

APPROXIMATIONS FOR ALL-TO-ALL UNIFORM TRAFFIC GROOMING ON UNIDIRECTIONAL RINGS*

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Traffic grooming in a WDM network consists of assigning to each request (lightpath) a wavelength with the constraint that a given wavelength can carry at most C requests or equivalently a request uses $1/C$ of the bandwidth. C is known as the grooming ratio. A request (lightpath) needs two SONET add-drop multiplexers (ADMs) at each end node; using grooming, different requests can share the same ADM. The so called traffic grooming problem consists of minimizing the total number of ADMs to be used (in order to reduce the overall cost of the network). Here we consider the traffic grooming problem in WDM unidirectional rings which has been recently shown to be APX-hard and for which no constant approximations are known. We furthermore suppose an all to all uniform unitary traffic. This problem has been optimally solved for specific values of the grooming ratio, namely $C = 2, 3, 4, 5, 6$. In this paper we present various simple constructions for the grooming problem providing approximation of the total number of ADMs with a small constant ratio. For that we use the fact that the problem corresponds to a partition of the edges of the complete graph into subgraphs, where each subgraph has at most C edges and where the total number of vertices has to be minimized.

Keywords: Traffic Grooming; WDM Networks; ADM; Unidirectional Rings; Approximation; Designs; Partition of graphs.

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1. Introduction

Traffic grooming is the generic term for packing low rate signals into higher speed streams (see the surveys ^{7,14,22,24,30,21}). By using traffic grooming, one can bypass the electronics in the nodes for which there is no traffic sourced or destined to it and therefore reduce the cost of the network. Typically, in a WDM (Wavelength Division Multiplexing) network, instead of having one SONET Add Drop Multiplexer (ADM) on every wavelength at every node, it may be possible to have ADMs only for the wavelength used at that node (the other wavelengths being optically routed without electronic switching).

In SONET/WDM networks, we assign to each request $\{i, j\}$ a fraction of the bandwidth offered by a wavelength along a path from node i to node j . If a given wavelength can carry at most C requests, we can assign to each request at most $\frac{1}{C}$ of the bandwidth. C is known as the grooming ratio. In the particular case of unidirectional rings, the routing is unique. Furthermore, if the traffic is symmetric, it can be easily shown (by exchanging wavelengths) that there always exists an optimal solution in which the same wavelength is given to a pair of symmetric requests. Then, without loss of generality, we will assign to each pair of symmetric requests, called a circle, a fraction of the bandwidth in the whole ring. In both cases, we need one ADM at node i and one at node j . Also, two requests with a common extremity and assigned to the same wavelength will share an ADM. For example, if requests $\{1, 2\}$ and $\{2, 3\}$ are assigned to two different wavelengths, then we need 4 ADMs, while if they are assigned to the same wavelength we will need only 3 ADMs.

The so called traffic grooming problem consists of minimizing the total number of ADMs to be used (in order to reduce the overall cost of the network). Here we study the problem for an unidirectional SONET ring with N nodes, a grooming ratio C , and an all-to-all uniform unitary traffic. This problem has been modeled as a graph partition problem in both ⁶ and ¹⁸. The set of requests is modeled by a graph I , where $I = K_N$ in the all-to-all case. To a wavelength w is associated a subgraph B_w in which each edge corresponds to a request and each node to an ADM. The grooming constraint, that a wavelength can carry at most C requests, corresponds to the fact that the number of edges $|E(B_w)|$ of each subgraph B_w is at most C . The objective is therefore to minimize the total number of vertices used in the subgraphs.

Problem 1.1 (Grooming on unidirectional cycle ⁶) *Given a number of nodes N and a grooming ratio C find a partition of the edges of the undirected graph $I = K_N$ into subgraphs B_w , $w = 1, \dots, W$ with $|E(B_w)| \leq C$ such that $\sum_{1 \leq w \leq W} |V(B_w)|$ is minimum. (This minimum will be denoted $A(C, N)$).*

The traffic grooming problem has recently been extensively studied on unidirectional WDM rings, primarily in the context of variable traffic requirements ^{10,13,18,25,28}, but the case of fixed traffic requirements has served as an important special case ^{3,4,5,6,7,8,14,16,17,19,20,22,26,29}. The problem has also been studied on

Table 1. Approximation factor of the different constructions.

C	8	9	12	15	16	28	32	48	64	192
$\rho_{\max}(C)$	$\frac{8}{5}$	$\frac{9}{5}$	2	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{32}{9}$	$\frac{9}{2}$	$\frac{16}{3}$	$\frac{19}{2}$
γ for Cons. 4.1	1.6	1.2	1.33	1.67	1.25	1.4	1.42	1.5	1.33	1.46
γ for Cons. 4.4	1.2	1.2	1.17	1.33	1.25	1.37	1.3	1.31	1.33	1.39
γ for Cons. 5.1	1.6	1.8	1	1.25	1.25	1.17	1.19	1.13	1.33	1.19

the path ².

With a general set of requests, $I \neq K_N$, the grooming problem has been proved NP-Complete in unidirectional ring with grooming factor $C \geq 1$ ^{10,23}. Then a first approximation algorithm for computing the total number of ADMs with approximation factor $\frac{C(1+\lceil C/2 \rceil)}{\lfloor \frac{1+\sqrt{1+8C}}{2} \rfloor}$, i.e. $\sim \sqrt{C}$, has been given in ¹⁸, and in ¹⁵ a $\log(C)$ -approximation algorithm has been obtained. More recently, the grooming problem has been proved APX-Hard in ¹ (i.e. there exists a constant c , such that Problem 1.1 can not be approximate within a factor c).

With the all-to-all set of requests, $I = K_N$, the extremal problem of finding the exact value of $A(C, N)$ is open and there is not even a conjecture for the extremal constructions. Optimal constructions for given grooming ratio C were obtained using tools of graph and design theory ^{7,11,12}, in particular for grooming ratio $C = 3$ ³, $C = 4$ ^{19,8}, $C = 5$ ⁵, $C = 6$ ⁴ and $C \geq N(N-1)/6$ ⁸. However it will be a very long and intractable task to find optimal constructions for all grooming ratio. Existing heuristic algorithms ^{17,26,29} as well as the approximation algorithm proposed in ^{15,18} are not satisfactory for the all-to-all case. Therefore, it is important to show that in this case we have approximation algorithms with a small approximation ratio.

