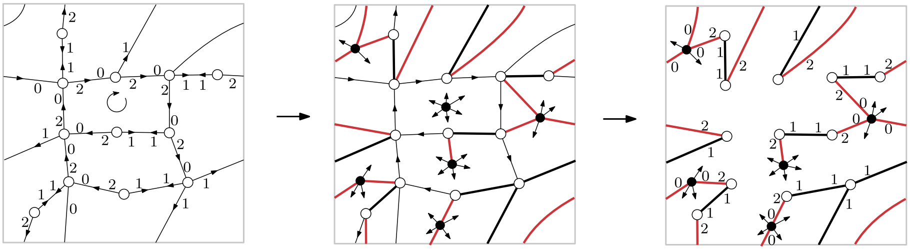


Fig. 18. Left: a bipartite toroidal face-rooted $2b$-angulation in $\hat{\mathcal{F}}_{2b}$ ($b = 3$ in the example), endowed with its unique balanced $\frac{b}{b-1}$-orientation in $\mathcal{O}^1_{2b}$. Right: the associated $\mathbb{N}$-bimobile.

Theorem 19 *(Toroidal d-angulations of essential girth d). For $d \geq 3$, there is a bijection between the map family $\mathcal{F}_d$ and the $\mathbb{N}$-bimobile family $\mathcal{U}_d^{Bal}$. Every non-root face of the map corresponds to a black vertex in the associated $\mathbb{N}$-bimobile.*

Two examples are given in Fig. 17 for $d = 3$ and $d = 5$.

We now give the statement for bipartite maps. Let $b \geq 2$, and $d = 2b$. We denote by $\hat{\mathcal{F}}_{2b}$ the subfamily of maps in $\mathcal{F}_{2b}$ that are bipartite. Proposition 13 ensures that a face-rooted d-toroidal map M is bipartite if and only if all the weights of the unique minimal balanced $\frac{d}{d-2}$-orientation of M are even. Hence, in the bijection of Theorem 19, $M \in \mathcal{F}_{2b}$ is bipartite if and only if all half-edge weights in the associated $\mathbb{N}$-bimobile are even. We formalize this simplification as follows.

We define a $\frac{b}{b-1}$-*mobile* as an $\mathbb{N}$-bimobile of genus 1, where every white vertex has weight b, every edge has weight $b - 1$ and every black vertex has degree $2b$. The family of $\frac{b}{b-1}$-mobiles is denoted by $\hat{\mathcal{U}}_b$. Note that for $T \in \hat{\mathcal{U}}_b$, the $\mathbb{N}$-bimobile T' obtained from T by doubling every half-edge weight is an element of $\mathcal{U}_{2b}$ (in particular, T must have excess $2b$). We say that T is *balanced* if T' is balanced and we denote by $\hat{\mathcal{U}}_b^{Bal}$ the subset of elements of $\hat{\mathcal{U}}_b$ that are balanced. Thus we obtain the following bijection:

Corollary 20 *(Bipartite toroidal $2b$-angulations of essential girth $2b$). For $b \geq 2$, there is a bijection between the map family $\hat{\mathcal{F}}_{2b}$ and the $\mathbb{N}$-bimobile family $\hat{\mathcal{U}}_b^{Bal}$. Every non-root face of the map corresponds to a black vertex in the associated $\mathbb{N}$-bimobile.*

An example is shown in Fig. 18 for $b = 3$.

4.6. Extension to toroidal maps of essential girth d with a root-face of degree d

We state here a generalization of Theorem 19 (resp. Corollary 20) to toroidal face-rooted maps of girth d (resp. bipartite maps of girth $2b$) with root-face degree d (resp. $2b$). Note that we allow here all faces, except the root face, to have degree larger than d. These results can be seen as toroidal counterparts of the bijections obtained in [8] for planar maps of girth d with a root-face of degree d.

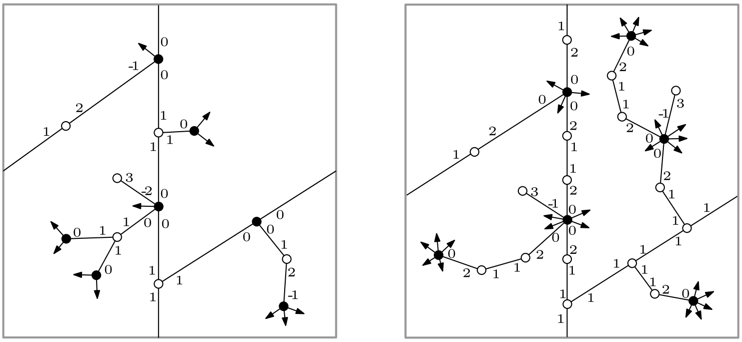


Fig. 19. Left: a $\mathbb{Z}$-bimobile in $\mathcal{V}_3^{Bal}$; apart from the root-face (which has degree 3), the corresponding toroidal map has 4 faces of degree 3, 2 faces of degree 4 and one face of degree 5. Right: a $\mathbb{Z}$-bimobile in $\hat{\mathcal{V}}_3^{Bal}$; apart from the root-face (which has degree 6), the corresponding toroidal bipartite map has 4 faces of degree 6 and 2 faces of degree 8.

Let $d \geq 1$. We denote by $\mathcal{L}_d$ the family of face-rooted toroidal maps of essential girth d, such that the root-face contour is a maximal d-angle.

We now define the mobiles that will be set in bijection with maps in $\mathcal{L}_d$. Recall that a $\mathbb{Z}$-bimobile is a bimobile with integer weights at the non-bud half-edges, which are in $\mathbb{Z}_{>0}$ (resp. in $\mathbb{Z}_{\leq 0}$) when the incident vertex is white (resp. black). We define a *toroidal* $\frac{d}{d-2}$*-$\mathbb{Z}$-mobile* as a $\mathbb{Z}$-bimobile of genus 1 with weights in $\{-2, \ldots, d\}$ such that every white vertex has weight d, every edge has weight $d-2$ and every black vertex of degree i has weight $-i+d$ (hence $i \geq d$). We denote by $\mathcal{V}_d$ the family of these $\mathbb{Z}$-bimobiles. (Note that for $d \leq 3$, an element of $\mathcal{V}_d$ has no white-white edge, while for $d \geq 3$, it has no black-black edge.) A counting argument similar to the one for proving Lemma 18 ensures that every $T \in \mathcal{V}_d$ has excess d. Consider T in $\mathcal{V}_d$ and a cycle C of T with a traversal direction. Let $w_L(C)$ (resp. $w_R(C)$) be the total weight of half-edges incident to vertices (black or white) of C on the left (resp. right) side of C. Let $s_L(C)$ (resp. $s_R(C)$) be the total number of half-edges, including buds, incident to black vertices of C on the left (resp. right) side of C. We define $\gamma_L(C) = w_L(C) + s_L(C)$, $\gamma_R(C) = w_R(C) + s_R(C)$, and the γ-score of C by $\gamma(C) = \gamma_R(C) - \gamma_L(C)$. Then T is called *balanced* if the γ-score of any non-contractible cycle of T is 0. We denote by $\mathcal{V}_d^{Bal}$ the subset of elements of $\mathcal{V}_d$ that are balanced (see the left-part of Fig. 19 for an example).

Theorem 21 *(Toroidal maps). For $d \geq 1$, there is a bijection between the map family $\mathcal{L}_d$ and the $\mathbb{Z}$-bimobile family $\mathcal{V}_d^{Bal}$. Every non-root face in the map corresponds to a black vertex of the same degree in the associated $\mathbb{Z}$-bimobile.*

The proof of Theorem 21 is delayed to Section 6.3.

We now give the statement for bipartite maps. Let $b \geq 1$. We denote by $\hat{\mathcal{L}}_{2b}$ the subfamily of maps in $\mathcal{L}_{2b}$ that are bipartite. We define a *toroidal* $\frac{b}{b-1}$*-$\mathbb{Z}$-mobile* as a $\mathbb{Z}$-bimobile of genus 1 with weights in $\{-1, \ldots, b\}$, all black vertices of even degree, such that every white vertex has weight b, every edge has weight $b-1$ and every black vertex of degree $2i$ has weight $-i+b$ (hence $i \geq b$). The family of these $\mathbb{Z}$-bimobiles is denoted by $\hat{\mathcal{V}}_b$. (Note that for $b \leq 2$, an element of $\hat{\mathcal{V}}_b$ has no white-white edge, while for $b \geq 2$, it has no black-black edge.) Note also that the $\mathbb{Z}$-bimobile T' obtained from an element T in $\hat{\mathcal{V}}_b$ by doubling every half-edge weights is an element of $\mathcal{V}_{2b}$ (in particular, T must have excess $2b$). We say that T is *balanced* if T' is balanced and denote by $\hat{\mathcal{V}}_b^{Bal}$ the subset of elements of $\hat{\mathcal{V}}_b$ that are balanced (see the right-part of Fig. 19 for an example).

Theorem 22 *(Bipartite toroidal maps). For $b \geq 1$, there is a bijection between the map family $\hat{\mathcal{L}}_{2b}$ and the $\mathbb{Z}$-bimobile family $\hat{\mathcal{V}}_b^{Bal}$. Every non-root face in the map corresponds to a black vertex of the same degree in the associated $\mathbb{Z}$-bimobile.*

The proof is again delayed to Section 6.3.

Similarly as for d-angulations, in the bijection of Theorem 21 the map M is bipartite if and only if the half-edge weights in the corresponding $\mathbb{Z}$-bimobile T are even, and upon dividing the weights by 2 the $\mathbb{Z}$-bimobile one obtains is the one associated to M by the bijection of Theorem 22 (which can thus be seen as a parity specialization of the bijection of Theorem 21).

5. Counting results

For $d \geq 1$, let $\mathcal{M}'_d$ (resp. $\mathcal{M}_d$) be the family of rooted (resp. face-rooted) toroidal maps of essential girth d with a root-face of degree d. In Section 5.1 we express the generating function of $\mathcal{M}'_d$ (with control on the face-degrees) in terms of generating functions of balanced toroidal $\frac{d}{d-2}$-$\mathbb{Z}$ mobiles. To do this, we rely on the bijections obtained so far (Theorems 21 and 22) and on a decomposition of maps in $\mathcal{M}'_d$ into a toroidal part and a planar part by cutting along a certain d-angle (the 'maximal' one) enclosing the root-face. Then, in Sections 5.2 and 5.3 we show that the generating function of $\frac{d}{d-2}$-$\mathbb{Z}$ mobiles can be expressed in certain specific cases (we show the approach on essentially simple triangulations and bipartite quadrangulations).

5.1. A general expression in terms of balanced mobiles

For $M \in \mathcal{M}_d$, recall that a d-angle of M is a contractible closed walk of length d, and it is called maximal if its enclosed area is not contained in the enclosed area of another d-angle.

Lemma 23. *Two distinct maximal d-angles of a map $M \in \mathcal{M}_d$ always have disjoint interiors.*

Proof. Let us first reformulate the definition of a d-angle. We define a *region* of M as given by $R = V' \cup E' \cup F'$ where V', E', F' are subsets of the vertex-set, edge-set and face-set of M, such that if $v \in V'$ then the edges incident to v are in E', and if $e \in E'$ then the faces incident to e are in F'. Note that the union (resp. intersection) of two regions is also a region. A *boundary-edge-side* of R is an incidence face/edge of M such that the face is in F' and the edge is not in E'. The *boundary-length* of R, denoted by $\ell(R)$, is the number of boundary-edge-sides of R. A *disk-region* is a region R homeomorphic to an open disk. A d-angle thus corresponds to the (cyclic sequence of) boundary-edge-sides of a disk-region R such that $\ell(R) = d$; and it is *maximal* if there is no other disk-region $\bar{R}$ of boundary-length d such that $R \subset \bar{R}$.

We thus have to show that for two distinct disk-regions R_1, R_2 both enclosed by maximal d-angles, we have $R_1 \cap R_2 = \emptyset$. It is easy to see that for any two regions S_1, S_2 we have $\ell(S_1) + \ell(S_2) = \ell(S_1 \cup S_2) + \ell(S_1 \cap S_2)$ (any incidence face/edge of M has the same contribution to $\ell(S_1) + \ell(S_2)$ as to $\ell(S_1 \cup S_2) + \ell(S_1 \cap S_2)$). Assume $R_1 \cap R_2 \neq \emptyset$. Since R_1 and R_2 are disk-regions, $R_1 \cap R_2$ is a disjoint union of disk-regions $D_1, \ldots, D_k$, and we have

$$2d = \ell(R_1) + \ell(R_2) = \ell(R_1 \cup R_2) + \sum_{i=1}^{k} \ell(D_i).$$

Since M has essential girth d, we have $\ell(D_i) \geq d$ for each $1 \leq i \leq k$. Hence we must have $k = 1$ (we use $\ell(R_1 \cup R_2) \geq 1$ to exclude the case $k = 2$). Since $R_1 \cap R_2$ is a disk-region, the union $R_1 \cup R_2$ must also be a disk-region, hence $\ell(R_1 \cup R_2) \geq d$. But $\ell(R_1 \cup R_2) = 2d - \ell(D_1) \leq d$, hence $\ell(R_1 \cup R_2) = d$. Thus $R_1 \cup R_2$ is enclosed by a d-angle, contradicting the fact that R_1 and R_2 are enclosed by maximal d-angles. $\square$

Every $M \in \mathcal{M}_d$ is rooted in a face f_0 of degree d, so f_0 is included in a maximal d-angle, and Lemma 23 ensures that M has a *unique* maximal d-angle enclosing the root-face. This d-angle is called the *root-d-angle*. Consider the operation of cutting along the root-d-angle C of M. This operation yields two maps (one on each side of C): a toroidal map L with a marked face of degree d and a planar map A with two marked faces f_0, f_1 each of degree d.

Recall that $\mathcal{L}_d$ is the subfamily of $\mathcal{M}_d$ where the root-face contour is a maximal d-angle; we denote by $\mathcal{L}'_d$ the family of rooted toroidal maps such that the underlying face-rooted map is in $\mathcal{L}_d$. Moreover we let $\mathcal{A}'_d$ be the family of planar maps of girth d with two marked faces f_0, f_1 of degree d, and a marked corner in f_0 (we consider f_1 as the outer face). Then the previous decomposition at the root-d-angle yields (see Fig. 20)

$$\mathcal{M}'_d \simeq \mathcal{L}'_d \times \mathcal{A}'_d. \tag{1}$$

Let $M_d \equiv M_d(z; x_d, x_{d+1}, \ldots)$ (resp. $L_d \equiv L_d(z; x_d, x_{d+1}, \ldots)$) be the generating function of maps in $\mathcal{M}'_d$ (resp. in $\mathcal{L}'_d$), with z dual to the number of vertices and x_i dual to the number of non-root faces of degree i. And let $A_d \equiv A_d(z; x_d, x_{d+1}, \ldots)$ be the generating function of maps in $\mathcal{A}'_d$, with z dual to the number of vertices not incident to f_1, and x_i dual to the number of non-marked faces of degree i. Then by (1) we have

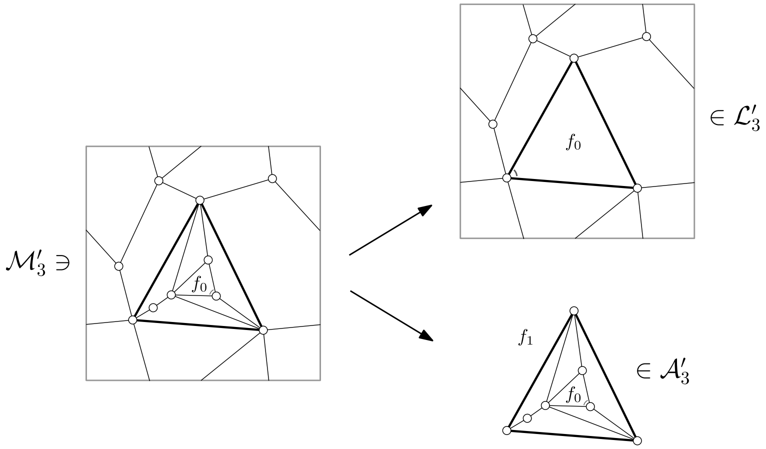


Fig. 20. Left: a map $M \in \mathcal{M}'_3$. Cutting along the root-3-angle of M, one obtains a map $L \in \mathcal{L}'_3$ and a map $A \in \mathcal{A}'_3$. One of the 3 vertices on the root-3-angle can be canonically chosen (i.e., the first of the 3 vertices that is reached in a left-to-right depth-first-search starting from the root-corner), and this vertex is taken as the one incident to the root-corner of L. The correspondence thus obtained is bijective.

$$M_d = L_d \cdot A_d. \tag{2}$$

The generating function A_d has already been computed bijectively in [8], it reads:

$$A_d = (1 + W_0)^d,$$

where W_0 is part of a finite set $W_{-1}, W_0, \dots, W_{d-1}$ of series (in the variables $z, x_d, x_{d+1}, \dots$) that are specified by the system[8]:

$$\begin{cases} W_j = z\, h_{j+2}(W_1, \dots, W_{d-1}) & \text{for all } j \text{ in } [-1 \mathinner{..} d-3], \\ W_j = [u^{j+2}] \displaystyle\sum_{i \geq d} x_i u^i (1 + W_0 + u^{-1} W_{-1} + u^{-2})^{i-1} & \text{for all } j \text{ in } \{d-2, d-1\}, \end{cases} \tag{3}$$

where h_j denotes the multivariate polynomial in the variables $w_1, w_2, \dots$ defined by:

$$h_j(w_1, w_2, \dots) = [t^j] \frac{1}{1 - \sum_{i>0} t^i w_i} = \sum_{r \geq 0} \sum_{\substack{i_1, \dots, i_r > 0 \\ i_1 + \cdots + i_r = j}} w_{i_1} \cdots w_{i_r}. \tag{4}$$

Regarding L_d, let $\mathcal{T}_d \equiv \mathcal{V}_d^{Bal}$ be the family of balanced toroidal $\frac{d}{d-2}$-$\mathbb{Z}$-mobiles and let $\mathcal{T}'_d$ be the family of objects in $\mathcal{T}_d$ where one of the d exposed half-edges is marked (see Section 4.2 for the definition of exposed half-edges). Then the bijection of Theorem 21 directly yields a bijection between $\mathcal{L}'_d$ and $\mathcal{T}'_d$ (indeed the bijection of Theorem 21 relies

[8] We use the usual bracket notation: if $P = \sum_k a_k u^k$, then $[u^k]P = a_k$.

on the general bijection given in Corollary 16, for which there is a natural 1-to-1 correspondence between the d corners in the root-face and the d exposed half-edges). Hence L_d is also the generating function of balanced toroidal $\frac{d}{d-2}$-$\mathbb{Z}$-mobiles with a marked exposed half-edge, with z dual to the number of white vertices and x_i dual to the number of black vertices of degree i.