In this paper, we will first present an asymptotical $1 + \frac{4C}{N} + o(\frac{1}{N})$ -approximation algorithm; unfortunately the construction is valid only for large N . Then we present a very simple construction using bipartite graphs which provides a $\gamma(C, N)$ -approximation for the total number of ADMs, where $\gamma(C, N)$ is at most $\sqrt{2} \frac{\sqrt{C}}{\lfloor \sqrt{C} \rfloor}$ and in many cases better (for example, for $C = 16$: $\gamma(16, N) = \frac{5}{4}$, and for $C = 64$: $\gamma(64, N) = \frac{4}{3}$). Then we show several improvements of this construction by using other bipartite graphs or tripartite graphs (in that case $\gamma(C, N)$ is of order $\sqrt{\frac{3}{2}}$) or multipartite graphs. Values of the approximation factor obtained with different constructions are given in Table 1 for realistic values of C .

2. Lower Bound

A tight lower bound for Problem 1.1 has been given in ^{6,8} and is recalled in Theorem 2.1. The idea consists in using in the partition subgraphs which, for a given

number of edges (less than C), have the minimum number of vertices. So let us denote by $\rho(G)$ the ratio of a subgraph G , $\rho(G) = \frac{|E(G)|}{|V(G)|}$, and by $\rho(m)$ the maximum ratio of a subgraph with m edges. Let finally $\rho_{\max}(C)$ denote the maximum possible ratio among all the subgraphs with $m \leq C$ edges, that is:

$$\rho_{\max}(C) = \max \{ \rho(G) \mid |E(G)| \leq C \} = \max_{m \leq C} \rho(m) \tag{2.1}$$

Recall that $A(C, N)$ is the minimum number of ADM's needed in an unidirectional ring with the all-to-all set of request ($I = K_N$) and with a grooming ratio C . $A(C, N) = \sum_{1 \leq w \leq W} |V(B_w)|$; so using $\rho_{\max}(C)|V(B_w)| \geq |E(B_w)|$ and $\sum_{1 \leq w \leq W} |E(B_w)| = \frac{N(N-1)}{2}$ we get the following lower bound:

Theorem 2.1 (Lower Bound ⁶) $A(C, N) \geq \frac{N(N-1)}{2\rho_{\max}(C)}$.

The value of $\rho_{\max}(C)$ has been evaluated in ⁶ and is recalled in Proposition 2.1.

Proposition 2.1 (⁶)

- If $\frac{x(x-1)}{2} \leq C \leq \frac{(x+1)(x-1)}{2}$, then $\rho_{\max}(C) = \frac{x-1}{2}$ and the value is attained for K_x .
- If $\frac{(x+1)(x-1)}{2} < C < \frac{(x+1)x}{2}$, then $\rho_{\max}(C) = \frac{C}{x+1}$ and the value is attained for any graph with C edges and $x + 1$ vertices.

Values of $\rho_{\max}(C)$ are given in Table 1 for realistic values of C . The following corollary gives also a lower bound easier to manipulate.

Corollary 2.1. $\rho_{\max}(C) \leq \sqrt{\frac{C}{2}}$ and so $A(C, N) \geq \frac{N(N-1)}{\sqrt{2C}}$.

Proof. From Proposition 2.1, we know that $\rho_{\max}(C) = \frac{x-1}{2}$ (case 1) or $\frac{C}{x+1}$ (case 2), and we can observe that $x = \lfloor \frac{1+\sqrt{1+8C}}{2} \rfloor$. Thus we have

- case 1 : $2\rho_{\max}(C) = \lfloor \frac{1+\sqrt{1+8C}}{2} \rfloor - 1 \leq \lfloor \frac{\sqrt{1+8C}-1}{2} \rfloor \leq \frac{\sqrt{8C}}{2} \leq \sqrt{2C}$ and so $\rho_{\max}(C) \leq \sqrt{\frac{C}{2}}$
- case 2 : $\rho_{\max}(C) = \frac{C}{\lfloor \frac{1+\sqrt{1+8C}}{2} \rfloor + 1} \leq \frac{C}{\frac{1+\sqrt{1+8C}}{2}} \leq \sqrt{\frac{C}{2}}$

So $A(C, N) \geq \frac{N(N-1)}{\sqrt{2C}}$. □

3. Asymptotic Construction

It has been shown in ⁶ that design theory can help to solve the grooming problem. In particular, a G -design of order N (see ¹¹ VI.24 or ⁹) is nothing else than a partition of the edges of K_N into subgraphs isomorphic to a given graph G . The interest of the existence of a G -design is shown by the following immediate proposition.

Proposition 3.1 ⁽⁶⁾ *If there exists a G -design of order N , where G is a graph with at most C edges and with ratio $\rho_{\max}(C)$, then $A(C, N) = \frac{N(N-1)}{2\rho_{\max}(C)}$.*

Necessary conditions 3.1 (Existence of a G -design) *If there exists a G -design, then*

- (i) $\frac{N(N-1)}{2}$ should be a multiple of $E(G)$
- (ii) $N - 1$ should be a multiple of the greatest common divisor of the degrees of the vertices of G .

It has been shown that these conditions are sufficient for $C = 3, 6, 10$: G being the complete graph K_3 , K_4 , or K_5 (in that case we have a “classical design”, see ¹¹ chapter II.3), and also for $C = 15$ ($N \geq 802$): G being K_6 . They are also sufficient for $C = 4, 5, 8$ ($N \neq 48$), 9 ($N \geq 235$) (see ¹¹ chapter VI.24). More generally, Wilson ²⁷ has shown that these necessary conditions are sufficient for any C when N is large enough. So we have the following Theorem:

Theorem 3.1 (see ⁶) *We have*

- $A(3, N) = \frac{N(N-1)}{2}$ when $N \equiv 1$ or $3 \pmod{6}$
- $A(4, N) = \frac{N(N-1)}{2}$ when $N \equiv 0$ or $1 \pmod{8}$
- $A(5, N) = \frac{2N(N-1)}{5}$ when $N \equiv 0$ or $1 \pmod{5}$
- $A(6, N) = A(7, N) = \frac{N(N-1)}{3}$ when $N \equiv 1$ or $4 \pmod{12}$
- $A(8, N) = \frac{5N(N-1)}{16}$ when $N \equiv 0$ or $1 \pmod{16}$, and $N \neq 48$
- $A(9, N) = \frac{5N(N-1)}{18}$ when $N \equiv 0$ or $1 \pmod{9}$, and $N \geq 235$
- $A(10, N) = \frac{N(N-1)}{4}$ when $N \equiv 1$ or $5 \pmod{20}$
- $A(15, N) = A(16, N) = \frac{N(N-1)}{5}$ when $N \equiv 1$ or $6 \pmod{15}$, and $N \geq 802$

Construction 3.1. For a given C , let $N_2 \geq N$ be the smallest integer such that there exists a G -design where G has at most C edges and a ratio $\rho_{\max}(C)$. We obtain a valid construction for N by removing $N_2 - N$ nodes and the corresponding edges from the optimal construction for N_2 .

In order to get an approximation factor of this solution, we need to know a lower bound for $A(C, N)$. A trivial lower bound is given by $A(C, N_1)$, where $N_1 \leq N$ is the biggest integer such that there exists a G -design.

The following lemma allows to find values of N_1 and N_2 that are near to each other.

Lemma 3.1. *Let $\alpha(C)$ be defined as follows :*

- If $\frac{x(x-1)}{2} \leq C \leq \frac{(x+1)(x-1)}{2}$, then $\alpha(C) = x(x-1)$.
- If $\frac{(x+1)(x-1)}{2} \leq C < \frac{(x+1)x}{2}$, then $\alpha(C) = 2C$

Let $N_1 = \alpha(C)t + 1$ and $N_2 = \alpha(C)(t + 1) + 1$ be such that $N_1 \leq N \leq N_2$.

There always exists a graph G with at most C edges and ratio $\rho_{\max}(C)$ which satisfies Conditions 3.1 for N_1 and N_2 .

Proof. When $\frac{x(x-1)}{2} \leq C \leq \frac{(x+1)(x-1)}{2}$, then $\rho_{\max}(C)$ is attained for K_x , and so let $G = K_x$. Both $N_1 - 1$ and $N_2 - 1$ are multiple of $\alpha(C) = x(x - 1)$; and so the number of edges of K_{N_1} (resp. K_{N_2}) $\frac{N_1(N_1-1)}{2}$ (resp. $\frac{N_2(N_2-1)}{2}$) is a multiple of $E(G) = \frac{x(x-1)}{2}$. Condition (ii) is also satisfied as the degree of a vertex of K_{N_1} (resp. K_{N_2}) $N_1 - 1$ (resp. $N_2 - 1$) is a multiple of $x - 1$ the degree of K_x .

When $\frac{(x+1)(x-1)}{2} \leq C < \frac{(x+1)x}{2}$, then $\rho_{\max}(C)$ is attained for any graph with C edges and $x + 1$ vertices. Let $r = \frac{(x+1)x}{2} - C$. So $0 < r < x$. Let G be the graph obtained from K_{x+1} by removing the edges of a path of length r . G has C edges and so Condition (i) is satisfied as $\frac{N_1(N_1-1)}{2} = (2Ct + 1)Ct$ and $\frac{N_2(N_2-1)}{2} = (2C(t + 1) + 1)C(t + 1)$ are multiples of $E(G)$. As $0 < r \leq x - 1$, G has a vertex which is not in the path that have been removed; this vertex has degree x , and the extremities of the path have degree $x - 1$, so the greatest common divisor of the degrees of the vertices of G is 1. Condition (ii) is trivially satisfied. \square

Proposition 3.2. *When N is large enough to satisfy Wilson’s Theorem, Construction 3.1 has an approximation factor $\gamma(C, N) \leq 1 + \frac{4C}{N} + o\left(\frac{1}{N}\right)$.*

Proof. Let $f(C, N)$ denotes the number of ADMs obtained by Construction 3.1 and let $\gamma(C, N) = \frac{f(C, N)}{A(C, N)}$ be its approximation factor. Let also N_1 and N_2 , with $N_1 \leq N \leq N_2$, be given by Lemma 3.1. We have $A(C, N_1) \leq A(C, N) \leq f(C, N) \leq A(C, N_2)$ and $\gamma(C, N) = \frac{f(C, N)}{A(C, N)} \leq \frac{A(C, N_2)}{A(C, N_1)} = \frac{N_2(N_2-1)}{N_1(N_1-1)}$.

Since $N_2 = \alpha(C)(t + 1) + 1 = N_1 + \alpha(C)$, we have $\gamma(C, N) \leq 1 + \frac{2\alpha(C)}{N_1} + \frac{\alpha(C)(\alpha(C)+1)}{N_1(N_1-1)}$.

Finally, as in both cases $\alpha(C) \leq 2C$ and $N - N_1 \leq \alpha(C)$, we obtain $\gamma(C, N) \leq 1 + \frac{4C}{N} + o\left(\frac{1}{N}\right)$. \square

Unfortunately, except for the small values of C given in Theorem 3.1, the values of N for which Wilson’s Theorem and so Proposition 3.2 applies are very large. Furthermore, it is not known how to implement Construction 3.1 in polynomial time. So there is a need to find simpler and general constructions.

4. Construction using Bipartite Graphs

In this section, we first present a simple construction which gives an upper bound on the number of ADM’s and we analyze it’s approximation factor. Then, we present some improvements of this construction.

Basically our construction consists of partitioning the edges of K_N into a maximum number of bipartite graphs, with at most C edges, plus some small complete graphs. A complete-bipartite graph with 2 sets of p nodes each has p^2 edges and a

ratio of $\frac{p}{2}$. Therefore choosing p^2 to be C or almost C we get a ratio near to $\frac{\sqrt{C}}{2}$. As we will see in the proof of Proposition 4.1, the number of ADMs due to bipartite graphs dominates the total cost of the construction, and so the number of ADMs will be around $\frac{N(N-1)}{\sqrt{C}}$. From Theorem 2.1 we know that the lower bound is larger than $\frac{N(N-1)}{2\rho_{\max}(C)}$. So our construction gives an approximation factor close to $\frac{2\rho_{\max}(C)}{\sqrt{C}} \leq \sqrt{2}$ by Corollary 2.1.

Several constructions are possible. We first present a basic construction (Construction 4.1) and then some improvements (Constructions 4.2 and 4.3) in order to have a precise approximation factor. We also give a variant in which C is the product of two numbers.