For a unicellular map M of positive genus, the *core* C of M is obtained from M by successively deleting leaves, until there is no leaf (so C has all its vertices of degree at least 2; the deleted edges form trees attached at vertices of C). In C we call *maximal chain* a path P whose extremities have degree larger than 2 and all non-extremal vertices of P have degree 2. Then the *kernel* K of M is obtained from C by replacing every maximal chain by an edge. In genus 1 it is known that the kernel of a unicellular map is either made of one vertex with two loops (double loop) or is made of 2 vertices and 3 edges joining them (triple edge).

Hence there are two types of toroidal mobiles, those where the associated kernel is the triple edge, called of type I, and those where the associated kernel is the double loop, called of type II. Let $G_d \equiv G_d(z; x_d, x_{d+1}, \ldots)$ (resp. $H_d \equiv H_d(z; x_d, x_{d+1}, \ldots)$ be the generating function of elements of type I (resp. type II) in $\mathcal{T}_d$ and with a marked half-edge in the associated kernel. And let $\widetilde{G}_d \equiv \widetilde{G}_d(z; x_d, x_{d+1}, \ldots)$ (resp. $\widetilde{H}_d \equiv \widetilde{H}_d(z; x_d, x_{d+1}, \ldots)$ be the generating function of elements of type I (resp. type II) in $\mathcal{T}_d$ and with a marked exposed half-edge. In all these generating functions, z is dual to the number of white vertices and x_i is dual to the number of black vertices of degree i. We have $L_d = (\widetilde{G}_d + \widetilde{H}_d)$, so by what precedes $M_d = A_d \cdot (\widetilde{G}_d + \widetilde{H}_d)$; and by a classical double-counting argument we have $\widetilde{G}_d = \frac{d}{6} G_d$ and $\widetilde{H}_d = \frac{d}{4} H_d$. Hence we obtain the following expression of M_d in terms of generating functions of balanced toroidal $\frac{d}{d-2}$-$\mathbb{Z}$-mobiles:

Proposition 24. *For $d \geq 1$, the generating function M_d is given by*

$$M_d = d \cdot A_d \cdot (\tfrac{1}{6} G_d + \tfrac{1}{4} H_d).$$

Very similarly we can obtain a general expression in the bipartite case. For $b \geq 1$, let $\hat{M}_{2b}$, $\hat{L}_{2b}$ and $\hat{A}_{2b}$ be the generating functions gathering (respectively) the terms of M_{2b}, L_{2b} and A_{2b} given by bipartite maps.

Then, specializing (1) to bipartite maps yields

$$\hat{M}_{2b} = \hat{L}_{2b} \cdot \hat{A}_{2b}.$$

In addition the generating function $\hat{A}_{2b}$ has been given an explicit expression in [8], it reads:

$$\hat{A}_{2b} = (1 + V_0)^{2b}$$

where V_0 is part of a finite set $\{V_0, \ldots, V_{b-1}\}$ of generating functions specified by the system:

$$\begin{cases} V_j = z\, h_{j+1}(V_1, \ldots, V_{b-1}) & \text{for all } j \text{ in } [0\,..\,b-2], \\ V_{b-1} = \sum_{i \geq b} x_{2i} \binom{2i-1}{i-b} (1+V_0)^{b+i-1} & \end{cases} \tag{5}$$

Let $\hat{G}_{2b} \equiv \hat{G}_{2b}(z; x_2, x_4, \ldots)$ (resp. $\hat{H}_{2b} \equiv \hat{H}_{2b}(z; x_2, x_4, \ldots)$) be the generating function of balanced toroidal $\frac{b}{b-1}$-$\mathbb{Z}$-mobiles of type I (resp. type II) with a marked half-edge in the associated kernel, with z dual to the number of white vertices and x_{2i} dual to the number of black vertices of degree $2i$. By the very same arguments as to prove Proposition 24, we can express $\hat{M}_{2b}$ in terms of generating functions of balanced toroidal $\frac{b}{b-1}$-$\mathbb{Z}$-mobiles:

Proposition 25. *For $b \geq 1$, the generating function $\hat{M}_{2b}$ is given by*

$$\hat{M}_{2b} = 2b \cdot \hat{A}_{2b} \cdot (\tfrac{1}{6}\hat{G}_{2b} + \tfrac{1}{4}\hat{H}_{2b}).$$

Propositions 24 and 25 ensure that the enumeration of rooted toroidal maps (resp. bipartite toroidal maps) of essential girth d (resp. $2b$) and root-face degree d (resp. $2b$), with control on the face-degrees, amounts to counting balanced toroidal $\frac{d}{d-2}$-$\mathbb{Z}$-mobiles (resp. $\frac{b}{b-1}$-$\mathbb{Z}$-mobiles) with control on the degrees of the black vertices. We show in the next two sections that this can be carried out for essentially simple triangulations and for essentially simple bipartite quadrangulations, yielding the two simple generating function expressions stated next:

Proposition 26 *(Essentially simple triangulations). Let t_n be the number of essentially simple rooted toroidal triangulations with n vertices. Then*

$$\sum_{n \geq 1} t_n z^n = \frac{r}{(1-3r)^2},$$

where $r \equiv r(z)$ is given by $r = z(1+r)^4$.

Proposition 27 *(Essentially simple quadrangulations). Let q_n be the number of rooted toroidal quadrangulations with n vertices (and also n faces) that are essentially simple and bipartite. Then*

$$\sum_{n \geq 1} q_n z^n = \frac{r^2}{(1+2r)(1-2r)^2},$$

where $r \equiv r(z)$ is given by $r = z(1+r)^3$.

Similar calculations could be carried out for bipartite quadrangulations and for essentially loopless triangulations. The expression for the series of rooted toroidal bipartite quadrangulations (counted by vertices) is $F(z) = \frac{r^2(1+3r)}{(1+r)(1-3r)^2}$ where $r \equiv r(z)$ is given by $r = z(1+3r)^2$. Bijective derivations of this formula have been given in [19,29]. And the expression for the series of rooted toroidal essentially loopless triangulations (counted by vertices) is $G(z) = \frac{r(1+2r)}{(1-4r)^2}$ where $r \equiv r(z)$ is given by $r = z(1+2r)^3$. By a classical substitution approach [28, Sec. 2.9] the series $F(z)$ can be related to the series of Proposition 27 (and similarly the series $G(z)$ can be related to the series of Proposition 26), so that one expression can be deduced from the other one (however via some algebraic manipulations, so a bijective derivation of one expression does not yield a bijective derivation of the other expression via this approach).

Calculations for toroidal d-angulations of essential girth $d \geq 5$ seem much more technical. In principle the line of approach we follow in the next two subsections is doable (see [16] where it is carried out for constellations and hypermaps of arbitrarily large face-degrees) and should at least yield an algebraic expression, but likely a complicated one, whereas it is to be expected that the final expression should be simple.[9] In this perspective it would be helpful to have a better combinatorial explanation of the simplicity of the generating function expressions obtained in Propositions 26 and 27.

5.2. Bijective derivation of Proposition 26

In this section we compute the generating function $T(z)$ of rooted toroidal triangulations that have essential girth 3 (or equivalently, that are essentially simple), with z dual to the number of vertices. Note that, for $d = 3$, a toroidal $\frac{d}{d-2}$-mobile T has all its edges of weight 1, hence all edges are black-white with weight 1 on the half-edge incident to the white extremity. Since white vertices have weight 3, they have degree 3. Hence, for $d = 3$ the toroidal $\frac{d}{d-2}$-mobiles identify to toroidal mobiles (edges are black-white, buds are at black vertices only) where every vertex (white or black) has degree 3, which we call *3-regular toroidal mobiles*, see Fig. 21 (1st drawing) for an example. Note that such mobiles must be of type I, since in type II the unique vertex in the kernel must have degree at least 4. A 3-regular toroidal mobile T is called *balanced* if every cycle of T has the same number of incident half-edges on the left side as on the right side. Let $N(z)$ be the generating function of balanced 3-regular toroidal mobiles with a marked half-edge in the associated kernel.

When setting $x_i = \delta_{i=3}$ in system (3), one obtains $W_0 = zW_1^2$ and $W_1 = (1 + W_0)^2$. Let $R \equiv R(z)$ and $S \equiv S(z)$ be given by $R = 1 + W_0$ and $S = W_1$. So R, S satisfy the system $\{R = 1 + zS^2, S = R^2\}$. Then by Proposition 24, we have:

[9] Indeed, combining the substitution approach in [15] to deal with girth constraints, together with the expressions obtained from the topological recursion approach for toroidal maps with no girth constraint [22, 18], it should be possible to show that when the face-degrees are bounded (i.e., for some fixed N, the face-degree variables x_{2i} are taken to be 0 for $i > N$), the generating function $\hat{M}_{2b}$ has a rational expression in terms of the series $V_0, \ldots, V_b$ and the variables $x_{2b}, \ldots, x_{2N}$, and a similar rationality property should hold for M_d.

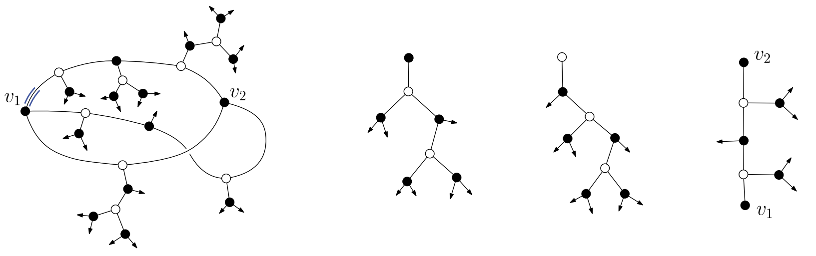


Fig. 21. From left to right: a toroidal 3-regular mobile T counted by $N_{\bullet\bullet}(z)$ (where the marked half-edge of the kernel is indicated); a rooted R-mobile; a rooted S-mobile; and a bi-rooted 3-regular mobile (the second branch of T, both roots are black).

$$T(z) = \frac{1}{2}R(z)^3 N(z).$$

Note that the generating function $N(z)$ splits as

$$N(z) = N_{\bullet\bullet}(z) + N_{\circ\bullet}(z) + N_{\bullet\circ}(z) + N_{\circ\circ}(z) = N_{\bullet\bullet}(z) + 2N_{\bullet\circ}(z) + N_{\circ\circ}(z),$$

depending on the colors of the two vertices v_1, v_2 of the kernel (with v_1 the one incident to the marked half-edge), where the second equality follows from $N_{\bullet\circ}(z) = N_{\circ\bullet}(z)$, since v_1 and v_2 play symmetric roles.

We now define a *rooted* mobile as a planar mobile with a marked vertex that is a leaf, called the *root* (it is allowed for a rooted mobile to be just made of a black vertex with a single incident bud). And we define a *bi-rooted* mobile as a mobile with two marked vertices v_1, v_2 that are leaves, called *primary root* and *secondary root*. A rooted or bi-rooted mobile is called *3-regular* if all its non-root vertices have degree 3.

An *R-mobile* (resp. *S-mobile*) is defined as a rooted 3-regular mobile where the root is black (resp. white), see 2nd and 3rd drawing in Fig. 21. By a decomposition at the root, one checks that R is the generating function of R-mobiles and S is the generating function of S-mobiles, with z dual to the number of non-root white vertices.

For a bi-rooted mobile, the path connecting the two roots is called the *spine*, the traversal direction being from the primary to the secondary root (see the fourth drawing of Fig. 21). For each non-extremal vertex v of the spine, the *balance* at v is defined as the number of half-edges (including buds) incident to v on the left side of the spine, minus the number of half-edges (including buds) incident to v on the right side of the spine. And the *balance* of the bi-rooted mobile is defined as the total balance over all non-extremal vertices of its spine. For a bi-rooted 3-regular mobile, the balance at each vertex of the spine is either $+1$ or -1, so that the sequence of balances along the spine is naturally encoded by a directed path with steps in $\{-1, +1\}$, and the final height of the path is the balance of the rooted bimobile, see Fig. 22.

Clearly a 3-regular toroidal mobile T (with a marked half-edge in the kernel) decomposes into an ordered triple of bi-rooted 3-regular mobiles (one for each edge of the kernel), and T is balanced if and only if the 3 bi-rooted mobiles have the same balance.

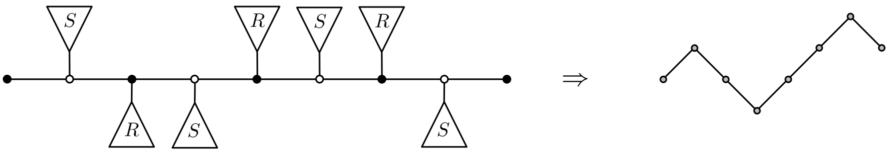


Fig. 22. Left: a bi-rooted mobile of balance 1 (generic notation for mobiles hanging from the spine). Right: the associated path with steps +1 or −1 (up or down), ending at height 1.