4.1. Basic construction

Construction 4.1.

Let $C = p^2 + p'$, $0 \leq p' \leq 2p$ ($p = \lfloor \sqrt{C} \rfloor$); let $N = qp + r$, $0 \leq r < p$, and let the vertices of K_N be $V = \cup_{i=1}^q V_i \cup V_{q+1}$ with $|V_i| = p$ and $|V_{q+1}| = r$.

We partition the edges of K_N into $\frac{q(q-1)}{2} K_{p,p}$ on $V_i \cup V_j$, $1 \leq i < j \leq q$, plus $q K_{p,r}$ on $V_i \cup V_{q+1}$, $1 \leq i \leq q$, plus $q K_p$ on V_i and one K_r on V_{q+1} .

Proposition 4.1. *Construction 4.1 is valid and uses $(q + 1)N$ ADMs.*

Proof. First all the subgraphs of the decomposition have at most $p^2 \leq C$ edges. Since a bipartite graph $K_{x,y}$ has $x + y$ vertices and a complete graph K_x has x vertices, the number of ADMs involved in the construction is: $2p\frac{q(q-1)}{2} + (p+r)q + qp + r = (q + 1)(qp + r) = (q + 1)N$ ADMs. □

Corollary 4.1. *When $C = p^2 + p'$, $0 \leq p' \leq 2p$, and $N = qp + r$, $0 \leq r < p$, Construction 4.1 provides a*

$$\frac{2\rho_{\max}(C)}{\lfloor \sqrt{C} \rfloor} + O\left(\frac{1}{N}\right) < \sqrt{2} \frac{\sqrt{C}}{\lfloor \sqrt{C} \rfloor} + O\left(\frac{1}{N}\right)\text{-approximation of the number of ADMs.}$$

Proof. Let $\gamma(C, N)$ be the approximation factor that is the ratio between the upper bound construction and the lower bound for a given grooming factor C . We know from Theorem 2.1 that $A(C, N) \geq \frac{N(N-1)}{2\rho_{\max}(C)}$. So

$$\gamma(C, N) = (q + 1)N \frac{2\rho_{\max}(C)}{N(N-1)} = 2\rho_{\max}(C) \frac{q + 1}{N - 1} \tag{4.1}$$

Since $C = p^2 + p'$, we have $p = \lfloor \sqrt{C} \rfloor$ and $q = \frac{N-r}{p} = \frac{N-r}{\lfloor \sqrt{C} \rfloor}$. Thus we obtain

$$\gamma(C, N) = \frac{2\rho_{\max}(C)}{\lfloor \sqrt{C} \rfloor} \left(1 + \frac{\lfloor \sqrt{C} \rfloor - r + 1}{N - 1} \right) \tag{4.2}$$

□

4.2. Improvements

The above construction is very simple and provides a better approximation factor than ¹⁵. The values of the approximation factor for some values of C are indicated in Table 1. A first improvement can be obtained by noting that some bipartite subgraphs of the decomposition have strictly less than C edges and therefore we can add to them some edges of the K_p 's and of the K_r . That is always the case for the $K_{p,r}$ as $pr < p^2 \leq C$ and also for the $K_{p,p}$ when $C > p^2$. Doing so we can get rid of the $O\left(\frac{1}{N}\right)$ in Corollary 4.1.

Construction 4.2. Let $C = p^2$ and $N = qp + r$, $0 < r < p$.

The construction consists of partitioning the edges of K_N into $\frac{q(q-1)}{2} K_{p,p}$ on $V_i \cup V_j, 1 \leq i < j \leq q$, plus q subgraphs on $V_i \cup V_{q+1}, 1 \leq i \leq q$ containing the pr edges of the $K_{p,r}$ between V_i and V_{q+1} plus the $\frac{p(p-1)}{2}$ of the K_p on V_i and some edges of the K_r on V_{q+1} .

Proposition 4.2. Let $C = p^2$ and $N = qp + r$, $0 < r < p$. If $\frac{r(r-1)}{2} \leq q\left(C - pr - \frac{p(p-1)}{2}\right)$, Construction 4.2 is valid and provides a $\frac{2\rho_{\max}(C)}{\sqrt{C}} \leq \sqrt{2}$ -approximation of the total number of ADMs.

Proof. The subgraphs $K_{p,p}$ have $p^2 = C$ edges. Each other subgraph contains the pr edges of the $K_{p,r}$ between V_i and V_{q+1} plus the $\frac{p(p-1)}{2}$ of the K_p on V_i . So we can still use $C - pr - \frac{p(p-1)}{2} > 0$ edges of the K_r on V_{q+1} ; and altogether we can use all the edges of K_r as $\frac{r(r-1)}{2} \leq q\left(C - pr - \frac{p(p-1)}{2}\right)$.

Construction 4.2 uses $q(q-1)p + q(p+r) = q(qp+r) = qN$ ADMs. So it has an approximation factor

$$\gamma(C, N) = qN \frac{2\rho_{\max}(C)}{N(N-1)} = \frac{2\rho_{\max}(C)}{p} \frac{N-r}{N-1}. \text{ Since } C = p^2, \text{ we have } \gamma(C, N) = \frac{2\rho_{\max}(C)}{\sqrt{C}} \left(1 - \frac{r-1}{N-1}\right), \text{ and since } 0 < r, \text{ we obtain } \gamma(C, N) \leq \frac{2\rho_{\max}(C)}{\sqrt{C}} \leq \sqrt{2}. \quad \square$$

This strategy allows us to win a small amount of ADMs (at most N). For example, when $C = 16$, $p = 4$, and $q = 4$, conditions of Construction 4.2 are satisfied for $r = 1$ and 2 , so $N = 17$ and $N = 18$. For $N = 17$ (resp. $N = 18$), Construction 4.1 uses $5 \times 17 = 85$ (resp. $5 \times 18 = 90$) ADMs and Construction 4.2 uses 68 (resp. 72) ADMs, that is a saving of 17 (resp. 18) ADMs.

When $C = p^2 + p'$, it is possible to improve Construction 4.2 by adding some edges of K_r and of the K_p 's to the subgraphs containing the bipartite graphs, thus reducing the total number of ADMs. In some cases, the subgraphs based on K_p or K_r may be completely absorbed as explained in the Construction 4.3 and Proposition 4.3.

Construction 4.3. Let $C = p^2 + p'$, $0 < p' \leq 2p$ and $N = qp + r$, $0 \leq r < p$.

The construction consists of partitioning the edges of K_N into $\frac{q(q-1)}{2}$ subgraphs on $V_i \cup V_j, 1 \leq i < j \leq q$ containing the p^2 edges of the $K_{p,p}$ between V_i and V_j plus some edges of one of the K_p , plus q subgraphs on $V_i \cup V_{q+1}, 1 \leq i \leq q$ containing the pr edges of the $K_{p,r}$ between V_i and V_{q+1} plus some edges of the K_r on V_{q+1} .

Proposition 4.3. *Let $C = p^2 + p'$, $0 < p' \leq 2p$ and $N = qp + r$, $0 \leq r < p$. If $(q-1)p' \geq p(p-1)$, then Construction 4.3 is valid and provides a $\frac{2\rho_{\max}(C)}{\lfloor \sqrt{C} \rfloor} \leq \sqrt{2} \frac{\sqrt{C}}{\lfloor \sqrt{C} \rfloor}$ -approximation of the total number of ADMs.*

Proof. The subgraphs on $V_i \cup V_j$, $1 \leq i < j \leq q$ use the p^2 edges of $K_{p,p}$ and p' edges of one of the K_p . Altogether we can use all the edges of the K_p as by the condition $\frac{q(q-1)}{2}p' \geq \frac{p(p-1)}{2}q$. In each $K_{p,r}$ we can use $p^2 + p' - pr = p(p-r) + p'$ edges of the K_r . Since we have q $K_{p,r}$ and that $qp' > (q-1)p' > p(p-1) > \frac{r(r-1)}{2}$, all edges of K_r are used.

Construction 4.3 uses $q(q-1)p + q(p+r) = q(qp+r) = qN$ ADMs. So it has the desired approximation factor. \square

Note that condition of Proposition 4.3 is satisfied as soon as $q \geq p(p-1)$, that is $N \geq p^2(p-1)$.

Remark that in some cases the approximation factor can be strictly larger than $\sqrt{2}$ due to the integer part of \sqrt{C} . For example if $C = 8$, $\lfloor \sqrt{C} \rfloor = 2$ but $\rho_{\max}(C) = \frac{8}{2} = 4$ and the approximation factor is $\frac{8}{2} = 4 > \sqrt{2}$. For $C = 15$, $\lfloor \sqrt{C} \rfloor = 3$, $\rho_{\max}(C) = \frac{15}{3} = 5$ and the approximation factor is $\frac{5}{3} > \sqrt{2}$.

4.3. Case $C = p_1p_2$

The next construction helps to deal with these cases where $C = p_1p_2$.

Construction 4.4. Let $C = p_1p_2 + p'$, $p_1 \leq p_2$; let also $N = qp_1p_2 + r$, $0 \leq r < p_1p_2$. Let the vertices of K_N be $V = \cup_{i=1}^q V_i \cup V_{q+1}$ with $|V_i| = p_1p_2$ and $|V_{q+1}| = r$.

We partition the edges of K_N into $\frac{q(q-1)}{2} K_{p_1p_2, p_1p_2}$ on $V_i \cup V_j$, $1 \leq i < j \leq q$, plus $q K_{p_1p_2, r}$ on $V_i \cup V_{q+1}$, $1 \leq i \leq q$, plus $q K_{p_1p_2}$ on V_i and one K_r on V_{q+1} .

Then we partition each $K_{p_1p_2, p_1p_2}$ into $p_1p_2 K_{p_1, p_2}$ and each $K_{p_1p_2, r}$, where $r = \alpha_1p_1 + \beta_1$, $0 \leq \beta_1 < p_1$, into $p_1\alpha_1 K_{p_1, p_2}$ plus $p_1 K_{\beta_1, p_2}$.

Finally, we partition each $K_{p_1p_2}$ into $\frac{p_2(p_2-1)}{2} K_{p_1, p_1}$ plus $p_2 K_{p_1}$, and each K_r into $\frac{\alpha_1(\alpha_1-1)}{2} K_{p_1, p_1}$ plus $\alpha_1 K_{p_1, \beta_1}$ and $\alpha_1 K_{p_1}$ and $1 K_{\beta_1}$.

Proposition 4.4. *Let $C = p_1p_2 + p'$, and $N = qp_1p_2 + r$, $0 \leq r < p_1p_2$, Construction 4.4 is valid and provides a $\frac{\rho_{\max}(C)(p_1+p_2)}{p_1p_2} + O(\frac{1}{N})$ -approximation of the total number of ADMs.*

Proof. As $\beta_1 < p_1 < p_2$, all the subgraphs of the decomposition, i.e. K_{p_1, p_2} , K_{β_1, p_2} , K_{p_1, p_1} , K_{p_1} , K_{p_1, β_1} , K_{β_1} , have at most $p_1p_2 \leq C$ edges, and so the construction is valid.

The total number of ADMs is $\frac{q(q-1)}{2}p_1p_2(p_1+p_2) + q(\alpha_1p_1(p_1+p_2) + p_1(\beta_1 + p_2)) + qp_1p_2^2 + \alpha_1(\alpha_1-1)p_1 + \alpha_1(2p_1 + \beta_1) + \beta_1$. Using $N = qp_1p_2 + r$ we get $\frac{N(N-1)(p_1+p_2)}{2p_1p_2} + O(N)$ ADMs and so the approximation factor. \square

Remark : We can also modify the construction like we did before to include the edges of the $K_{p_1 p_2}$ or K_r in the bipartite subgraphs, and therefore get rid in many cases of the $O\left(\frac{1}{N}\right)$ in the approximation factor.

Note that when $p_1 = p_2$, Construction 4.4 is exactly Construction 4.1. However, we have more possible choices for p_1 , p_2 , and p' , and so for many values Construction 4.4 is better than Construction 4.1. Of course we have interest to choose p' as small as possible, but also to choose p_1 and p_2 in order to minimize $\frac{p_1+p_2}{p_1 p_2}$; that can be achieved by choosing p_1 and p_2 near to each other but not necessarily equals.