Hence, if for $i \in \mathbb{Z}$ we let $K^{(i)}_{\bullet\bullet}(z)$, $K^{(i)}_{\bullet\circ}(z)$, $K^{(i)}_{\circ\circ}(z)$ be the generating functions of bi-rooted 3-regular mobiles of balance i where v_1, v_2 are black/black (resp. black/white, white/white), and with z dual to the number of non-root white vertices, then we find

$$N_{\bullet\bullet}(z) = \sum_{i\in\mathbb{Z}} K^{(i)}_{\bullet\bullet}(z)^3, \quad N_{\bullet\circ}(z) = z\sum_{i\in\mathbb{Z}} K^{(i)}_{\bullet\circ}(z)^3, \quad N_{\circ\circ}(z) = z^2\sum_{i\in\mathbb{Z}} K^{(i)}_{\circ\circ}(z)^3.$$

For $i \in \mathbb{Z}$, let $p_{n,i}$ be the number of walks of length n with steps in $\{-1, 1\}$, starting at 0 and ending at i (note that $p_{n,i} = 0$ if $i \neq n \bmod 2$). We also define the generating function of walks ending at i as

$$P^{(i)}(t) = \sum_{n\geq 0} p_{n,i} t^{\lfloor n/2\rfloor}.$$

We clearly have for $i \in \mathbb{Z}$,

$$K^{(i)}_{\circ\circ}(z) = 0 \text{ for } i \text{ even}, \quad K^{(i)}_{\circ\circ}(z) = R \cdot P^{(i)}(t)\Big|_{t=zRS} \text{ for } i \text{ odd}.$$

$$K^{(i)}_{\bullet\bullet}(z) = 0 \text{ for } i \text{ even}, \quad K^{(i)}_{\bullet\bullet}(z) = zS \cdot P^{(i)}(t)\Big|_{t=zRS} \text{ for } i \text{ odd}.$$

$$K^{(i)}_{\bullet\circ}(z) = 0 \text{ for } i \text{ odd}, \quad K^{(i)}_{\bullet\circ}(z) = P^{(i)}(t)\Big|_{t=zRS} \text{ for } i \text{ even}.$$

Let $B(t) = P^{(0)}(t)$ be the generating function of bridges (walks ending at 0), and let $U(t)$ be the generating function of non-empty Dyck walks (i.e., bridges of positive length never visiting negative values). Then $U \equiv U(t)$ is classically given by

$$U = t \cdot (1 + U)^2,$$

and then (looking at the first return to 0 for non-empty bridges), $B \equiv B(t)$ satisfies the equation $B = 1 + 2t(1 + U)B$, so that

$$B = \frac{1}{1 - 2t \cdot (1 + U)}.$$

Then we have $P^{(i)}(t) = P^{(-i)}(t)$ for $i < 0$, and for $i > 0$ we have (by a classical decomposition at the last visits to $0, 1, \ldots, i - 1$, see [31])

$$P^{(i)}(t) = B \cdot (1+U)^i \cdot t^{\lfloor i/2 \rfloor}.$$

Hence we have

$$\begin{aligned} N_{\circ\circ}(z) &= z^2 R^3 \sum_{\substack{i \in \mathbb{Z} \\ i \text{ odd}}} P^{(i)}(t)^3 \Big|_{t=zRS} \\ &= 2z^2 R^3 \frac{B^3 \cdot (1+U)^3}{1 - t^3(1+U)^6}\Big|_{t=zRS} = 2z^2 R^3 \frac{B^3 \cdot (1+U)^3}{1-U^3}\Big|_{t=zRS} \end{aligned}$$

The last expression can be written in terms of U uniquely. Indeed, all involved quantities can be expressed in terms of U: we have

$$t = \frac{U}{(1+U)^2} = \frac{1}{U + 2 + U^{-1}}, \quad B = \frac{1}{1 - 2t(1+U)} = \frac{1+U}{1-U},$$

and $t = zRS = zR^3 = (R-1)/R = 1 - 1/R$, so that

$$R = \frac{1}{1-t} = \frac{(1+U)^2}{(U^2+U+1)}, \quad z = \frac{R-1}{R^4} = \frac{(U^2+U+1)^3 U}{(U+1)^8}.$$

Overall we find

$$N_{\circ\circ}(z) = \frac{2U^2(U^2+U+1)^2}{(U-1)^4(1+U)^4} = \frac{2(U+1+U^{-1})^2}{(U-2+U^{-1})^2(U+2+U^{-1})^2}.$$

Similarly as in [19], we obtain an expression that is rational in $U + U^{-1}$ and so it is also rational in t since $U + U^{-1} = 1/t - 2$, and then rational in R since $t = 1 - 1/R$. Finally, we obtain

$$N_{\circ\circ}(z) = \frac{2(R-1)^2}{(3R-4)^2 R^2}.$$

Similarly we find

$$\begin{aligned} N_{\bullet\bullet}(z) &= 2z^3 S^3 \frac{B^3 \cdot (1+U)^3}{1 - t^3(1+U)^6}\Big|_{t=zRS} = 2z^3 S^3 \frac{B^3 \cdot (1+U)^3}{1-U^3}\Big|_{t=zRS} \\ &= \frac{2(U+1+U^{-1})^2}{(U-2+U^{-1})^2(U+2+U^{-1})^3} = \frac{2(R-1)^3}{(3R-4)^2 R^3} \end{aligned}$$

and

$$\begin{aligned} N_{\bullet\circ}(z) &= zB^3\Big(\frac{2}{1-U^3} - 1\Big)\Big|_{t=zRS} \\ &= \frac{(U+1+U^{-1})^2(U-1+U^{-1})}{(U-2+U^{-1})^2(U+2+U^{-1})^2} = \frac{(1-R)(2R-3)}{(3R-4)^2 R^2} \end{aligned}$$

We can now conclude the proof of Proposition 26. The sum of the 3 contributions above (with the 3rd contribution multiplied by 2) gives $N(z) = \frac{2(R-1)}{(4-3R)^2R^3}$, so that we obtain

$$T(z) = \frac{1}{2}N(z)R^3 = \frac{(R-1)}{(4-3R)^2},$$

which gives the stated expression upon writing r for $R-1$ (so that r is given by $r = z(1+r)^4$).

5.3. Bijective derivation of Proposition 27

We now compute the generating function $Q(z)$ (with z dual to the number of vertices) of rooted toroidal quadrangulations that are bipartite and essentially simple (we will overrule here some notation from the previous section). For $b = 2$, a toroidal $\frac{b}{b-1}$-mobile T has all its edges of weight 1, hence all edges are black-white with weight 1 on the half-edge incident to the white extremity. Since white vertices have weight 4, they have degree 4. Hence, for $b = 2$ the toroidal $\frac{b}{b-1}$-mobiles identify to toroidal mobiles where black vertices have degree 4 and white vertices have degree 2, which we call $(4,2)$-*regular toroidal mobiles*. Such a mobile is called *balanced* if, for every cycle, it has the same number of incident half-edges (including buds) on the left side as on the right side. Let $N_{\mathrm{I}}(z)$ (resp. $N_{\mathrm{II}}(z)$) be the generating function of toroidal balanced $(4,2)$-regular mobiles of type I (resp. type II), with z dual to the number of white vertices.

When setting $x_i = \delta_{i=4}$ in system (5), one obtains $V_0 = zV_1$ and $V_1 = (1+V_0)^3$. Let $R \equiv R(z)$ be given by $R = 1 + V_0$, so R satisfies $R = 1 + zR^3$. Then by Proposition 25, we have:

$$Q(z) = R(z)^4 \cdot \Big(\frac{2}{3}N_{\mathrm{I}}(z) + N_{\mathrm{II}}(z)\Big).$$

A rooted or bi-rooted (planar) mobile is called $(4,2)$-*regular* if the root-vertex is black and all the non-root vertices have degree 4 if black and degree 2 if white. Rooted $(4,2)$-regular mobile are shortly called R-mobiles; again it is easy to check that $R(z)$ is the generating function of R-mobiles, with z dual to the number of white vertices. For a bi-rooted $(4,2)$-regular mobile the balance at each black vertex of the spine is in $\{-2, 0, +2\}$, so that the sequence of balances along the spine is now encoded by a path with increments in $\{-1, 0, +1\}$, the final value of the path giving half of the total balance (see Fig. 23). Let $p_{n,i}$ be the number of such paths of length n ending at i, and let $P^{(i)}(t) = \sum_{n\geq 0} p_{n,i}t^n$ be the generating function for walks ending at i, and let $B \equiv B(t) = P^{(0)}(t)$ be the generating function of those ending at 0, called *bridges*.

A mobile counted by $N_{\mathrm{II}}(z)$ (see Fig. 24 for an example) clearly decomposes into a pair of bi-rooted $(4,2)$-regular mobiles both of balance 0 (one bi-rooted mobile for each of the two edges of the kernel), which gives

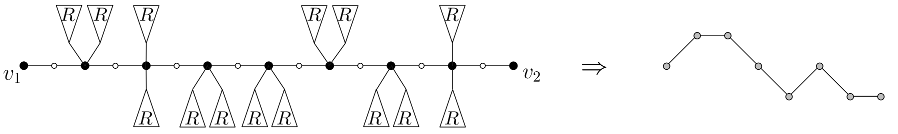


Fig. 23. Left: a bi-rooted (4, 2)-regular mobile of balance −1 (generic notation for the mobiles hanging from the spine). Right: the associated path with steps in $\{-1, 0, 1\}$, ending at height −1.

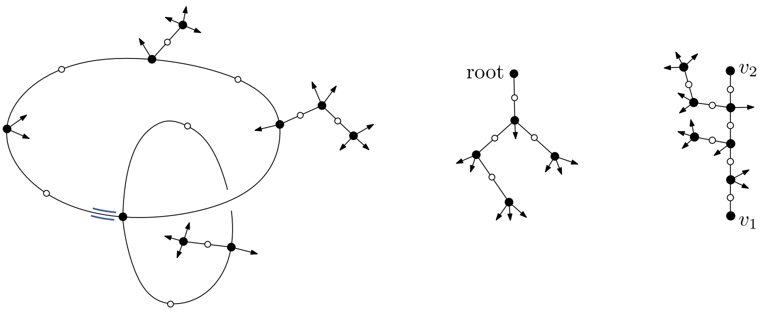


Fig. 24. From left to right: a toroidal (4, 2)-regular mobile M counted by $N_{\mathrm{II}}(z)$; an R-mobile; and a bi-rooted (4, 2)-regular mobile (the second branch of M).

$$N_{\mathrm{II}}(z) = z^2 B^2\Big|_{t=zR^2}.$$

Let $C \equiv C(t)$ be the generating function of walks counted by $B(t)$ that never visit values in $\mathbb{Z}_{<0}$ (called *Motzkin excursions*), and let $U(t) := tC(t)$. Note that $U \equiv U(t)$ is given by the equation

$$U = t \cdot (1 + U + U^2).$$

Again our aim will be to express all generating functions rationally in terms of U. We have

$$t = \frac{1}{U + 1 + U^{-1}},$$

and moreover we have $t = zR^2 = (R-1)/R = 1 - 1/R$, which gives

$$R = \frac{1}{1-t} = \frac{U^2+U+1}{U^2+1}, \quad z = \frac{R-1}{R^3} = \frac{(U^2+1)^2 U}{(U^2+U+1)^3}.$$

Note that B satisfies the equation $B = 1 + (t + 2tU)B$ (obtained by looking at the first return to 0), which gives

$$B = \frac{1}{1-t-2tU} = \frac{U^2+U+1}{(1-U)(1+U)}.$$

We thus obtain the following expression for $N_{\mathrm{II}}(z)$ in terms of U:

$$N_{\mathrm{II}}(z) = \frac{(U^2+1)^4U^2}{(U^2+U+1)^4(U-1)^2(U+1)^2}$$
$$= \frac{(U+U^{-1})^4}{(U+1+U^{-1})^4(U-2+U^{-1})(U+2+U^{-1})}.$$

We obtain an expression that is rational in $U+U^{-1}$ and so it is also rational in t since $U+U^{-1} = 1/t-1$, and then rational in R since $t = 1-1/R$. Finally, we obtain

$$N_{\mathrm{II}}(z) = \frac{(R-1)^2}{(2R-1)(3-2R)R^4}.$$

Regarding mobiles counted by $N_{\mathrm{I}}(z)$, note that the two vertices v_1, v_2 of the kernel κ have to be black (since white vertices have degree 2), and moreover, for $i \in \{1,2\}$, v_i has exactly one corner (denoted ν_i) that carries an attached R-mobile. Note that 3 situations can arise in a counterclockwise walk (of length 6 since κ has 3 edges) around the unique face of κ: ν_2 is either (a) just after ν_1, (b) or 3 steps after ν_1, (c) or just before ν_1. Let $N_{\mathrm{I}}^{(a)}(z)$, $N_{\mathrm{I}}^{(b)}(z)$, $N_{\mathrm{I}}^{(c)}(z)$ be the respective contributions to $N_{\mathrm{I}}(z)$. The first and last situations are clearly symmetric (up to exchanging the roles of v_1 and v_2), hence $N_{\mathrm{I}}^{(a)}(z) = N_{\mathrm{I}}^{(c)}(z)$.

In case (b), the mobile is made of 3 bi-rooted mobiles (one for each branch connecting v_1 to v_2) of the same excess $i \in \mathbb{Z}$, plus two attached R-mobiles (those at $\{\nu_1, \nu_2\}$). Hence

$$N_{\mathrm{I}}^{(b)}(z) = 3R(z)^2 \sum_{i\in\mathbb{Z}} z^3 P^{(i)}(t)^3 \Big|_{t=zR^2}$$

where the factor 3 accounts for the choice of the marked half-edge of κ, the factor $R(z)^2$ accounts for the attached R-mobiles at v_1 and v_2, and each of the 3 factors $zP^{(i)}(t)\big|_{t=zR^2}$ accounts for each of the 3 branches connecting v_1 to v_2.

Note that $P^{(i)}(t) = P^{-i}(t)$ for $i<0$, and for $i>0$ a decomposition at the last visits to 0, to $1,\ldots,i-1$, ensures that

$$P^{(i)}(t) = B(t)\cdot U(t)^i.$$

Hence we have

$$N_{\mathrm{I}}^{(b)}(z) = 3z^3R(z)^2B(t)^3\sum_{i\in\mathbb{Z}} U(t)^{3|i|}\Big|_{t=zR^2} = 3z^3R(z)^2B(t)^3\Big(1+\frac{2U(t)^3}{1-U(t)^3}\Big)\Big|_{t=zR^2}.$$

Again we can express everything rationally in terms of U, and find

$$N_{\mathrm{I}}^{(b)}(z) = \frac{3(U+U^{-1})^4(U-1+U^{-1})}{(U-2+U^{-1})^2(U+1+U^{-1})^5(U+2+U^{-1})} = \frac{3(R-1)^3(2-R)}{(2R-1)R^5(3-2R)^2}.$$

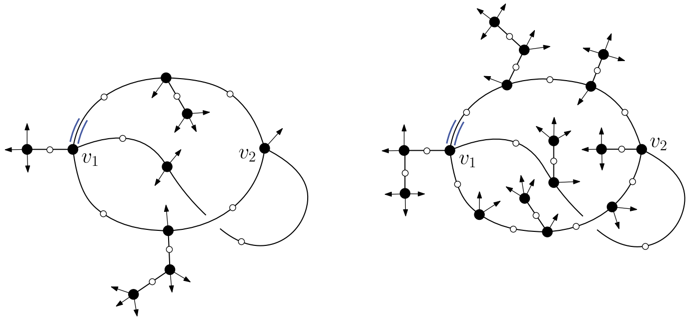


Fig. 25. Left: a toroidal $(4,2)$-regular mobile counted by $N_{\mathrm{I}}^{(a)}(z)$; Right: a toroidal $(4,2)$-regular mobile counted by $N_{\mathrm{I}}^{(b)}(z)$.

Finally, in case (a), it is easy to see that two of the bi-rooted mobiles from v_1 to v_2 have same balance $i \in \mathbb{Z}$, while the bi-rooted mobile for the remaining branch has balance $i-1$ (see 1st drawing of Fig. 25 for an example).

Hence

$$\begin{aligned} N_{\mathrm{I}}^{(a)}(z) &= 3R(z)^2 \sum_{i\in\mathbb{Z}} z^3 P^{(i)}(t)^2 P^{(i-1)}(t)\Big|_{t=zR^2} \\ &= 3R(z)^2 \sum_{i\in\mathbb{Z}} z^3 B(t)^3 U^{2|i|+|i-1|}(t)\Big|_{t=zR^2} \\ &= 3R(z)^2 z^3 B(t)^3 \frac{U(t)+U(t)^2}{1-U(t)^3}\Big|_{t=zR^2} \end{aligned}$$

Again we rewrite the expression in terms of U and then R, finding

$$N_{\mathrm{I}}^{(a)}(z) = \frac{3(U+U^{-1})^4}{(U-2+U^{-1})^2(U+1+U^{-1})^5(U+2+U^{-1})} = \frac{3(R-1)^4}{(2R-1)R^5(3-2R)^2}.$$

We thus get

$$N_{\mathrm{I}}(z) = 2N_{\mathrm{I}}^{(a)}(z) + N_{\mathrm{I}}^{(b)}(z) = \frac{3(R-1)^3}{R^4(2R-1)(3-2R)^2}.$$

We thus obtain

$$Q(z) = R(z)^4 \cdot \Big(\frac{2}{3}N_{\mathrm{I}}(z) + N_{\mathrm{II}}(z)\Big) = \frac{(R-1)^2}{(2R-1)(3-2R)^2},$$

which concludes the proof of Proposition 27, upon writing $r = R-1$ (so that r is given by $r = z(1+r)^3$).

6. Proofs of the bijections

6.1. Proof of Theorem 14

6.1.1. The Bernardi-Chapuy bijection

Similarly as in the planar case [7], the proof of Theorem 14 is by a reduction (in the dual setting) to the bijection of Bernardi and Chapuy [6]. In a *rooted map* (of genus $g \geq 0$), the convention adopted here is to indicate the root-corner by an artificial ingoing half-edge $\hat{h}$, see the top-left drawing in Fig. 26. For M a rooted map, an orientation of M is called a *co-left orientation* if for any edge e of M there is a (necessarily unique) sequence of half-edges $h_1, h_1', \ldots, h_k$ such that

- h_1 is the ingoing part of e, and $h_k = \hat{h}$,
- for every $i \in [1..k-1]$, h_i and h_i' are opposite on the same edge, with h_i the ingoing part and h_i' the outgoing part; in addition all the half-edges between h_i' and h_{i+1} (excluded) in clockwise order around their common incident vertex are outgoing.

For $g \geq 0$, let $\mathcal{R}^g$ be the family of co-left orientations of rooted maps of genus g. Bernardi and Chapuy give in [6, Section 7] a bijection between $\mathcal{T}_1^g$ (mobiles of genus g and excess 1) and $\mathcal{R}^g$.

We first describe the mapping Ψ from $\mathcal{T}_1^g$ to $\mathcal{R}^g$. For $T \in \mathcal{T}_1^g$, the *partial closure* of T is the figure obtained as follows (see the middle drawing in the top-row of Fig. 26):

- for each edge $e = (u, v) \in T$, with u the black extremity and v the white extremity, insert an ingoing bud in the corner just after e in counterclockwise order around u (since T has excess 1, there are one more ingoing buds than outgoing buds);
- match the outgoing and ingoing buds according to a walk (with the face on our right) around the unique face in T, considering outgoing buds as opening parentheses and ingoing buds as closing parentheses; each matched pair yields a new directed edge, and the unique unmatched ingoing bud is called *exposed* (in [6, Section 7] they call *balanced blossoming mobile* the mobile T plus the unique exposed ingoing bud).