For example, let $C = 32$. We can write $32 = 5 \times 5 + 7$, or $32 = 4 \times 8$, or $32 = 5 \times 6 + 2$. For $C = 5 \times 5 + 7$, Construction 4.4 or 4.1 give approximation factor $\frac{2}{5}\rho_{\max}(C)$; if we choose $32 = 4 \times 8$ in Construction 4.4 we get an approximation factor $\frac{12}{32}\rho_{\max}(C)$ which is better since $\frac{12}{32} < \frac{2}{5}$. But we can do better using $32 = 5 \times 6 + 2$ in Construction 4.4 getting approximation factor $\frac{11}{30}\rho_{\max}(C)$.

Let now $C = 8 = 2 \times 4$, $\rho_{\max}(C) = \frac{8}{5}$. With Construction 4.4 we get approximation factor is $\frac{8}{5} \times \frac{6}{8} = 1.2$ to be compared with $\frac{8}{5} = 1.6$ from Construction 4.1. Similarly, when $C = 15 = 3 \times 5$, $\rho_{\max}(C) = \frac{5}{2}$ and Construction 4.4 gives approximation factor $\frac{5}{2} \times \frac{8}{15} = \frac{4}{3}$ to be compared with $\frac{5}{3}$ from Construction 4.1.

5. Construction with Multipartite Graphs

In the previous section we have shown that using a partition of K_N into small bipartite graphs, it is possible to obtain a $\frac{2\rho_{\max}(C)}{\lfloor\sqrt{C}\rfloor} + O\left(\frac{1}{N}\right)$ -approximation of the total number of ADMs. We will now show that using small multipartite graphs it is possible to drastically improve the approximation factor.

5.1. Construction with tripartite graphs

We will first use the optimal decomposition of K_N obtained in³ for a grooming factor $C = 3$, and reported here in Theorem 5.1, to obtain a $\frac{\rho_{\max}(C)}{\lfloor\sqrt{\frac{C}{3}}\rfloor} + O\left(\frac{1}{N}\right)$ -approximation algorithm.

Theorem 5.1 (Theorem 4 of³) *Let $n \geq 2$. There exists a partition of K_n using*

- if $n \equiv 1, 3 \pmod{6}$, $\frac{n(n-1)}{6} K_3$
- if $n \equiv 5 \pmod{6}$, $\frac{n(n-1)-8}{6} K_3$ and $2 P_3$
- if $n \equiv 0, 4 \pmod{12}$, $\frac{n(n-1)}{6} - \frac{n}{4} K_3$ and $\frac{n}{4} K_{1,3}$
- if $n \equiv 2 \pmod{6}$, $\frac{n(n-1)-2}{6} - \lceil\frac{n-2}{4}\rceil K_3$, $\lceil\frac{n-2}{4}\rceil K_{1,3}$ and 1 edge
- if $n \equiv 6, 10 \pmod{12}$, $\frac{n(n-1)}{6} - \frac{n+2}{4} K_3$, $\frac{n-2}{4} K_{1,3}$ and $1 P_4$

where P_3 is a path with 3 vertices, P_4 a path with 4 vertices and $K_{1,3}$ a complete bipartite graph between a set of size 1 and a set of size 3 (also call a claw or a 3-star).

Construction 5.1. Let $C = 3p^2 + p'$, $0 \leq p' < 6p + 3$ and $N = qp + r$, $1 \leq r \leq p$, and let the vertices of K_N be $\cup_{i=1}^q V_i \cup V_{q+1}$, with $|V_i| = p$ and $|V_{q+1}| = r$.

Consider K_{q+1} ; replace each node i of K_{q+1} by the set of nodes V_i , $1 \leq i \leq q + 1$, and each edge $\{i, j\}$ by the corresponding $K_{p,p}$ or $K_{p,r}$ constructed on $V_i \cup V_j$. Now consider the partition of K_{q+1} given by Theorem 5.1. To each subgraph of the partition we associate in K_N a multipartite subgraph of K_N ; for example to a K_3 (i, j, k) will correspond a tripartite $K_{p,p,p}$ built on $V_i \cup V_j \cup V_k$. All these subgraphs plus the q K_p built on V_i , $1 \leq i \leq q$ and the K_r built on V_{q+1} form a partition of K_N .

Proposition 5.1. *Construction 5.1 is valid and provides a $\frac{\rho_{\max}(C)}{\left\lfloor \sqrt{\frac{C}{3}} \right\rfloor} + O\left(\frac{1}{N}\right)$ -approximation of the total number of ADMs.*

Proof. All the subgraphs of the partition of K_{q+1} contains at most 3 edges and so the corresponding subgraphs in K_N have at most $3p^2 \leq C$ edges. The K_p and K_r have at most $\frac{p(p-1)}{2} < C$ edges. So Construction 5.1 is valid.

We can count exactly the number of ADMs.

- When $q + 1 \equiv 1, 3 \pmod 6$, each node of K_{q+1} appears in $\frac{q}{2} K_3$. So each node of K_N appears in $\frac{q}{2} + 1$ subgraphs (the $+1$ coming from the K_p or K_r to which it belongs) and so we have $\frac{q+2}{2}N$ ADMs.
- When $q + 1 \equiv 5 \pmod 6$ and according to the proof of Theorem 5.1 of ³, the two P_3 of the partition of K_{q+1} contain the edges $x - u, u - y$ and $x - v, v - y$. Nodes different from x and y appear in $\frac{q}{2}$ subgraphs of the partition and node x and y in $\frac{q}{2} + 1$ subgraphs. As nodes x and y are replaced by at most p vertices, all nodes of K_N appear in $\frac{q}{2}$ subgraphs except at most $2p$ of them which appear in one more. So altogether we have at most $\frac{q+2}{2}N + 2p$ ADMs.
- When $q + 1 \equiv 0, 4 \pmod{12}$, $\frac{q+1}{4}$ nodes of K_{q+1} appears in $\frac{q-3}{2} K_3$ and one $K_{1,3}$ (namely the central vertices of the $K_{1,3}$) and $\frac{3(q+1)}{4}$ appear in $\frac{q-1}{2} K_3$ and one $K_{1,3}$. As the nodes are replaced by at most p vertices, each node of K_N appear in $\frac{q+1}{2}$ subgraphs and at most $\frac{3p(q+1)}{4}$ in one more subgraph. So altogether we have $\frac{q+1}{2}N + p\frac{3p(q+1)}{4}$ ADMs.
- When $q + 1 \equiv 2 \pmod 6$, a similar analysis gives that we have at most $\frac{q+1}{2}N + 3p \left\lceil \frac{q-1}{4} \right\rceil + 2p$ ADMs.
- When $q + 1 \equiv 6, 10 \pmod{12}$, we have at most $\frac{q+1}{2}N + 3p\frac{q-1}{4} + 2p$ ADMs.