Then, $M := \Psi(T)$ is obtained from the partial closure of T by erasing all the white vertices of T, all the edges of T, and declaring the single exposed ingoing bud as the root of the obtained oriented map M, see the top-row of Fig. 26.

Conversely, for M an oriented map in $\mathcal{R}^g$ (whose vertices are considered as black), $T = \Phi(M)$ is obtained as follows (see the bottom-row of Fig. 26):

- Insert a white vertex in each face of M,
- For each ingoing half-edge h of M (including the root half-edge), create a new edge connecting the vertex incident to h to the white vertex in the face on the left of h (looking from h toward the vertex incident to h),

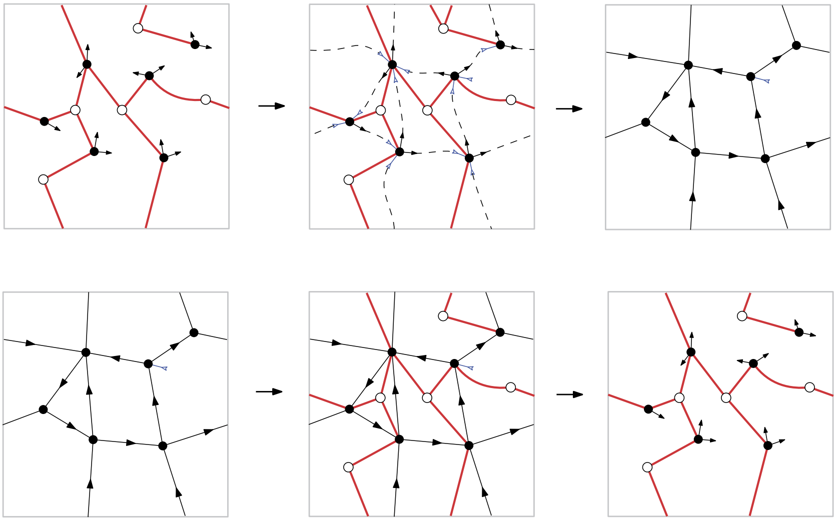


Fig. 26. The Bernardi-Chapuy bijection between $\mathcal{T}_1^g$ and $\mathcal{R}^g$ ($g = 1$ in the example). The top-row shows the mapping Ψ from $\mathcal{T}_1^g$ to $\mathcal{R}_g$. The bottom-row shows the mapping Φ from $\mathcal{R}_g$ to $\mathcal{T}_1^g$.

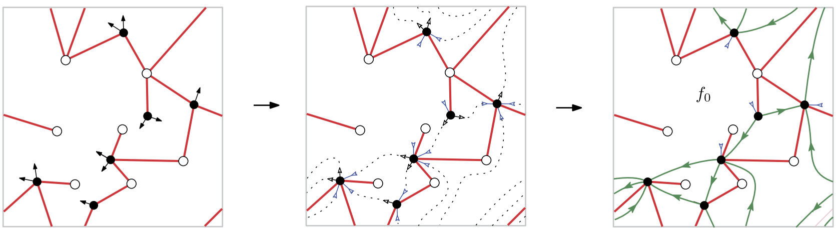


Fig. 27. The partial closure of a mobile of excess 4.

- Delete all the ingoing half-edges, and declare the outgoing half-edges as buds.

6.1.2. Deducing the bijectivity of Φ_+

We now explain how the bijectivity of Φ_+ in Theorem 14 can be deduced from properties of the bijections Ψ/Φ and properties of the relevant oriented maps. A first remark is that, for $d \geq 1$ and $T \in \mathcal{T}_d^g$, the *partial closure* of T can be performed exactly in the same way as for $d = 1$. One obtains a map (made of T, the new white vertices, and the new edges created by matching outgoing buds with ingoing buds) with d unmatched ingoing buds incident to a same face, see Fig. 27 for an example.

For $d \geq 1$, let $\mathcal{R}_d^g$ be the subfamily of $\mathcal{R}^g$ where the root-vertex has d outgoing half-edges and a single ingoing half-edge (the root). For $M \in \mathcal{R}_d^g$ let $\iota(M)$ be the underlying vertex-rooted oriented map (i.e., we delete the root ingoing half-edge but record that the incident vertex is distinguished), and let $\mathcal{S}_d^g$ be the family of vertex-rooted oriented maps of genus g that is the image of $\mathcal{R}_d^g$ by the mapping ι. For two oriented maps M, M' in $\mathcal{R}_d^g$ we write $M \sim M'$ if $\iota(M) = \iota(M')$, so that $\mathcal{S}_d^g \equiv \mathcal{R}_d^g/\sim$. Moreover let $\mathcal{U}_d^g$ be the subfamily of mobiles in $\mathcal{T}_1^g$ that are associated to maps in $\mathcal{R}_d^g$. Let $T' \in \mathcal{U}_d^g$ and let

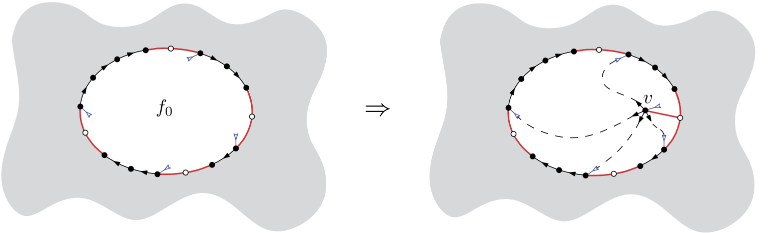


Fig. 28. Lifting a mobile in $\mathcal{T}_d^g$ to a mobile in $\mathcal{U}_d^g$.

$M' = \Psi(T')$, with v the root-vertex of M'. Since v has indegree 1 and outdegree d in M', the vertex v is a leaf in T' —it is incident to a single edge e— with d attached buds. If we delete v together with the attached edge and buds we clearly obtain a mobile in $\mathcal{T}_d^g$; we denote by $\iota(T')$ this mobile.

For two mobiles T', U' in $\mathcal{U}_d^g$ we write $T' \sim U'$ if $\iota(T') = \iota(U')$. Conversely, for $T \in \mathcal{T}_d^g$, let G be the partial closure of T; and let f_0 be the face of G containing the d unmatched ingoing buds. It is easy to see that f_0 has exactly d corner that are at a white vertex; indeed there is one such corner before each unmatched ingoing bud in a clockwise walk around f_0 (i.e., walking with the interior of f_0 on the right). Then we obtain all the mobiles T' such that $\iota(T') = T$ as follows (see Fig. 28): choose a white corner c in f_0, and then attach an edge at c (inside f_0) connected to a new black vertex v, and attach d buds at v.

From the preceding discussion, it is clear that the bijection Ψ/Φ between $\mathcal{U}_d^g$ and $\mathcal{R}_d^g$ respects the equivalence relations $\sim$, i.e. $\Phi(M') \sim \Phi(N')$ for $M' \sim N'$ and $\Psi(T') \sim \Psi(U')$ for $T' \sim U'$. Since $\mathcal{S}_d^g \equiv \mathcal{R}_d^g/\sim$ and $\mathcal{T}_d^g \equiv \mathcal{U}_d^g/\sim$, we conclude that Ψ/Φ induces a bijection between $\mathcal{T}_d^g$ and $\mathcal{S}_d^g$.

Moreover the duality property of $\mathcal{R}^g$ (see [6, Lemma 8.1]) implies that $\mathcal{O}_d^g$ is the image of $\mathcal{S}_d^g$ by duality (for $M \in \mathcal{S}_d^g$ and M^* the dual face-rooted map, every edge $e^* \in M^*$ is directed from the left-side to the right-side of the dual edge $e \in M$). Hence Φ induces a bijection between $\mathcal{O}_d^g$ and $\mathcal{T}_d^g$ for every $d \geq 1$, which one can check to be precisely the bijection Φ_+ described in Section 4.2.

6.2. Proof of Theorem 19

In this section we prove Theorem 19, which will follow from two lemmas: the first one (Lemma 29) ensuring that the bijection Φ_+ preserves the balancedness property, and the second one (Lemma 31) ensuring that the maps in $\mathcal{F}_d$ identify to the face-rooted toroidal maps endowed with a balanced $\frac{d}{d-2}$-orientation in $\mathcal{O}_d^1$.

6.2.1. Balanced specialization of Φ_+

For M a face-rooted map of genus g (whose vertices are considered as white), we define the *star-completion* of M as the map $M^\star$ obtained from M by adding a black vertex v_f inside each non-root face f, and connecting v_f to every vertex around f (via

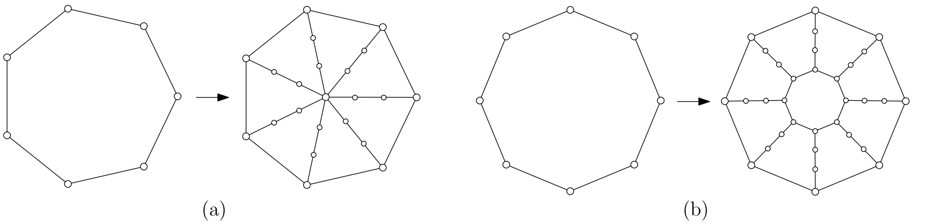


Fig. 29. Operations within each face for the mapping from M to M' used in the proof of Lemma 28. (a) shows the case of odd d; (b) shows the case of even d.

every corner around f), so that v_f has degree $\deg(f)$ in $M^\star$. The edges of $M^\star$ belonging to M are called *M-edges* and the edges incident to black vertices are called *star-edges*.

Let $d \geq 3$. Let M be a face-rooted toroidal d-angulation endowed with a $\frac{d}{d-2}$-orientation X. We extend X to an $\mathbb{N}$-biorientation $X^\star$ of $M^\star$ as follows: for each half-edge h of $M^\star$, if h is part of a M-edge, then it has the same weight (thus the same orientation) as in X, if h is part of a star-edge, then it has weight 0 if it is incident to a white vertex, and weight 1 if it is incident to a black vertex (thus star-edges are fully oriented from the black vertex toward the white vertex).

Lemma 28. *Let M be a d-toroidal map endowed with a $\frac{d}{d-2}$-orientation X. Then X is balanced if and only if $X^\star$ is balanced. Moreover, if the γ-score of two non-contractible non-homotopic cycles of $M^\star$ is* 0*, then $X^\star$ is balanced.*

Proof. We start with the case of d odd, which is a bit easier. Let M' be the d-angulation obtained from M where in each face f of M we insert a new vertex v_f, called a *star-vertex*, connected to every corner around f via a path of length $\frac{d-1}{2}$, called a *connection-path*, see Fig. 29(a). Any $\frac{d}{d-2}$-orientation X of M can be extended to a $\frac{d}{d-2}$-orientation X' of M': for each connection-path $e_1, \ldots, e_{(d-1)/(2)}$ (which is traversed starting from the star-vertex extremity), we give weight $2i-1$ (resp. $d-2i-1$) to the first (resp. second) traversed half-edge of e_i.

Note that the connection-paths have weight 0 at the incident vertex of M, hence for any non-contractible cycle C of M, the γ-score of C is the same for X as for X'. Hence, if X' is balanced, then so is X and the converse also holds by Lemma 7. Note also that any star-edge e of $M^\star$ corresponds to a connection-path of M'. Accordingly any non-contractible cycle C of $M^\star$ naturally induces a non-contractible cycle C' in M'. In addition, since the half-edges at the star-vertex extremity in connection-paths have weight 1, for any non-contractible cycle C of $M^\star$, we have $\gamma^{X^\star}(C) = \gamma^{X'}(C')$. So again if X' is balanced, then so is $X^\star$ and the converse also holds by Lemma 7. So X is balanced if and only if $X^\star$ is balanced.

Moreover, if the γ-score of two non-contractible non-homotopic cycles C_1, C_2 of $M^\star$ is 0, then the γ-score of the two corresponding cycles in X' is 0, so by Lemma 7, X' is balanced and so is $X^\star$.

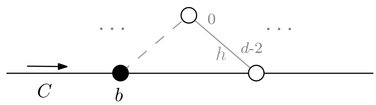


Fig. 30. Each black vertex b on C corresponds (1-to-1) to a C-adjacent half-edge h in $\mathcal{H}_L^{\circ}(C)$ (of weight $d-2$).

For d even, the augmentation from M to M' is done differently; we insert a d-gon D_f inside every face f, we set a one-to-one correspondence between the corners in clockwise order around f and the vertices in clockwise order around D_f, and we connect any matched pair by a path of length $d/2-1$, called a *connection path*, see Fig. 29(b). Similarly as before, every $\frac{d}{d-2}$-orientation X of M induces a $\frac{d}{d-2}$-orientation X' of M': we give weight $d/2-1$ to every half-edge of D_f, and for each connection-path $e_1,\ldots,e_{d/2-1}$ (which is traversed starting from the star-vertex extremity), we give weight $2i$ (resp. $d-2i-2$) to the first (resp. second) traversed half-edge of e_i.

Similarly as in the odd case, the half-edges of connection-paths incident to vertices of M have weight 0, hence for any non-contractible cycle C of M, the γ-score of C is the same for X as for X'. Hence X is balanced if and only if X' is balanced. For C a non-contractible cycle of $M^\star$ together with a traversal direction, let C' be the induced cycle of M', with the convention that when C passes by a star-vertex v_f, then C' takes the left side of the corresponding d-gon. Let f be a face of M such that C passes by the corresponding star-vertex v_f, and let $n_L(f)$ (resp. $n_R(f)$) be the number of star-edges on the left (resp. right) of C at v_f. Then the contribution to $\gamma_L^{X'}(C')$ within f is $2n_L(f)$, while the contribution to $\gamma_R^{X'}(C')$ within f is $d-2$ (due to the two half-edges of D_f incident to C' on its right side). Hence the contribution to $\gamma^{X'}(C')$ within f is $d-2-2n_L(f) = n_R(f)-n_L(f)$. Since $\gamma^{X^\star}(C)$ and $\gamma^{X'}(C')$ have the same contribution within f, we conclude that $\gamma^{X^\star}(C) = \gamma^{X'}(C')$. From here, the lemma is proved in the same way as in the odd case. $\square$

Lemma 29. *The mapping Φ_+ specializes into a bijection between face-rooted toroidal d-angulations endowed with a balanced $\frac{d}{d-2}$-orientation in $\mathcal{O}_d^1$, and the family $\mathcal{U}_d^{Bal}$ of $\mathbb{N}$-bimobiles.*

Proof. As already mentioned, the bijection Φ_+ specializes into a bijection between face-rooted toroidal d-angulations endowed with a $\frac{d}{d-2}$-orientation in $\mathcal{O}_d^1$, and the $\mathbb{N}$-bimobile family $\mathcal{U}_d$. We show here the "balanced version" of this bijection.

Let M be a face-rooted toroidal d-angulation endowed with a $\frac{d}{d-2}$-orientation X in $\mathcal{O}_d^1$. Let T be the corresponding $\mathbb{N}$-bimobile in $\mathcal{U}_d$ given by Φ_+. Let C be a (non-contractible) cycle of T together with a traversal direction. Note that C is also a non-contractible cycle of $M^\star$. Consider the extension $X^\star$ of X to $M^\star$. Clearly, for any black vertex u on C, the contribution of u to the left (resp. right) γ-score of C is the same in $M^\star$ as in T. We let $\mathcal{H}_L^{\circ}(C)$ be the set of half-edges of M that are on the left of C and incident to a white vertex on C. A half-edge h in $\mathcal{H}_L^{\circ}(C)$, with v its incident vertex, is called C-

adjacent if the next half-edge in $M^\star$ in ccw order around v is on C; it is called C*-internal* otherwise. Then the local rules of Fig. 15 ensure that $\gamma_L^T(C)$ gives the total contribution to $\gamma_L^{X^\star}(C)$ by C-internal half-edges in $\mathcal{H}_L^\circ(C)$. We let $A_L(C)$ be the total contribution to $\gamma_L^{X^\star}(C)$ by C-adjacent half-edges in $\mathcal{H}_L^\circ(C)$. As shown in Fig. 30, each black vertex on C yields a contribution $d-2$ to $A_L(C)$, hence $A_L(C) = (d-2)n_\bullet(C)$. We conclude that $\gamma_L^{X^\star}(C) = \gamma_L^T(C) + (d-2)n_\bullet(C)$. Similarly we have $\gamma_R^{X^\star}(C) = \gamma_R^T(C) + (d-2)n_\bullet(C)$, so that $\gamma^{X^\star}(C) = \gamma^T(C)$ for any non-contractible cycle C of T. Hence, if X is balanced, then, by Lemma 28, so is $X^\star$ and so is T. Conversely, if T is balanced, then it has two non-contractible non-homotopic cycles with γ-score equal to zero. Hence, by what precedes, $X^\star$ has also γ-score equal to zero on these two cycles. Then Lemma 28 ensures that $X^\star$ is balanced, and so is X. $\square$

6.2.2. Properties of rightmost walks

Consider a face-rooted d-toroidal map M.