In all the cases the total number of ADMs is $\frac{N(N-1)}{2p} + O(N)$ giving the proposition as $p = \left\lfloor \sqrt{\frac{C}{3}} \right\rfloor$. □

Note that we can in some cases get rid of the $O\left(\frac{1}{N}\right)$ like we did in Constructions 4.2 or 4.3 and in particular if $C = 3p^2 + p'$, $0 < p' < 6p + 3$ for all values of N large enough.

5.2. Alternative constructions with tripartite graphs

We could have also used instead of the partition of K_{q+1} of Theorem 5.1, a covering of the edges of K_{q+1} with K_3 's. Indeed it is known that the edges of K_n can be covered by $\lceil \frac{n}{3} \lceil \frac{n-1}{2} \rceil \rceil$ K_3 's (see ¹¹ chapter VI.11). So we obtain the following construction.

Construction 5.2. Let $C = 3p^2 + p'$, $0 \leq p' < 6p + 3$ and $N = qp + r$, $1 \leq r \leq p$, and let the vertices of K_N be $\cup_{i=1}^q V_i \cup V_{q+1}$, with $|V_i| = p$ and $|V_{q+1}| = r$.

Replace each vertex of K_{q+1} by the set of nodes V_i , $1 \leq i \leq q + 1$ and each edge by the corresponding $K_{p,p}$ or $K_{p,r}$. From a covering of the edges of K_{q+1} with K_3 's we obtain a covering of the edges of K_N with $K_{p,p,p}$ or $K_{p,p,r}$ plus the K_p on the V_i , $1 \leq i \leq q$, and the K_r on V_{q+1} .

Proposition 5.2. *Construction 5.2 is valid and gives a $\frac{\rho_{\max}(C)}{\lfloor \sqrt{\frac{C}{3}} \rfloor} + O(\frac{1}{N})$ -approximation of the number of ADMs.*

Proof. The subgraphs in the covering have at most $3p^2$ edges and so the construction is valid. Each node of K_N belongs to at most $\lceil \frac{q}{2} \rceil + 1$ subgraphs and so the partition of K_N uses $\frac{N(N-1)}{2p} + O(N)$ ADMs. □

In some cases we can also use other partitions based on partitions in tripartite graphs.

Construction 5.3.

Let $C = 3p^2$ and $N = 3^a p$, $a \geq 1$.

From the existence of 3-GDD of type u^3 (see ¹¹ chapter IV.4), that is a partition of the tripartite graph $K_{u,u,u}$ into K_3 , $u \geq 1$, we know that $K_{up,up,up}$ can be partition into u^2 $K_{p,p,p}$. Thus, we partition the edges of K_N as follows

- (1) If $N = 3p$ (i.e. $a = 1$), partition K_{3p} into one $K_{p,p,p}$ and 3 K_p .
- (2) Otherwise
 - (a) Partition the edges of K_N into 3 $K_{3^{a-1}p}$ and one $K_{3^{a-1}p, 3^{a-1}p, 3^{a-1}p}$
 - (b) Partition $K_{3^{a-1}p, 3^{a-1}p, 3^{a-1}p}$ into $(3^{a-1})^2$ $K_{p,p,p}$
 - (c) Repeat the process on each $K_{3^{a-1}p}$

One can check that we have partitioned K_N into $\sum_{i=0}^{a-1} 3^i (3^{a-i-1})^2 = \frac{3^a(3^a-1)}{6} = \frac{N(N-p)}{6p^2}$ $K_{p,p,p}$ and $3^a = \frac{N}{p}$ K_p .

Proposition 5.3. *Construction 5.3 uses $\frac{N(N+p)}{2p}$ ADMs and provide a $\frac{\rho_{\max}(C)}{\sqrt{\frac{C}{3}}} + O(\frac{1}{N})$ -approximation of the total number of ADMs.*

Proof. Construction 5.3 uses $3p \frac{N(N-p)}{6p^2} + p \frac{N}{p} = \frac{N(N+p)}{2p}$ ADMs. Thus it has approximation factor $\gamma(C, N) = \frac{N(N+p)}{p} \frac{\rho_{\max}(C)}{N(N-1)} = \frac{\rho_{\max}(C)}{\sqrt{\frac{C}{3}}} \left(1 + \frac{p+1}{N-1}\right)$. □

Remark that Construction 5.3 gives the same approximation ratio than Construction 5.1, but it can be better for some particular values of C and N . For example, when $C = 12$ and $N = 18$ Construction 5.3 uses 90 ADMs, while Construction 5.1 uses 106 ADMs.

5.3. Construction with multipartite graphs

Finally, we can use partitions with 4 partite (resp. 5 partite) graphs using partitions or coverings of K_{q+1} with K_4 (resp. K_5). For example it is known that the edges of K_{q+1} can be covered with $\left\lceil \frac{q+1}{4} \left\lceil \frac{q}{3} \right\rceil \right\rceil$ K_4 's for $q + 1 > 19$ or $\left\lceil \frac{q+1}{5} \left\lceil \frac{q}{4} \right\rceil \right\rceil$ K_5 's for $q + 1 > 429$ (see ¹¹ chapter VI.11.4). Replacing each vertex i of K_{q+1} by the set V_i and the edges by the corresponding $K_{p,p}$ or $K_{p,r}$, we get respectively a covering of the edges of K_N with 4-partite subgraphs $K_{p,p,p,p}$ or $K_{p,p,p,r}$ (or 5-partite subgraphs). We obtain respectively a total number of $\frac{N(N-1)}{3p} + O(N)$ ADMs, or $\frac{N(N-1)}{4p} + O(N)$ ADMs. We summarize the results in the next proposition.

Proposition 5.4. *Let $C = 6p^2 + p'$, $0 \leq p' < 12p + 6$, and $N = qp + r$, $1 \leq r \leq p$. We have a $\frac{\rho_{\max}(C)}{\left\lceil \sqrt{\frac{3C}{8}} \right\rceil} + O\left(\frac{1}{N}\right)$ approximation.*

Let $C = 10p^2 + p'$, $0 \leq p' < 20p + 10$, and $N = qp + r$, $1 \leq r \leq p$. We have a $\frac{\rho_{\max}(C)}{\left\lceil \sqrt{\frac{2C}{5}} \right\rceil} + O\left(\frac{1}{N}\right)$ approximation.

Using $\rho_{\max}(C) \leq \sqrt{\frac{C}{2}}$, we have a $\sqrt{\frac{4}{3}} + O\left(\frac{1}{N}\right)$ -approximation or a $\sqrt{\frac{5}{4}} + O\left(\frac{1}{N}\right)$ -approximation.

We can also give a construction analogous to Construction 5.1 using the optimal construction for a grooming factor $C = 6$ presented in ⁴. The results are a little better but give the same order of approximation.

We can also generalize Construction 5.3 when $C = 6p^2$ and $N = 4^a p$, using a partition of $K_{up,up,up,up}$ into $u^2 K_{p,p,p,p}$. We will obtain a $\sqrt{\frac{4}{3}} + O\left(\frac{1}{N}\right)$ -approximation. Similarly, when $C = 10p^2$ and $N = 5^a p$ we will obtain a $\sqrt{\frac{5}{4}} + O\left(\frac{1}{N}\right)$ -approximation, and more generally, when $C = \frac{\alpha(\alpha-1)}{2} p^2$ and $N = \alpha^a p$ we will obtain a $\sqrt{\frac{\alpha}{\alpha-1}} + O\left(\frac{1}{N}\right)$ -approximation. Unfortunately, such constructions apply only for a few values of N . Also, the constructions that we have presented in Section 4 and 5.1 or 5.2 are more interesting in practice.

6. Comparison between Constructions

We have presented various constructions, but none of them is always better than the others. According to the values of C and N , one has to choose the most efficient construction. To illustrate that, we have written a program that computes for any C and N the values of all constructions, and we have reported some results in

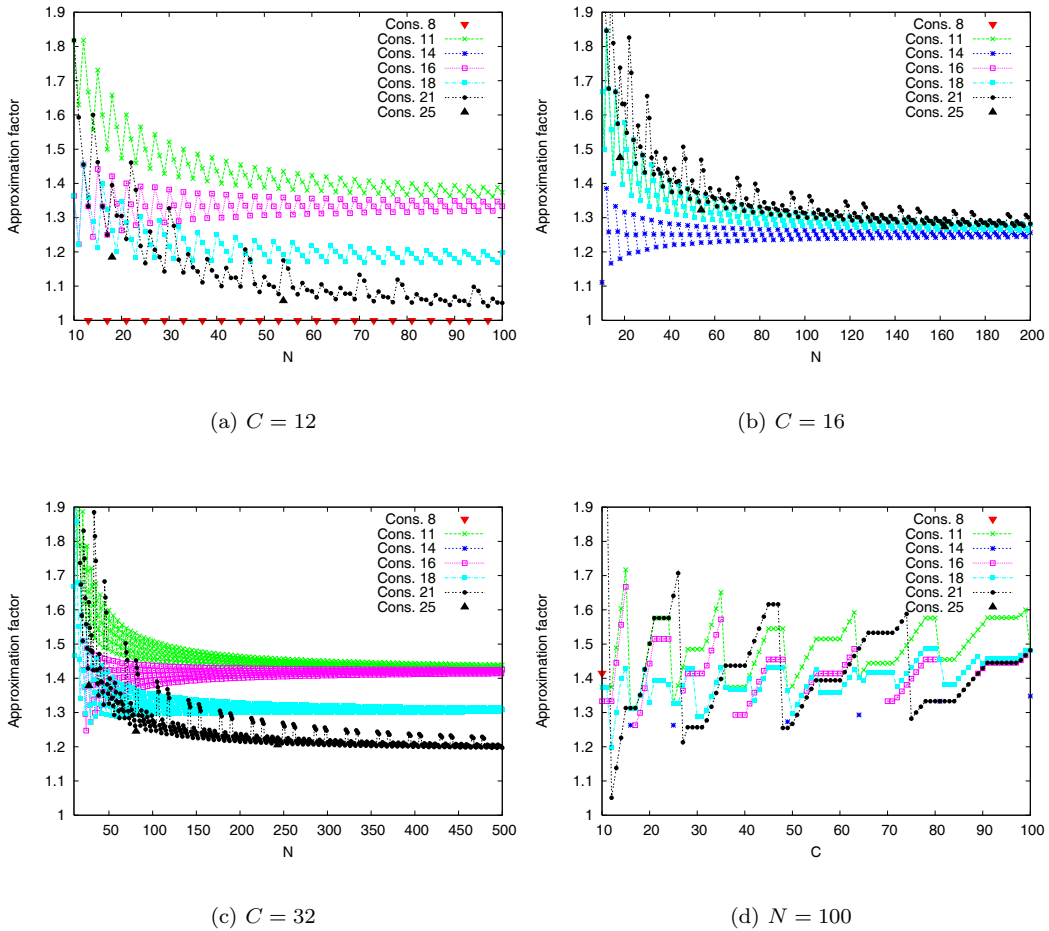


Fig. 1. Approximation ratio for given C (a, b, c) or given N (d).

Figure 1. The results of the program show that for small values of C and N there is no absolute winner. For given C and large value of N , Figure 1 confirms the asymptotic results of Table 1. For example, when $C = 16$, Figure 1(b) shows that all the constructions are equivalent with a slight advantage for Construction 4.2. However, we know from Theorem 3.1 that starting from $N = 802$, Construction 3.1 will always be better. For $C = 12, 32$, Construction 5.1 is the best (except for spare values where Constructions 3.1 and 5.3 apply). For given N , Figure 1(d) shows again that there is no absolute winner and the importance of divisibility condition on C like C near to p^2 (see the isolated point of Construction 4.2 for $C = 64$) or $3p^2$ or $6p^2$.

7. Conclusion

In this paper, using tools of design theory, we have given different constructions with a small approximation factor for all-to-all traffic grooming in unidirectional

ring. These simple constructions might also be used to compute good solutions for very dense set of requests, i.e. instances that are almost all-to-all, for which only $O(\log C)$ -approximation algorithms are known so far. The traffic grooming problem being APX-Hard ¹, this work represents an important step toward the conception of tight approximation algorithms for practical instances.

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