We have the following crucial lemma regarding rightmost walks in $\frac{d}{d-2}$-orientations of M:

Lemma 30. *In a balanced $\frac{d}{d-2}$-orientation of M, any rightmost walk of M eventually loops on the contour of a d-angle W with the (contractible) interior of W on its right side.*

Proof. Let W be the looping part of a rightmost path. Note that W is a non-repetitive closed walk, and it cannot cross itself, otherwise it is not a rightmost walk. However W may have repeated vertices but in that case W intersects itself tangentially on the left side.

Let $(e_1, \dots, e_p)$ be the cyclic list of edges in W. Suppose by contradiction that there is an oriented subwalk $W' = e_i, \dots, e_{(i+k') \bmod p}$ of W (possibly $W' = W$) that forms a closed walk (i.e., the head of the last edge is the same as the tail of the first edge of W') enclosing on its left side a region R homeomorphic to an open disk. Let v be the starting and ending vertex of W'. Let H be the planar map obtained from M by keeping $R \cup W'$, where W' (which may visit vertices repeated times, but only 'from the outside') is turned into a cycle of length k', the outer cycle of H. Let n', m', f' be the numbers of vertices, edges and faces of G. By Euler's formula, $n' - m' + f' = 2$. All the inner faces of H have degree d and the outer face has degree k', so $2m' = d(f'-1) + k'$. Since W' is a subwalk of a rightmost walk, all the half-edges that are not in H and incident to a vertex $v' \neq v$ on W' have weight zero. The first half-edge of W' has non-zero weight. Thus, as we are considering a $\frac{d}{d-2}$-orientation, we have $(d-2)m' \geq d(n'-1) + 1$. Combining these three (in)equalities gives $k' \leq -1$, a contradiction.

We have the following crucial property:

Claim. *The right side of W encloses a region homeomorphic to an open disk.*

Proof of the claim. We consider two cases depending on the fact that W is a cycle (i.e., with no repetition of vertices) or not.

- *W is a cycle*
  Suppose by contradiction that W is a non-contractible cycle C. Let k be its length. Since W is a rightmost walk, all the half-edges incident to the right side of C have weight 0. Since we are considering a $\frac{d}{d-2}$-orientation of M, the sum of the weights of all edges of W is $(d-2)k$ and the sum of the weights of all the half-edges incident to vertices of W is dk. So finally the sum of the weights of all the half-edges incident to the left side of C is $2k$ and we have $\gamma(C) = -2k < 0$. So the orientation is not balanced, a contradiction.
  Thus W is a contractible cycle. By previous arguments, the contractible cycle W does not enclose a region homeomorphic to an open disk on its left side. So W encloses a region homeomorphic to an open disk on its right side, as claimed.
- *W is not a cycle*
  Since W cannot cross itself nor intersect itself tangentially on the right side, it has to intersect tangentially on the left side. Such an intersection can be on a single vertex or a path, as depicted on Fig. 31(i). The edges of W incident to this intersection are noted as on figure (i)–(iv), where W is going periodically through a, b, c, d in this order. By previous arguments, the (green) subwalk of W from a to b does not enclose regions homeomorphic to open disks on its left side. So we are not in the case depicted on Fig. 31(ii). Moreover if this (green) subwalk encloses a region homeomorphic to an open disk on its right side, then this region contains the (red) subwalk of W from c to d, see Fig. 31(iii). Since W cannot cross itself, this (red) subwalk necessarily encloses regions homeomorphic to open disks on its left side, a contradiction. So the (green) subwalk of W starting from a has to form a non-contractible curve before reaching b. Similarly for the (red) subwalk starting from c and reaching d. Since W is a rightmost walk and cannot cross itself, we are, without loss of generality, in the situation of Fig. 31(iv) (with possibly more tangent intersections on the left side). In any case, W encloses a region homeomorphic to an open disk on its right side. $\diamond$

The claim ensures that W encloses a region R homeomorphic to an open disk on its right side. Since W is a rightmost walk, there is no outgoing half-edge in R whose incident vertex is on W. Hence, by Claim 1, we conclude that W has length d. $\square$

Recall that $\mathcal{F}_d$ is the family of face-rooted d-toroidal maps such that the root-face contour is a maximal d-angle.

Lemma 31. *A face-rooted toroidal d-angulation M has a balanced $\frac{d}{d-2}$-orientation in $\mathcal{O}_d^1$ if and only if $M \in \mathcal{F}_d$. In that case, M has a unique balanced $\frac{d}{d-2}$-orientation in $\mathcal{O}_d^1$, which is the minimal one.*

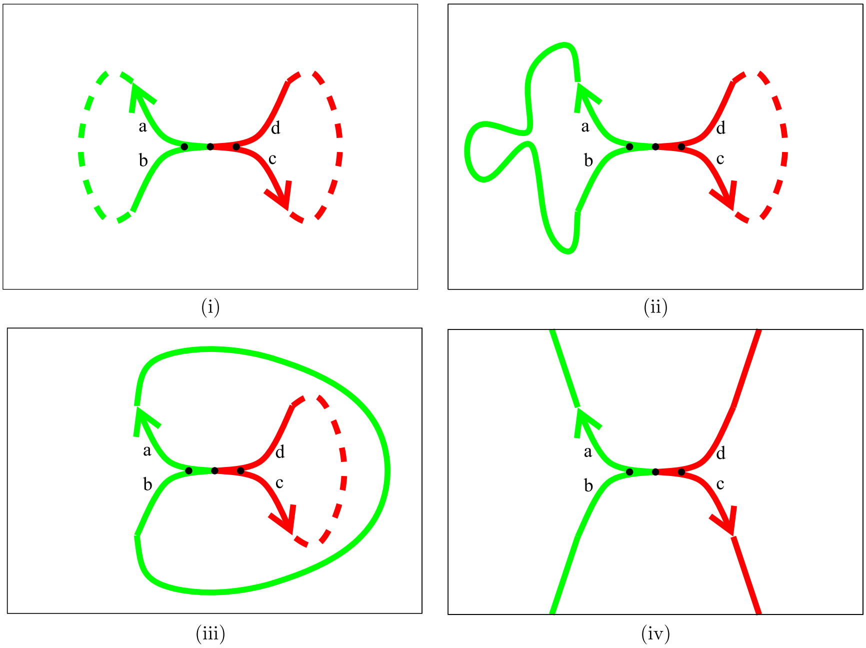


Fig. 31. Case analysis for the proof of the claim in Lemma 30.

Proof. ($\Longrightarrow$) Suppose that M has a balanced $\frac{d}{d-2}$-orientation O in $\mathcal{O}_d^1$. Then, by Lemma 5, M has essential girth d.

Suppose by contradiction that the contour C_0 of the root-face f_0 is not a maximal d-angle. Consider a maximal d-angle $C_{\max}$ whose interior strictly contains the interior of C_0. Consider an outgoing half-edge h of $C_{\max}$ and the rightmost walk W started from h. By Claim 1, all the half-edges that are in the interior of $C_{\max}$ and incident to it have weight zero, i.e. they are ingoing at their incident vertex. So it is not possible that W enters in the interior of $C_{\max}$. So W does not loop on C_0, a contradiction. So $M \in \mathcal{F}_d$.

($\Longleftarrow$) Suppose that $M \in \mathcal{F}_d$. By Proposition 8, M admits a balanced $\frac{d}{d-2}$-orientation. Then by Corollary 4, M has a (unique) balanced $\frac{d}{d-2}$-orientation $D_{\min}$ that is minimal.

Let us prove that $D_{\min} \in \mathcal{O}_d^1$. Consider an outgoing half-edge h of $D_{\min}$ and the rightmost walk W starting from h. By Lemma 30, W ends on a d-angle W' with its interior R on the right side. Consider the $(d-2)$-expansion M' of M and the orientation $D'_{\min}$ of M' corresponding to $D_{\min}$ (see Section 2 for the definition of β-expansion). Let S be the set of faces corresponding to the region R in M'. The set S is such that every edge on the boundary of S has a face in S on its right. Since $D'_{\min}$ is minimal, S contains the face of M' corresponding to the root face f_0. Since $M \in \mathcal{F}_d$, the contour of f_0 is a maximal d-angle. So W' is indeed the contour of f_0 with f_0 on its right. So $D_{\min} \in \mathcal{O}_d^1$.

Moreover, suppose by contradiction, that M has a balanced $\frac{d}{d-2}$-orientation D in $\mathcal{O}_d^1$ that is different from $D_{\min}$. By unicity of the balanced $\frac{d}{d-2}$-orientation that is minimal (Corollary 4), we have that D is non-minimal, a contradiction to Lemma 17. $\square$

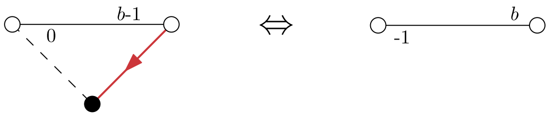


Fig. 32. The local rules of the 1-to-1 correspondence σ to turn a transferable b-regular orientation of $M^\star$ into a $\frac{b}{b-1}$-$\mathbb{Z}$-orientations of M (the half-edge directions on the M-edges are not indicated, these are determined by the status of the weights, either in $\mathbb{Z}_{>0}$ or in $\mathbb{Z}_{\leq 0}$); after applying these rules the star-vertices and star-edges of $M^\star$ are to be deleted.

6.3. Proof of Theorems 22 and 21

6.3.1. Proof of Theorem 22 for $b \geq 2$

We start by giving some terminology and results for $b \geq 1$, before continuing with $b \geq 2$ in the rest of the section.

Let $b \geq 1$. Let $\mathcal{E}_{2b}$ be the family of face-rooted toroidal maps with root-face of degree exactly $2b$ and with all face-degrees even and at least $2b$.

Recall from Section 4.3, that a $\mathbb{Z}$-biorientation has the weights at outgoing half-edges that are in $\mathbb{Z}_{>0}$ while the weights at ingoing half-edges are in $\mathbb{Z}_{\leq 0}$. In an $\mathbb{N}$-biorientation all the ingoing half-edges have weight 0.

For $M \in \mathcal{E}_{2b}$, we define a $\frac{b}{b-1}$-*$\mathbb{Z}$-orientation* of M as a $\mathbb{Z}$-biorientation of M with weights in $\{-1, \dots, b\}$, such that each vertex has weight b, each edge has weight $b-1$, and each face f has weight $-\frac{1}{2}\deg(f)+b$. Recall that the weight of a face f is the sum of the weights of the ingoing half-edges that have f on their left (traversing the half-edge toward its incident vertex).

The bijection Φ_+ specializes into a bijection between maps in $\mathcal{E}_{2b}$ endowed with a $\frac{b}{b-1}$-$\mathbb{Z}$-orientation in $\mathcal{O}^1_{2b}$ and the family $\hat{\mathcal{V}}_b$ of toroidal $\frac{b}{b-1}$-$\mathbb{Z}$-mobiles, as defined in Section 4.5. Showing Theorem 22 for $b \geq 2$ thus amounts to proving the following statement:

Proposition 32. *Let $b \geq 2$ and M be a map in $\mathcal{E}_{2b}$. Then M admits a $\frac{b}{b-1}$-$\mathbb{Z}$-orientation in $\mathcal{O}^1_{2b}$ whose associated mobile by Φ_+ is in $\hat{\mathcal{V}}_b^{Bal}$ if and only if M is in $\hat{\mathcal{L}}_{2b}$.*

In that case M admits a unique such orientation.

The rest of this section is devoted to proving Proposition 32. Similarly as in the planar case [8] we work with closely related orientations called b-regular orientations.

Let $b \geq 2$ and $M \in \mathcal{E}_{2b}$. Let $M^\star$ be the star-completion of M, as defined in Section 6.2.1. A *b-regular orientation* of $M^\star$ is defined as an $\mathbb{N}$-biorientation of $M^\star$ such that every M-edge has weight $b-1$, every star-edge has weight 1 (hence is a simply oriented edge), every M-vertex has weight b, and every star-vertex u has weight (i.e., outdegree) $\frac{1}{2}\deg(u)+b$ (hence indegree $\frac{1}{2}\deg(u)-b$).

A b-regular orientation of $M^\star$ is called *transferable* if for each star-edge e directed out of its incident M-vertex v, the M-edge ϵ just after e in clockwise order around v is of weight $b-1$ at v (and thus weight 0 at the other half-edge).

For a transferable b-regular orientation X of $M^\star$, the *induced* $\frac{b}{b-1}$-$\mathbb{Z}$-orientation $Y = \sigma(X)$ of M is obtained by applying the weight-transfer rules of Fig. 32 to each star-edge going toward its black extremity, and then deleting the star-edges and black vertices.

Lemma 33. *The mapping σ is a bijection from the transferable b-regular orientations of $M^\star$ to the $\frac{b}{b-1}$-$\mathbb{Z}$-orientations of M. In addition a transferable b-regular orientation X is in $\mathcal{O}^1_{2b}$ if and only if $\sigma(X)$ is in $\mathcal{O}^1_{2b}$.*

Proof. The bijectivity of the mapping is straightforward. And the second statement follows from the observation that if X is a transferable b-regular orientation, then the rightmost walk P_e starting at any edge $e \in X$ will only pass by M-edges after reaching an M-vertex w for the first time (indeed, when entering an M-vertex v, the rightmost outgoing edge to leave v can not be a star-edge since the orientation is transferable). Once it has reached an M-edge e', it will follow a rightmost walk $P_{e'}$ that consists only of M-edges, the rightmost walk $P_{e'}$ being exactly the same in X as in $\sigma(X)$, hence $P_{e'}$ eventually loops around the root-face contour in X if and only if the same holds in $\sigma(X)$. □

Let X be a b-regular orientation of $M^\star$. For C a non-contractible cycle of $M^\star$ traversed in a given direction, let $w_L(C)$ (resp. $w_R(C)$) be the total weight of half-edges incident to an M-vertex of C from the left side (resp. right side) of C, and let $o_L(C)$ (resp. $o_R(C)$) be the total number of outgoing star-edges incident to a star-vertex on C on the left side (resp. right side) of C, and let $\iota_L(C)$ (resp. $\iota_R(C)$) be the total number of ingoing star-edges incident to a star-vertex on C on the left side (resp. right side) of C. Let $\hat{\gamma}_L(C) = 2w_L(C) + o_L(C) - \iota_L(C)$, $\hat{\gamma}_R(C) = 2w_R(C) + o_R(C) - \iota_R(C)$. We define the *$\hat{\gamma}$-score* of C as $\hat{\gamma}(C) = \hat{\gamma}_R(C) - \hat{\gamma}_L(C)$. Then X is called *$\hat{\gamma}$-balanced* if the $\hat{\gamma}$-score of any non-contractible cycle C of $M^\star$ is 0.

Lemma 34. *Consider two b-regular orientations X, X' of $M^\star$ and C a non-contractible cycle of $M^\star$ traversed in a given direction. The cycle C has the same γ-score in X and X' if and only if it has the same $\hat{\gamma}$-score in X and X'.*

Proof. Let $s_L(C)$ (resp. $s_R(C)$) be the total number of star-edges incident to a star-vertex of C on the left (resp. right) side of C. Note that we have

$$\hat{\gamma}^X_L(C) = 2\gamma^X_L(C) - s_L(C), \quad \hat{\gamma}^X_R(C) = 2\gamma^X_R(C) - s_R(C).$$

Hence $\hat{\gamma}^X(C) = 2\gamma^X(C) + (s_L(C) - s_R(C))$, where we note that the quantity $s_L(C) - s_R(C)$ only depends on $M^\star$ and C (not on the orientation X). □

Let $\hat{\mathcal{M}}_{2b}$ be the subfamily of maps in $\mathcal{E}_{2b}$ that are bipartite and of essential girth $2b$.

Lemma 35. *Let M be a map in $\mathcal{E}_{2b}$. If $M^\star$ admits a b-regular orientation, then M has essential girth $2b$. Moreover $M^\star$ admits a $\hat{\gamma}$-*balanced *b-regular orientation if and only if $M \in \hat{\mathcal{M}}_{2b}$ (i.e., is bipartite of essential girth $2b$).*

Proof. Assume $M^\star$ is endowed with a b-regular orientation X, and let us show that M has essential girth $2b$. Since the root-face has degree $2b$, the essential girth is at most $2b$, hence we just have to show that the essential girth is at least $2b$. Consider a contractible closed walk C in M. Let M_C be the planar map obtained by keeping C and its interior, where C is 'unfolded' into a simple cycle, taken as the outer face contour. Since all inner face-degrees in M_C are even, the outer face degree is also even, so that the length of C is an even number, denoted $2k$, and we have to prove that $b \leq k$. Let v, f, e be respectively the numbers of vertices, edges, and faces that are strictly inside C. By Euler's formula applied to M_C we have $v - e + f = 1$. Let $2S$ be the sum of the degrees of the faces inside C. Note that $2S = 2e + 2k$, i.e., $S = e + k$. Consider the extension of the b-regular orientation X to the interior of C. The total weight over all M-vertices strictly inside C is bv and the total weight (outdegree) over all star-vertices strictly inside C is $S + bf$. On the other hand the total weight over all edges that are strictly inside C is $(b-1)e + 2S$. Note also that the total weight over all edges strictly inside C must be at least the total weight over all vertices strictly inside C. Hence we must have $bv + S + bf \leq (b-1)e + 2S$, so that $b(v - e + f) \leq S - e$. But we have seen that $v - e + f = 1$ and $S = e + k$, hence we obtain $b \leq k$.

Now we show that if $M^\star$ can be endowed with a $\hat{\gamma}$-balanced b-regular orientation X, then M is bipartite. Let C be a non-contractible cycle of M, and let k be the length of C. We also denote by C the corresponding cycle in $M^\star$ (that is going through M-vertices only). We have $\hat{\gamma}(C) = w_R(C) - w_L(C)$. Since the orientation is $\hat{\gamma}$-balanced, we have $\hat{\gamma}(C) = 0$ and thus $w_R(C) = w_L(C)$. Since every vertex on C has weight b and every edge on C has weight $b-1$ we have $w_L(C) + w_R(C) + k\,(b-1) = k\,b$. So finally $k = 2\,w_R(C)$ is even. Since all face-degrees of M are even and all non-contractible cycles have even length, we conclude that M is bipartite. So $M \in \hat{\mathcal{M}}_{2b}$.

It now remains to show that if $M \in \hat{\mathcal{M}}_{2b}$ then $M^\star$ admits a $\hat{\gamma}$-balanced b-regular orientation. Our strategy is the toroidal counterpart of the one for planar maps given in [8, Prop. 47]. We define the *$2b$-angular lift of M* as the bipartite toroidal $2b$-angulation M' obtained by the following process. We first fix for each $\ell \geq b$ an arbitrary planar map M_ℓ of girth $2b$, where the outer face has degree 2ℓ and its contour is a cycle, and all inner faces have degree $2b$. Then in each non-root face f of M, with $\ell = \deg(f)/2$, we insert a copy Q_f of M_ℓ strictly inside f, we set a one-to-one correspondence between the corners in clockwise order around f and the outer vertices of Q_f in clockwise order around Q_f, and we connect any matched pair by a path of length $b-1$, called a *connection path*, see Fig. 33 for an example. Since M is bipartite and all the faces inserted inside each face of M have even degree, then M' is bipartite as well. And similarly as in the planar case [8] it is easy to check that M' has essential girth $2b$. Hence, by Proposition 13, M' can be endowed with a balanced $\frac{b}{b-1}$-orientation X'. For P a connection-path within

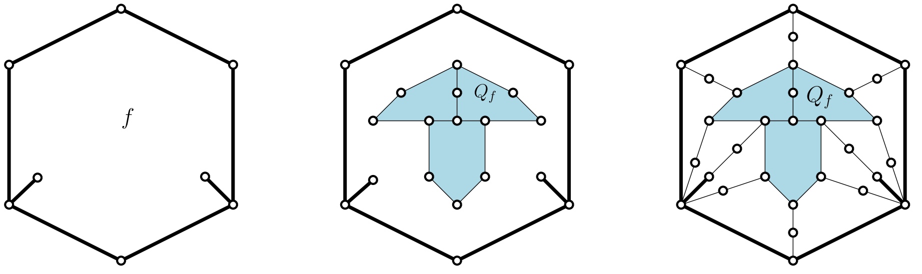


Fig. 33. Insertion operations to obtain a $2b$-angular lift (case $b = 3$ here): in each face f of degree $2p$ ($p = 5$ here) a map Q_f having outer degree $2p$, inner face degrees $2b$, and girth $2b$, is inserted inside f, and each outer vertex of Q_f is connected to each corner around f by a path of length $b - 1$ called a *connection-path*.

a face f of M, let h the extremal half-edge of P touching a vertex of f and let h' be the extremal half-edge of P touching a vertex of Q_f. Then it is easy to see that the respective weights of $\{h, h'\}$ are either $\{0, 1\}$ or $\{1, 0\}$. The connection-path P is called *outgoing* (resp. *ingoing*) in the first (resp. second) case. By a simple counting argument using Euler's formula, one can check that for each non-root face f of M of degree $2k$, the number of connection-paths inside f that are outgoing (resp. ingoing) is $k + b$ (resp. $k - b$). Hence, if for each non-root face f of M we contract Q_f into a black vertex u_f and turn every outgoing (resp. ingoing) connection-path within f into an edge of weight 1 directed out of u_f (resp. toward u_f), we obtain a b-regular orientation X of $M^\star$. It now remains to show that X is $\hat{\gamma}$-balanced.

Let e_1, e_2 be a pair of star-edges incident to a same star-vertex u, and let f be the face of M corresponding to u, and v_1, v_2 the respective white extremities of e_1, e_2. We let $P(e_1, e_2)$ denote an arbitrarily selected path of M', with v_1, v_2 as extremities and staying strictly in (the area corresponding to) f in between. For each non-contractible cycle C of $M^\star$, we call the *canonical lift of C to M'*, the cycle C' of M' obtained as follows: C' has the same M-edges as in C, and for each star-vertex u on C, with e_1, e_2 the edges before and after u along C, we replace the pair e_1, e_2 by the path $P(e_1, e_2)$. Since X' is balanced we have $\gamma^{X'}(C') = 0$. We want to deduce from it that $\hat{\gamma}^X(C) = 0$.

Let S be the set of star-vertices on C. For every $u \in S$, let f be the corresponding face of M, let v_1 (resp. e_1) be the vertex (resp. edge) before u along C, and let v_2 (resp. e_2) be the vertex (resp. edge) after u along C. Let $o_L(u)$ (resp. $\iota_L(u)$) be the number of outgoing (resp. ingoing) edges incident to u on the left side of C, and let $o_R(u)$ (resp. $\iota_R(u)$) be the number of outgoing (resp. ingoing) edges incident to u on the right side of C. And let $w_L(u)$ (resp. $w_R(u)$) be the total weight in X' of half-edges incident to vertices of the path $P(e_1, e_2) \backslash \{v_1, v_2\}$ on its left (resp. right) side. Note that we have:

$$\hat{\gamma}_L^X(C) - 2\gamma_L^{X'}(C') = \sum_{u \in S} o_L(u) - \iota_L(u) - 2w_L(u)$$

$$\hat{\gamma}_R^X(C) - 2\gamma_R^{X'}(C') = \sum_{u \in S} o_R(u) - \iota_R(u) - 2w_R(u).$$

For $u \in S$ and f the corresponding face of M, the cycle C splits the contour of f into a left portion denoted $P_L(u)$ and a right portion denoted $P_R(u)$. Let $C_L(u)$ (resp. $C_R(u)$) be the closed walk formed by the concatenation of $P_L(u)$ and $P(e_1, e_2)$ (resp. of $P_R(u)$ and $P(e_1, e_2)$). Let $\ell_L(u)$ be the length of $P_L(u)$, let $\ell(u)$ be the length of $P(e_1, e_2)$, and let $\ell_R(u)$ be the length of $P_R(u)$; note that $\ell_L(u) = o_L(u) + \iota_L(u) + 1$ and $\ell_R(u) = o_R(u) + \iota_R(u) + 1$. Note also that $\iota_L(u) + w_L(u)$ is the total weight of half-edges inside $C_L(u)$ and incident to a vertex on $C_L(u)$. By a simple counting argument based on the Euler relation, this number is equal to $\frac{1}{2}(\ell_L(u) + \ell(u)) - b$. This gives the equation

$$o_L(u) - \iota_L(u) - 2w_L(u) = 2b - 1 - \ell(u).$$

Similarly we obtain

$$o_R(u) - \iota_R(u) - 2w_R(u) = 2b - 1 - \ell(u).$$

Summing over $u \in S$ we find

$$\hat{\gamma}_L^X(C) - 2\gamma_L^{X'}(C') = \hat{\gamma}_R^X(C) - 2\gamma_R^{X'}(C'),$$

hence $\hat{\gamma}^X(C) = 2\gamma^{X'}(C') = 0$. Hence X is $\hat{\gamma}$-balanced. □

Lemma 36. *Consider $M \in \mathcal{E}_{2b}$ such that $M^\star$ admits a b-regular orientation X. If the $\hat{\gamma}$-score of two non-contractible non-homotopic cycles of $M^\star$ is 0, then X is $\hat{\gamma}$-balanced.*

Proof. Let C_1, C_2 be two non-homotopic non-contractible cycles of $M^\star$, each given with a traversal direction, such that $\hat{\gamma}(C_1) = \hat{\gamma}(C_2) = 0$.

By Lemma 35, M has essential girth $2b$. We now show that M has to be bipartite. For $C \in \{C_1, C_2\}$, let $n_{\circ\circ}(C)$ be the number of M-edges on C, and let $\mathcal{V}_\bullet(C)$ (resp. $\mathcal{V}_\circ(C)$) be the set of black (resp. white) vertices on C and $n_\bullet(C) = |\mathcal{V}_\bullet(C)|,\ n_\circ(C) = |\mathcal{V}_\circ(C)|$. For each $u \in \mathcal{V}_\bullet(C)$, let $c_L(u)$ (resp. $c_R(u)$) be the number of corners of $M^\star$ incident to u on the left (resp. right) of C, and let $\kappa_L(C) = \sum_{u \in \mathcal{V}_\bullet(C)} c_L(u)$, and $\kappa_R(C) = \sum_{u \in \mathcal{V}_\bullet(C)} c_R(u)$, and $\kappa(C) = \kappa_L(C) + \kappa_R(C)$; note that $\kappa(C)$ is the total degree of faces corresponding to the black vertices on C, hence $\kappa(C)$ is an even integer. The *left length* of C is defined as

$$\ell_L(C) = n_{\circ\circ}(C) + \kappa_L(C).$$

It corresponds to the length of the closed walk of edges of M that coincides with C at M-edges, and takes the left boundary of the corresponding face of M each time C passes by a black vertex. Since all face-degrees of M are even and C_1, C_2 are non-contractible non-homotopic cycles, it is enough to show that $\ell_L(C)$ is even for $C \in \{C_1, C_2\}$ to prove that M is bipartite. Recall that $w_L(C)$ (resp. $w_R(C)$) denotes the total weight of half-edges incident to white vertices of C on the left (resp. right) side of C, and $\iota_L(C)$ (resp.

$\iota_R(C)$) denotes the total number of ingoing edges at black vertices on C on the left (resp. right) side of C. It is easy to see that the property $\hat{\gamma}(C) = 0$ rewrites as $\eta_L(C) = \eta_R(T)$, where

$$\eta_L(C) = 2w_L(C) - 2\iota_L(C) + \kappa_L(C), \qquad \eta_R(C) = 2w_R(C) - 2\iota_R(C) + \kappa_R(C).$$

Let $e(C)$ be the number of edges on C (which is also the length of C). Let $\Sigma(C)$ be the total weight of half-edges incident to vertices in $\mathcal{V}_\circ(C)$ minus the total ingoing degree of vertices in $\mathcal{V}_\bullet(C)$. Then it is easy to see that

$$\Sigma(C) = w_L(C) - \iota_L(C) + w_R(C) - \iota_R(C) + (b-1) \cdot n_{\circ\circ}(C).$$

Moreover, since X is b-regular we have

$$\begin{aligned}\Sigma(C) &= b \cdot n_\circ(C) + b \cdot n_\bullet(C) - \frac{1}{2}\kappa(C) = b \cdot e(C) - \frac{1}{2}\kappa(C)\\ &= b(n_{\circ\circ}(C) + 2n_\bullet(C)) - \frac{1}{2}\kappa(C).\end{aligned}$$

The equality between the two expressions of $\Sigma(C)$ yields $\eta_L(C) + \eta_R(C) = 2n_{\circ\circ}(C) + 4bn_\bullet(C)$, which gives $\eta_L(C) = n_{\circ\circ}(C) + 2bn_\bullet(C)$. Since $\eta_L(C) = 2w_L(C) - 2\iota_L(C) + \kappa_L(C)$, we conclude that

$$\ell_L(C) = n_{\circ\circ}(C) + \kappa_L(C) = (\eta_L(C) - 2bn_\bullet(C)) + (\eta_L(C) - 2w_L(C) + 2\iota_L(C)),$$

so that $\ell_L(C)$ is even. This concludes the proof that M is bipartite.

We now prove that X is $\hat{\gamma}$-balanced. By Lemma 35, $M^\star$ has a $\hat{\gamma}$-balanced b-regular orientation X'. Then, C_1 and C_2 have the same $\hat{\gamma}$-score (which is zero) in X as in X', hence, by Lemma 34, they have the same γ-score in X as in X'. By Corollary 4, X and X' are γ-equivalent. Thus, again by Lemma 34, X is $\hat{\gamma}$-balanced. $\square$

Lemma 37. *Let $M \in \hat{\mathcal{M}}_{2b}$. Then $M^\star$ has a unique minimal $\hat{\gamma}$-balanced b-regular orientation. This orientation is transferable. Moreover, it is in $\mathcal{O}_{2b}^1$ if and only if $M \in \hat{\mathcal{L}}_{2b}$ (the root-face contour is a maximal $2b$-angle).*

Proof. By Lemma 35, $M^\star$ admits a $\hat{\gamma}$-balanced b-regular orientation X. By Lemma 34, a b-regular orientation is $\hat{\gamma}$-balanced if and only if it is γ-equivalent to X. Hence, by Corollary 4, M admits a unique b-regular orientation X_0 that is minimal and $\hat{\gamma}$-balanced.

The argument to ensure that X_0 is transferable is the same as given in the planar case [8, Lemma 50]. Suppose by contradiction that there is a star-edge $\epsilon = \{b, w\}$ going toward its black extremity b, and such that the M-edge $e = \{w, w'\}$ just after ϵ in clockwise order around w has weight different from $b - 1$. Thus e has strictly positive weight at w'. Then let ϵ' be the star-edge just after ϵ in counterclockwise order around b. Note that ϵ' has to be directed toward b, otherwise (ϵ', ϵ, e) would form a face S distinct from the root-face, such that every edge on the boundary of S has a face in S on its right, contradicting the minimality of X_0. Let $e' = (w', w'')$ be the M-edge just after ϵ' in clockwise order around w'. Since the edges e and ϵ' contribute by at least 2 to the weight of w', the edge e' can not have weight $b-1$ at w', hence it has positive weight at w''. Continuing iteratively in counterclockwise order around b we obtain that b has only ingoing edges, a contradiction. So X_0 is transferable.

Let us now characterize when X_0 is in $\mathcal{O}_{2b}^1$.

Suppose that the root-face contour is not a maximal $2b$-angle, let C be a maximal $2b$-angle whose interior contains the root-face. By a counting argument similar to the proof of Claim 1, all half-edges incident to a vertex on C and in the interior of C have weight 0, hence a rightmost walk starting from an edge on C can never loop on the root-face contour. Hence X_0 is not in $\mathcal{O}_{2b}^1$.

Conversely assume that M is in $\hat{\mathcal{L}}_{2b}$. Let e be an outgoing half-edge of X_0 and let P_e be the rightmost path starting at e. Since X_0 is transferable, it is easy to see that once P_e has reached an M-vertex (which occurs after traversing at most two edges), it will only take M-edges. Hence the cycle C formed when P_e eventually loops is a right cycle of M-edges. By the same line of arguments as in Section 6.2.2, this cycle has to be of length $2b$, with a contractible region on its right. This region has to contain the root-face since X_0 is minimal. Since the root-face contour is a maximal $2b$-angle, we conclude that C is actually the root-face contour. Hence X_0 is in $\mathcal{O}_{2b}^1$. □

Lemma 38. *Let M be a map in $\mathcal{E}_{2b}$. Let Y be a $\frac{b}{b-1}$-$\mathbb{Z}$-orientation of M in $\mathcal{O}_{2b}^1$, let $X = \sigma^{-1}(Y)$ be the associated b-regular orientation of $M^\star$, and let T be the associated mobile in $\hat{\mathcal{V}}_b$. Then X is $\hat{\gamma}$-balanced if and only if T is balanced.*

Proof. Recall that the rules to obtain the mobile associated to Y are the ones of Fig. 15.

Let C be a non-contractible cycle of T given with a traversal direction; note that C is also a non-contractible cycle of $M^\star$ (it is convenient here to see T and $M^\star$ as superimposed). Let $n_\bullet(C)$ be the number of black vertices on C, and let $n_{\bullet\circ}(C)$ (resp. $n_{\circ\bullet}(C)$) be the number of black-white edges e on C where the black extremity is traversed before (resp. after) the white extremity when traversing e (along the traversal direction of C). Note that all the black-white edges on C have weights $(0, b-1)$ (the weights can not be $(-1, b)$ since the white extremity is a leaf in that case). Note also that $n_{\bullet\circ}(C) = n_\bullet(C) = n_{\circ\bullet}(C)$, since every black vertex is preceded and followed by white vertices along C. Let $w_L^T(C)$ (resp. $w_R^T(C)$) be the total weight of half-edges in T that are incident to a vertex (white or black) of C on the left side (resp. right side) of C. Let $s_L(C)$ (resp. $s_r(C)$) be the total number of half-edges, including the buds, that are incident to a black vertex of C on the left (resp. right) side of C (note that this quantity is the same for T as for X). Let $w_L^X(C)$ (resp. $w_R^X(C)$) be the total weight in X of half-edges at M-vertices of C, on the left (resp. right) side of C. Let $\iota_L^X(C)$ (resp. $\iota_R^X(C)$) be the total number of ingoing edges at black vertices on the left (resp. right) side of C. We have $\gamma_L^T(C) = 2w_L^T(C) + s_L(C)$, $\gamma_R^T(C) = 2w_R^T(C) + s_R(C)$, and $\gamma^T(C) = \gamma_R^T(C) - \gamma_L^T(C)$. Moreover, we have $\hat{\gamma}_L^X(C) = 2w_L^X(C) + s_L(C) - 2\iota_L^X(C)$, $\hat{\gamma}_R^X(C) = 2w_R^X(C) + s_R(C) - 2\iota_R^X(C)$, and $\hat{\gamma}^X(C) = \hat{\gamma}_R^X(C) - \hat{\gamma}_L^X(C)$.

The quantity $w_L^T(C)$ decomposes as $w_L^{\circ,T}(C) + w_L^{\bullet,T}(C)$, where the first (resp. second) term gathers the contribution from the half-edges at white (resp. black) vertices. Clearly $w_L^{\bullet,T}(C) = -\iota_L^X(C)$. We let $\mathcal{H}_L^\circ(C)$ be the set of half-edges of $M^\star$ that are on the left of C and incident to a white vertex on C. A half-edge h in $\mathcal{H}_L^\circ(C)$, with v its incident vertex, is called *C-adjacent* if the next half-edge in $M^\star$ in ccw order around v is on C; it is called *C-internal* otherwise. Then the combined effect of the transfer rule of Fig. 32 and of the local rules of Fig. 15 ensure that $w_L^{\circ,T}(C)$ gives the total contribution to $w_L^X(C)$ by C-internal half-edges in $\mathcal{H}_L^\circ(C)$. Let $A_L(C)$ be the total contribution to $w_L^X(C)$ by C-adjacent half-edges in $\mathcal{H}_L^\circ(C)$. Then, very similarly as in the proof of Lemma 29 (see Fig. 30), each black vertex on C yields a contribution $b-1$ to $A_L(C)$, so that $A_L(C) = (b-1)n_\bullet(C)$. We conclude that $w_L^X(C) - \iota_L^X(C) = w_L^T(C) + (b-1)n_{\bullet\circ}(C)$. Very similarly we have $w_R^X(C) = w_R^T(C) + \iota_R(C) + (b-1)n_{\circ\bullet}(C)$. Hence $\gamma^T(C) = \hat{\gamma}^X(C)$.

This implies that if X is $\hat{\gamma}$-balanced then T is balanced. Now, suppose that T is balanced. Then $\gamma^T(C) = 0$ for any non-contractible cycle C of T. Let $\{C_1, C_2\}$ be two such distinct cycles. They are not homotopic since T is unicellular. By what precedes we have $\hat{\gamma}^X(C_1) = 0$ and $\hat{\gamma}^X(C_2) = 0$. Hence X is $\hat{\gamma}$-balanced by Lemma 36. □

We are now able to prove Proposition 32.

Proof of Proposition 32. Suppose that $M \in \mathcal{E}_{2b}$ admits a $\frac{b}{b-1}$-$\mathbb{Z}$-orientation $Y \in \mathcal{O}_{2b}^1$ whose associated mobile by Φ_+ is in $\hat{\mathcal{V}}_b^{Bal}$, and let $X = \sigma^{-1}(Y)$. Then X is $\hat{\gamma}$-balanced (according to Lemma 38), is transferable and in $\mathcal{O}_{2b}^1$ (according to Lemma 33), and is minimal (according to Lemma 17). Lemma 35 implies that $M \in \hat{\mathcal{M}}_{2b}$. Hence, according to Lemma 37, M is in $\hat{\mathcal{L}}_{2b}$ and moreover Y is unique (it has to be the image by σ of the unique minimal $\hat{\gamma}$-balanced b-regular orientation of $M^\star$).

Conversely let us prove the existence part, for $M \in \hat{\mathcal{L}}_{2b}$. By Lemma 37, let X be the minimal $\hat{\gamma}$-balanced b-regular orientation of $M^\star$ that is moreover transferable and in $\mathcal{O}_{2b}^1$. Let $Y = \sigma(X)$ so that $Y \in \mathcal{O}_{2b}^1$ by Lemma 33. And Lemma 38 ensures that the mobile associated to Y by Φ_+ is in $\hat{\mathcal{V}}_b^{Bal}$. □

6.3.2. Proof of Theorem 21 for $d \geq 2$

We start by giving some terminology and results for $d \geq 1$, before continuing with $d \geq 2$ in the rest of the section.

Let $d \geq 1$. Let $\mathcal{H}_d$ be the family of face-rooted toroidal maps with root-face degree d and with all faces of degree at least d. For $M \in \mathcal{H}_d$, we define a $\frac{d}{d-2}$-$\mathbb{Z}$-orientation of M as a $\mathbb{Z}$-biorientation with weights in $\{-2, \ldots, d\}$ such that all vertices have weight d, all edges have weight $d-2$, and every face f has weight $-\deg(f) + d$.

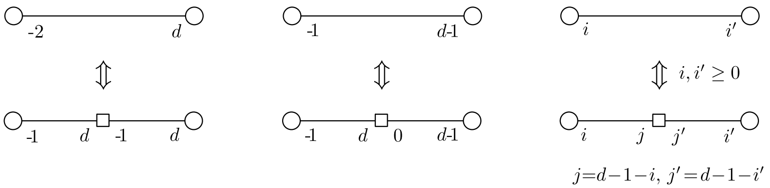


Fig. 34. The local rule in the 1-to-1 correspondence ι between the $\frac{d}{d-2}$-$\mathbb{Z}$-orientations of M and the $\frac{d}{d-1}$-$\mathbb{Z}$-orientation of M_2 (the half-edge direction is not indicated, it is determined by the status of the weight it carries, either in $\mathbb{Z}_{>0}$ or in $\mathbb{Z}_{\leq 0}$).

The bijection Φ_+ specializes into a bijection between maps in $\mathcal{H}_d$ endowed with a $\frac{d}{d-2}$-$\mathbb{Z}$-orientation in $\mathcal{O}_d^1$ and the family $\mathcal{V}_d$ of toroidal $\frac{d}{d-2}$-$\mathbb{Z}$-mobiles. Showing Theorem 22 for $d \geq 2$ thus amounts to proving the following statement:

Proposition 39. *Let $d \geq 2$ and let M be a map in $\mathcal{H}_d$. Then M admits a $\frac{d}{d-2}$-$\mathbb{Z}$-orientation in $\mathcal{O}_d^1$ whose associated mobile by Φ_+ is in $\mathcal{V}_d^{Bal}$ if and only if M is in $\mathcal{L}_d$. In that case M admits a unique such orientation.*

Let $d \geq 1$ and let $M \in \mathcal{H}_d$. We denote by M_2 the (bipartite) map obtained from M by inserting a new vertex on each edge. Note that $M \in \mathcal{L}_d$ if and only if $M_2 \in \hat{\mathcal{L}}_{2d}$. Applying (as done in the planar case in [8, Lem. 55]) the rules of Fig. 34 to each edge of a $\frac{d}{d-2}$-$\mathbb{Z}$-orientation Z of M, we obtain a $\frac{d}{d-1}$-$\mathbb{Z}$-orientation $Y = \iota(Z)$ of M_2. The mapping ι is clearly bijective. Moreover, Z is in $\mathcal{O}_d^1$ if and only if $\iota(Z)$ is in $\mathcal{O}_{2d}^1$.

From now on we assume that $d \geq 2$. We first prove the analogue of Lemma 38:

Lemma 40. *Let $d \geq 2$ and M be a map in $\mathcal{H}_d$. Let Z be a $\frac{d}{d-2}$-$\mathbb{Z}$-orientation of M in $\mathcal{O}_d^1$, let $X = \sigma^{-1}(\iota(Z))$ be the associated d-regular orientation of $M_2^\star$, and let T be the associated mobile (by Φ_+) in $\mathcal{V}_d$. Then X is $\hat{\gamma}$-balanced if and only if T is balanced.*

Proof. Let C be a non-contractible cycle of T. Let $n_\bullet(C)$ be the number of black vertices on C. Let $n_{\bullet\circ}(C)$ (resp. $n_{\circ\bullet}(C)$) be the number of black-white edges e on C where the black extremity is traversed before (resp. after) the white extremity when traversing e (along the traversal direction of C), and let $n_{\bullet\bullet}(C)$ be the number of black-black edges along C (note that $n_{\bullet\bullet}(C) = 0$ for $d > 2$). As in the last section it is easy to see that $n_{\bullet\circ}(C) = n_{\circ\bullet}(C)$. Note that black-white edges on C can have weights $(0, d-2)$ or $(-1, d-1)$ (but not $(-2, d)$ since the white extremity would be a leaf).

Let $w_L^T(C)$ (resp. $w_R^T(C)$) be the total weight of half-edges in T incident to a vertex, white or black, of C on the left side (resp. right side) of C. Let $s_L^T(C)$ (resp. $s_R^T(C)$) be the total number of half-edges, including the buds, that are incident to a black vertex of C on the left (resp. right) side of C. Note that C identifies to a cycle of $M_2^\star$, which we also call C by a slight abuse of notation (the only difference to keep in mind is that, for each black-black or white-white edge e on C seen as a cycle of T, in $M_2^\star$ there is a white square vertex in the middle of e). Let $w_L^X(C)$ (resp. $w_R^X(C)$) be the total weight (in X) of half-edges at white vertices (round or square) of C on the left (resp. right)

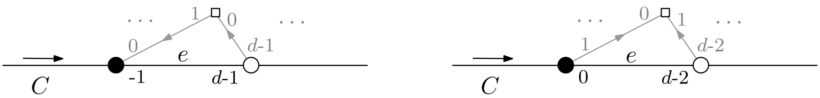


Fig. 35. The situation at an edge e counted by $n_{\bullet\circ}(C)$ (the weights and orientations of the two C-adjacent edges, shown in gray, are determined by the combined effect of the transfer rule of Fig. 32, the rules of Fig. 34, and the local rules in Fig. 15). If e has weights $(-1, d-1)$ it has contribution $d-1$ to $w_L^X(C)$ and contribution 1 to $\iota_L^X(C)$. If e has weights $(0, d-2)$ it has contribution $d-2$ to $w_L^X(C)$ and contribution 0 to $\iota_L^X(C)$. Hence it always has a contribution $d-2$ to $w_L^X(C) - \iota_L(X)$.

side of C. Let $\iota_L^X(C)$ (resp. $\iota_R^X(C)$) be the total number of ingoing edges (in X) at black vertices on the left (resp. right) side of C. Let $s_L^X(C)$ (resp. $s_R^X(C)$) be the total number of edges incident to a black vertex on the left (resp. right) side of C. We have $\gamma_L^T(C) = w_L^T(C) + s_L^T(C)$, $\gamma_R^T(C) = w_R^T(C) + s_R^T(C)$, and $\gamma^T(C) = \gamma_R^T(C) - \gamma_L^T(C)$. And we have $\hat{\gamma}_L^X(C) = 2w_L^X(C) + s_L^X(C) - 2\iota_L^X(C)$, $\hat{\gamma}_R^X(C) = 2w_R^X(C) + s_R^X(C) - 2\iota_R^X(C)$, and $\hat{\gamma}^X(C) = \hat{\gamma}_R^X(C) - \hat{\gamma}_L^X(C)$.

The quantity $w_L^T(C)$ decomposes as $w_L^{\circ,T}(C) + w_L^{\bullet,T}(C)$ where the first (resp. second) term gathers the contributions from the white (resp. black) vertices. We let $\mathcal{H}_L(C)$ be the set of half-edges of $M_2^\star$ that are on the left of C and incident to a vertex on C (including white square vertices on C, i.e., seeing C as a cycle in $M_2^\star$). The set $\mathcal{H}_L(C)$ partitions as $\mathcal{H}_L(C) = \mathcal{H}_L^\circ(C) \cup \mathcal{H}_L^\diamond(C) \cup \mathcal{H}_L^\bullet(C)$ whether the incident vertex is white round, white square, or black. A half-edge h of $\mathcal{H}_L^\circ(C)$ (resp. $\mathcal{H}_L^\bullet(C)$) is called *C-adjacent* if the next half-edge of $M_2^\star$ after h in ccw order (resp. cw order) around the vertex incident to h is on C; it is called *C-internal* otherwise. By the combined effect of the transfer rule of Fig. 32, the rules of Fig. 34, and the local rules in Fig. 15, the quantity $w_L^{\circ,T}(C)$ represents the total contribution to $w_L^X(C)$ of the C-internal half-edges in $\mathcal{H}_L^\circ(C)$, while $w_L^{\bullet,T}(C)$ represents the total contribution to $-\iota_L^X(C)$ of the C-internal half-edges in $\mathcal{H}_L^\bullet(C)$.

We let $A_L(C)$ be the total contribution to $w_L^X(C) - \iota_L^X(C)$ of C-adjacent half-edges from $\mathcal{H}^\circ(C) \cup \mathcal{H}^\bullet(C)$. An important observation (see Fig. 35) is that an edge of T counted by $n_{\bullet\circ}(C)$ always gives a contribution $d-2$ to $A_L(C)$ (whether it has weights $(-1, d-1)$ or $(0, d-2)$), hence $A_L(C) = (d-2)n_{\bullet\circ}(C)$. Finally, the total contribution to $w_L^X(C) - \iota_L^X(C)$ by half-edges in $\mathcal{H}^\diamond(C)$ is $n_{\bullet\bullet}(C)$. Indeed the only contribution is a contribution by one to $w_L^X(C)$ for each black-black edge e on C (there is a white square vertex in the middle of e, with an outgoing edge on each side, recalling that black-black edges in T occur only for $d = 2$). We thus have

$$w_L^X(C) - \iota_L^X(C) = w_L^T(C) + (d-2)n_{\bullet\circ}(C) + n_{\bullet\bullet}(C),$$

and similarly we have

$$w_R^X(C) - \iota_R^X(C) = w_R^T(C) + (d-2)n_{\circ\bullet}(C) + n_{\bullet\bullet}(C).$$

Moreover we have

$$s_L^X(C) = 2s_L^T(C) + n_\bullet(C) - 2n_{\bullet\bullet}(C), \qquad s_R^X(C) = 2s_R^T(C) + n_\bullet(C) - 2n_{\bullet\bullet}(C).$$

With these equalities, and using the fact that $n_{\bullet\circ}(C) = n_{\circ\bullet}(C)$, we easily deduce $2\gamma^T(C) = \hat{\gamma}^X(C)$, and in particular $\gamma^T(C) = 0$ if and only if $\hat{\gamma}^X(C) = 0$.

From there, very similarly as in the end of the proof of Lemma 38, we conclude that X is balanced if and only if T is balanced, which concludes the proof. □

We are now able to prove Proposition 39:

Proof of Proposition 39. Suppose that $M \in \mathcal{H}_d$ admits a $\frac{d}{d-2}$-$\mathbb{Z}$-orientation $Z \in \mathcal{O}^1_{2b}$ whose associated mobile by Φ_+ is in $\mathcal{V}_d^{Bal}$, and let $X = \sigma^{-1}(\iota(Z))$. Then X is $\hat{\gamma}$-balanced (according to Lemma 40), is transferable and in $\mathcal{O}^1_{2d}$ (by Lemma 33 and since ι preserve the property of being in $\mathcal{O}^1_{2d}$), and minimal (according to Lemma 17). Lemma 35 implies that $M_2 \in \hat{\mathcal{M}}_{2d}$. Hence, according to Lemma 37, M_2 is in $\hat{\mathcal{L}}_{2d}$, so that M is in $\mathcal{L}_d$, and moreover Z is unique (it has to be the image by $\iota^{-1} \circ \sigma$ of the unique minimal $\hat{\gamma}$-balanced d-regular orientation of $M_2^\star$).

Conversely let us prove the existence part, for $M \in \mathcal{L}_d$. By Lemma 37, let X be the minimal $\hat{\gamma}$-balanced d-regular orientation of $M_2^\star$, that is moreover transferable and in $\mathcal{O}^1_{2d}$. Let $Z = \iota^{-1}(\sigma(X))$ so that $Z \in \mathcal{O}^1_{2d}$ by Lemma 33 and since ι preserve the property of being in $\mathcal{O}^1_{2d}$. And Lemma 40 ensures that the mobile associated to Z by Φ_+ is in $\mathcal{V}_d^{Bal}$. □

6.3.3. Proof of Theorem 22 for $b = 1$

Before proving Theorem 22 for $b = 1$ let us make a simple observation. We have proved Theorem 22 for $b \geq 2$ and Theorem 21 for $d \geq 2$. For $b \geq 1$ a $\mathbb{Z}$-bimobile in $\mathcal{V}_{2b}^{Bal}$ is called *even* if all its half-edge weights are even. The mapping consisting in doubling the half-edge weights gives a bijection between $\hat{\mathcal{V}}_b^{Bal}$ and even $\mathbb{Z}$-bimobiles in $\mathcal{V}_{2b}^{Bal}$. Moreover the toroidal map (obtained by performing Ψ_+) associated to a bimobile in $\hat{\mathcal{V}}_b^{Bal}$ is the same as the toroidal map associated to the weight-doubled bimobile. Hence, if we call ϕ_d, for $d \geq 2$, the bijection in Theorem 21 and $\hat{\phi}_b$, for $b \geq 2$, the bijection in Theorem 22 then we have already obtained:

> 'For $b \geq 2$ and $M \in \mathcal{L}_{2b}$, we have that $\phi_{2b}(M)$ is even if and only if M is bipartite, and in that case $\phi_{2b}(M)$ is equal to $\hat{\phi}_b(M)$ upon doubling the half-edge weights'.

Note that if we can establish (as stated next) the similar bipartiteness condition for $b = 1$ then we will have Theorem 22 for $b = 1$ (as the bipartite specialization of Theorem 21 for $d = 2$).

Lemma 41. *Let $M \in \mathcal{L}_2$ and let $T = \phi_2(M)$. Then T is even if and only if M is bipartite.*

Proof. Assume T is even, and let T' be obtained from T after dividing by 2 the half-edge weights. Note that $T' \in \hat{\mathcal{V}}_1^{Bal}$ and in particular the degrees of all black vertices of T' are even, so that all face-degrees of M are even. Since the weight of a white vertex of T' is 1, in T' all white vertices are leaves. Consider two distinct cycles C_1, C_2 of T' and

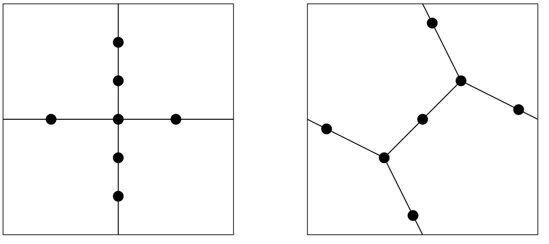


Fig. 36. The two types of toroidal unicellular maps.

$C \in \{C_1, C_2\}$. Since white vertices are leaves, the cycle C is made only of black vertices and black-black edges, with zero weights on both half-edges. Let k be the length of C. Let $w_L(C)$ (resp. $w_R(C)$) be the total weight of half-edges of T incident to (black) vertices of C on the left (resp. right) side of C. Let $s_L(C)$ (resp. $s_R(C)$) be the total number of half-edges (including buds) incident to black vertices of C on the left (resp. right) side of C. Since T' is balanced we have $2\,w_L(C) + s_L(C) = 2\,w_R(C) + s_R(C)$. Let $\kappa(C)$ be the total degree of faces corresponding to vertices of C, so $\kappa(C) = s_L(C) + s_R(C) + 2k$. Since all the half-edges on C have weight 0, the total weight of vertices of C is $w_L(C) + w_R(C) = \sum_{u\in C}(-\frac{1}{2}\deg(u) + 1) = -\frac{1}{2}\kappa(C) + k$. By combining the three equalities, we obtain that $s_L(C) = -2w_L(C)$. So $s_L(C)$ is even. So $s_L(C_i)$ is even for $i \in \{1, 2\}$. For $i \in \{1, 2\}$, let W_i be the walk of M that is "just on the left" of C_i (seeing M and T as superimposed), i.e. obtained by following the left boundary of the corresponding face of M each time C_i passes by a black vertex. By the local rules of Φ_+ shown in Fig. 15, the length of W_i is precisely equal to $s_L(C_i)$ and thus is even. All the faces of M are even, and the two walks W_i are non-homotopic to a contractible cycle and non-homotopic to each other. Thus M is bipartite.

Conversely, assume that M is bipartite. Note that there are 3 types of edges in $T \in \mathcal{V}_2^{Bal}$: those of weights $(-2, 2)$ that connect a black vertex to a white leaf, those of weights $(-1, 1)$ that connect a black vertex to a white vertex of degree 2 (incident to two such edges), and those of weights $(0, 0)$ that connect two black vertices. We call *odd* the edges of weights $(-1, 1)$. To prove that T is even we thus have to show that T has no odd edges. Let Γ be the subgraph of T induced by the odd edges. Since M is bipartite, all its faces have even degree and thus all black vertices of T have even weight (since for $d = 2$ the weight of a black vertex of T is 2 minus the degree of the associated face). Moreover every white vertex is incident to either no odd edge or to two odd edges. Hence Γ is an Eulerian subgraph of T. There are two types of toroidal unicellular maps since two cycles of a toroidal unicellular map may intersect either on a single vertex (square case) or on a path (hexagonal case), as depicted on Fig. 36. If T is hexagonal, then Γ is exactly one of the cycles of T. If T is square, then Γ can be either one of the cycles of T or the union of the two cycles of T. One easily checks that in all cases, there exists a cycle C of T that has exactly one incident edge in Γ on each side. We endow C with a traversal direction.

Recall from Section 4.6, that $\gamma_L(C) = w_L(C) + s_L(C)$, $\gamma_R(C) = w_R(C) + s_R(C)$. Moreover since $T \in \mathcal{V}_2^{Bal}$, we have $\gamma_L(C) = \gamma_R(C)$. Note that white vertices of C have all their weight on C. Let $n_\bullet(C)$ be the number of black vertices on C. Let $\kappa(C)$ be the total degree of faces corresponding to the black vertices on C. So black vertices of C have total weight $-\kappa(C) + 2n_\bullet(C)$. Let $n_{\bullet\circ}(C)$ (resp. $n_{\circ\bullet}(C)$) be the number of black-white (resp. white-black) edges on C while following the traversal direction of C. Clearly $n_{\bullet\circ}(C) = n_{\circ\bullet}(C)$. The total weight of half-edges on C incident to a black vertex is precisely $-n_{\bullet\circ}(C) - n_{\circ\bullet}(C) = -2n_{\bullet\circ}(C)$. So $w_L(C) + w_R(C) = -\kappa(C) + 2n_\bullet(C) + 2n_{\bullet\circ}(C)$. Note that we have $\kappa(C) = s_L(C) + s_R(C) + 2n_\bullet(C)$. Combining the equalities gives $w_L(C) = -s_L(C) + n_{\bullet\circ}(C)$.

Let W be the walk of M that is "just on the left" of C (seeing M and T as superimposed), i.e. obtained by following the left boundary of the corresponding face of M each time C passes by a black vertex. Since M is bipartite, the length of W is even, and according to the local rules of Φ_+ shown in Fig. 15, it is equal to $s_L(C) + n_{\bullet\circ}(C)$. So $w_L(C) = (s_L(C) + n_{\bullet\circ}(C)) - 2s_L(C)$ is even. So C is incident to an even number of edges of Γ on its left side, a contradiction. □

6.3.4. Proof of Theorem 21 for $d = 1$

Recall from Section 6.3.2 that $\mathcal{H}_1$ denotes the family of face-rooted toroidal maps with root-face degree 1 (i.e. a loop). Moreover, for $M \in \mathcal{H}_1$, a $\frac{1}{-1}$-$\mathbb{Z}$-orientation of M is a $\mathbb{Z}$-biorientation with weights in $\{-2, -1, 0, 1\}$ such that all vertices have weight 1, all edges have weight -1, and every face f has weight $-\deg(f) + 1$. Note that there are just two types of edges in such an orientation, with weights $(-1, 0)$ or $(-2, 1)$ (see the first row of Fig. 37).

The bijection Φ_+ specializes into a bijection between maps in $\mathcal{H}_1$ endowed with a $\frac{1}{-1}$-$\mathbb{Z}$-orientation in $\mathcal{O}_1^1$ and the family $\mathcal{V}_1$ of toroidal $\frac{1}{-1}$-$\mathbb{Z}$-mobiles. Showing Theorem 22 for $d = 1$ thus amounts to proving the following statement:

Proposition 42. *Let M be a map in $\mathcal{H}_1$. Then M admits a $\frac{1}{-1}$-$\mathbb{Z}$-orientation in $\mathcal{O}_1^1$ whose associated mobile by Φ_+ is in $\mathcal{V}_1^{Bal}$ if and only if M is in $\mathcal{L}_1$. In that case M admits a unique such orientation.*

For a map $M \in \mathcal{H}_1$, let M_2 (resp. M_4) be the map obtained from M by subdividing every edge into a path of length 2 (resp. 4). If M is endowed with a $\frac{1}{-1}$-$\mathbb{Z}$-orientation Z let $\tau(Z)$ be the (transferable) 2-regular orientation of $M_4^\star$ obtained from M using the rules of Fig. 37, i.e., applying the rule of Fig. 34 to obtain a $\frac{1}{0}$-$\mathbb{Z}$-orientation of M_2, then doubling the weights to get to an even $\frac{2}{0}$-$\mathbb{Z}$-orientation of M_2, then applying the rule of Fig. 34 to get to a $\frac{2}{1}$-$\mathbb{Z}$-orientation of M_4, and finally applying the mapping σ^{-1} to get to a transferable 2-regular orientation of $M_4^\star$. Note that Z is in $\mathcal{O}_1^1$ if and only if $\tau(Z)$ is in $\mathcal{O}_4^1$. Note that τ is injective but not a bijection since when doubling the weights to obtain a $\frac{2}{0}$-orientation of M_2 we have only even weights.

We first prove the analogue of Lemma 40:

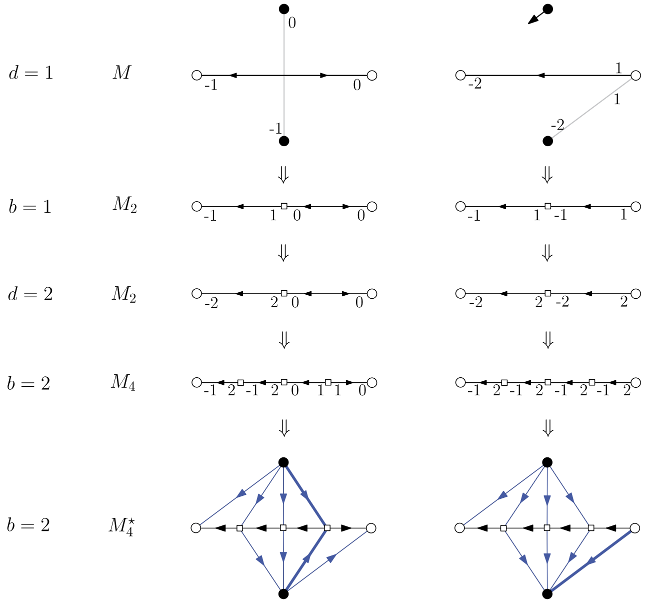


Fig. 37. The mapping τ from $\frac{1}{-1}$-orientations in $\mathcal{O}_1^1$ to (certain) transferable 2-regular orientations in $\mathcal{O}_4^1$. In the top row, we show the corresponding mobile-edge; in the bottom-row we show (in bolder form) on which star-edges we lift the mobile-edge.

Lemma 43. *Let M be a map in $\mathcal{H}_1$, let Z be a $\frac{1}{-1}$-$\mathbb{Z}$-orientation of M in $\mathcal{O}_1^1$, let $X = \tau(Z)$ be the associated 2-regular orientation of $M_4^\star$, and let T be the associated mobile (by Φ_+) in $\mathcal{V}_1$. Then X is $\hat{\gamma}$-balanced if and only if T is balanced.*

Proof. Let C be a non-contractible cycle of T given with a traversal direction. We call *canonical lift* of C the (non-contractible) cycle C' of $M_4^\star$ obtained by keeping the bolder edges as shown in the bottom-row of Fig. 37.

Let e be a black-black edge on C, where the half-edge of weight -1 is traversed before the half-edge of weight 0. Looking at the left part of Fig. 37 it is clear that e has contribution 5 to $s_L^X(C')$, contribution 1 to $s_R^X(C')$, contribution 1 to $w_L^X(C')$, contribution 1 to $w_R^X(C')$, contribution 2 to $\iota_L^X(C')$, and contribution 0 to $\iota_R^X(C')$. Hence e has contribution 3 to $\gamma_L^X(C') = 2(w_L^X(C') - \iota_L^X(C')) + s_L^X(C')$, and contribution 3 to $\gamma_R^X(C') = 2(w_R^X(C') - \iota_R^X(C')) + s_R^X(C')$, hence has zero contribution to $\gamma^X(C')$. Similarly a black-black edge where the half-edge of weight -1 is traversed after the half-edge of weight 0 has zero contribution to $\gamma^X(C')$.

Now let e be a black-white edge on C where the black extremity is traversed before the white extremity. Then it is easy to see (again looking at Fig. 37) that e has contribution 3 to $s_L^X(C')$, contribution 0 to $s_R^X(C')$, contribution 1 to $w_L^X(C')$, contribution 0 to $w_R^X(C')$, contribution 3 to $\iota_L^X(C')$, and contribution 0 to $\iota_R^X(C')$. Hence it has contribution -1 to $\gamma_L^X(C')$ and contribution 0 to $\gamma_R^X(C')$, hence contribution -1 to $\gamma^X(C')$. Symmetrically a black-white edge whose black extremity is traversed after the white extremity has contribution 1 to $\gamma^X(C')$. Now the numbers of black-white edges of both types on C are clearly equal, so that the total contribution of black-white edges on C to $\gamma^X(C')$ is zero.

On the other hand, let h be a half-edge of T not on C but incident to a vertex on C, and let δ be the weight of h (by convention $\delta = 0$ if h is a bud). Then it is easy to see (still looking at Fig. 37) that if h is on the left (resp. right) side of C and incident to a black vertex, then it has contribution 4 to $s_L^X(C')$ (resp. to $s_R^X(C')$) and contribution 2δ to $-\iota_L^X(C')$. And if h is on the left (resp. right) side of C and incident to a white vertex, then it has contribution 2δ to $w_L^X(C')$ (resp. to $w_R^X(C')$). From what precedes we conclude that $\gamma^X(C') = 4\gamma^T(C)$, and in particular $\gamma^X(C') = 0$ if and only if $\gamma^T(C) = 0$.

From there, very similarly to the end of the proof of Lemma 38, we conclude that X is balanced if and only if T is balanced, which concludes the proof. $\square$

We are now able to prove Proposition 42:

Proof of Proposition 42. Suppose that M admits a $\frac{1}{-1}$-$\mathbb{Z}$-orientation Z in $\mathcal{O}_1^1$ whose associated mobile by Φ_+ is in $\mathcal{V}_1^{Bal}$. By Lemma 43, we have $X = \tau(Z)$ is a $\hat{\gamma}$-balanced 2-regular orientation of $M_4^\star$. Since X is in $\mathcal{O}_4^1$, by Lemma 17, we have that X is minimal. By Lemma 35, we have $M_4 \in \hat{\mathcal{M}}_4$. Then, by Lemma 37, we have $M_4 \in \hat{\mathcal{L}}_4$, hence $M \in \mathcal{L}_1$. In addition we have uniqueness of the orientation Z, since Z has to be preimage under the injective mapping τ of the unique minimal $\hat{\gamma}$-balanced 2-regular orientation of $M_4^\star$.

Conversely we prove the existence part, for $M \in \mathcal{L}_1$. Then $M_4 \in \hat{\mathcal{L}}_4$ and by Lemma 37, $M_4^\star$ admits a transferable $\hat{\gamma}$-balanced 2-regular orientation X in $\mathcal{O}_4^1$. Consider Y_4 the $\frac{2}{1}$-$\mathbb{Z}$-orientation of M_4 in $\mathcal{O}_4^1$ such that $Y = \sigma(X)$.

Consider Z' the $\frac{2}{0}$-$\mathbb{Z}$-orientation of M_2 in $\mathcal{O}_2^1$ such that $Z' = \iota^{-1}(Y) = \iota^{-1}(\sigma(X))$. By Lemma 40, the mobile T' associated to Z' is in $\mathcal{V}_2^{Bal}$. Since $M \in \mathcal{L}_1$, we have $M_2 \in \hat{\mathcal{L}}_2$, hence all the weights of T' are even according to Lemma 41. So all the weights of Z' are even. Let Y' be the $\frac{1}{0}$-$\mathbb{Z}$-orientation of M_2 in $\mathcal{O}_2^1$ obtained by dividing all the weights of Z' by two. Consider Z the $\frac{1}{-1}$-$\mathbb{Z}$-orientation of M in $\mathcal{O}_1^1$ such that $Z = \iota^{-1}(Y')$. Note that $X = \tau(Z)$. Let $T \in \mathcal{V}_1$ be the mobile associated to Z. Lemma 43 then ensures that T is in $\mathcal{V}_1^{Bal}$. $\square$

Acknowledgments

The authors thank Olivier Bernardi and the members of the ANR project "Gato" for very interesting and helpful discussions; they also thank the two anonymous referees for their insightful reports.

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