2-Distance Coloring of Sparse Graphs

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Received March 16, 2012; Revised November 19, 2013

Published online 27 February 2014 in Wiley Online Library (wileyonlinelibrary.com).
DOI 10.1002/jgt.21782

Abstract: For graphs of bounded maximum average degree, we consider the problem of 2-distance coloring, that is, the problem of coloring the vertices while ensuring that two vertices that are adjacent or have a common neighbor receive different colors. We prove that graphs with maximum average degree less than $\frac{7}{3}$ and maximum degree Δ at least 4 are 2-distance $(\Delta + 1)$-colorable, which is optimal and improves previous results from Dolama and Sopena, and from Borodin et al. We also prove that graphs with maximum average degree less than $\frac{12}{5}$ (resp. $\frac{5}{2}$, $\frac{18}{7}$) and maximum degree Δ at least 5 (resp. 6, 8) are list 2-distance $(\Delta + 1)$-colorable, which improves previous results from Borodin et al., and from Ivanova. We prove that any graph with maximum average degree m less than $\frac{14}{5}$ and with large enough maximum degree Δ (depending only on m) can be list 2-distance $(\Delta + 1)$-colored. There exist graphs with arbitrarily large maximum degree and maximum average degree less than 3 that cannot be 2-distance $(\Delta + 1)$-colored: the question of what happens between $\frac{14}{5}$ and 3 remains open. We prove also that any graph with maximum average degree $m < 4$ can be list 2-distance $(\Delta + C)$-colored (C depending only on m). It is optimal as there exist graphs with arbitrarily large maximum degree

Contract grant sponsor: ANR Grant EGOS (2012-2015); Contract grant number: 12 JS02 002 01.

and maximum average degree less than 4 that cannot be 2-distance colored with less than $\frac{3\Delta}{2}$ colors. Most of the above results can be transposed to injective list coloring with one color less. J. Graph Theory 77: 190–218, 2014

Keywords: *2-distance coloring; maximum average degree; square coloring*

1. INTRODUCTION

A 2-*distance k-coloring* of a graph G is a coloring of the vertices of G with k colors such that two vertices that are adjacent or have a common neighbor receive distinct colors. We define $\chi^2(G)$ as the smallest k such that G admits a 2-distance k-coloring. This is equivalent to a proper vertex-coloring of the square of G, which is defined as a graph with the same set of vertices as G, where two vertices are adjacent if and only if they are adjacent or have a common neighbor in G. For example, the cycle of length 5 cannot be 2-distance colored with less than 5 colors as any two vertices are either adjacent or have a common neighbor: indeed, its square is the clique of size 5.

The study of $\chi^2(G)$ on planar graphs was initiated by Wegner in 1977 [19], and has been actively studied because of his conjecture. The *maximum degree* of a graph G is denoted $\Delta(G)$.

Conjecture 1 (Wegner [19]). *If G is a planar graph, then:*

- $\chi^2(G) \leq 7$ *if* $\Delta(G) = 3$
- $\chi^2(G) \leq \Delta(G) + 5$ *if* $4 \leq \Delta(G) \leq 7$
- $\chi^2(G) \leq \lfloor \frac{3\Delta(G)}{2} \rfloor + 1$ *if* $\Delta(G) \geq 8$

This conjecture remains open.

Note that any graph G satisfies $\chi^2(G) \geq \Delta(G) + 1$. Indeed, if we consider a vertex of maximal degree and its neighbors, they form a set of $\Delta(G) + 1$ vertices, any two of which are adjacent or have a common neighbor. Hence, at least $\Delta(G) + 1$ colors are needed for a 2-distance coloring of G. It is therefore natural to ask when this lower bound is reached. For that purpose, we can study, as suggested by Wang and Lih [18], what conditions on the sparseness of the graph can be sufficient to ensure the equality holds. The sparseness of a planar graph can, for example, be measured by its girth. The *girth* of a graph G, denoted $g(G)$, is the length of a shortest cycle.

Conjecture 2 (Wang and Lih [18]). *For any integer* $k \geq 5$, *there exists an integer* $M(k)$ *such that for every planar graph G satisfying* $g(G) \geq k$ *and* $\Delta(G) \geq M(k)$, $\chi^2(G) = \Delta(G) + 1$.

Conjecture 2 was proved in [5, 8, 12, 13] to be true for $k \geq 7$ and false for $k \in \{5, 6\}$. More precisely, the following is known.

Theorem 1 (Borodin et al. [5, 6, 8]). *There exist planar graphs G with* $g(G) = 6$ *such that* $\chi^2(G) > \Delta(G) + 1$ *for arbitrarily large* $\Delta(G)$ *[5].*

For any planar graph G, $\chi^2(G) = \Delta(G) + 1$ *in each of the following cases:*

(1) *[6]* $\Delta(G) \geq 3$ *and* $g(G) \geq 22$
(2) *[8]* $\Delta(G) \geq 4$ *and* $g(G) \geq 15$

(3) *[8]* $\Delta(G) \geq 5$ *and* $g(G) \geq 13$
(4) *[8]* $\Delta(G) \geq 6$ *and* $g(G) \geq 12$
(5) *[8]* $\Delta(G) \geq 7$ *and* $g(G) \geq 11$
(6) *[8]* $\Delta(G) \geq 9$ *and* $g(G) \geq 10$
(7) *[5]* $\Delta(G) \geq 15$ *and* $g(G) \geq 8$
(8) *[5]* $\Delta(G) \geq 30$ *and* $g(G) \geq 7$

An extension of 2-distance k-coloring is the 2-*distance* k-*list-coloring*, where instead of having the same set of k colors for the whole graph, every vertex is assigned some set of k colors and has to be colored from it. Given a graph G, we call $\chi^2_\ell(G)$ the minimal integer k such that a 2-distance k-list-coloring exists. Obviously, 2-distance coloring is a subcase of 2-distance list-coloring (where the same color list is assigned to every vertex), so for any graph G, $\chi^2_\ell(G) \geq \chi^2(G)$. Kostochka and Woodall [15] even conjectured that it is actually an equality.

Conjecture 3 (Kostochka and Woodall [15]). *Any graph G satisfies* $\chi^2_\ell(G) = \chi^2(G)$.

However, this strong conjecture was recently disproved [16].

Borodin, Ivanova, and Neustroeva [9] strengthened Theorem 1 by extending the cases (2)–(8) to list-coloring. Ivanova [14] improved the lower-bounds into the following theorem.

Theorem 2 (Ivanova [14]). *If G is a planar graph, then* $\chi^2_\ell(G) = \Delta(G) + 1$ *in each of the following cases:*

(1) $\Delta(G) \geq 5$ *and* $g(G) \geq 12$
(2) $\Delta(G) \geq 6$ *and* $g(G) \geq 10$
(3) $\Delta(G) \geq 10$ *and* $g(G) \geq 8$
(4) $\Delta(G) \geq 16$ *and* $g(G) \geq 7$

Another way to measure the sparseness of a graph is through its maximum average degree as defined below. The average degree of a graph G, denoted $\mathrm{ad}(G)$, is $\frac{\sum_{v\in V} d(v)}{|V|} = \frac{2|E|}{|V|}$. The *maximum average degree* of a graph G, denoted $\mathrm{mad}(G)$, is the maximum of $\mathrm{ad}(H)$ on every subgraph H of G. See [10] for a first use of this measure in the context of sparse graphs coloring.

Intuitively, this measures the sparseness of a graph because it states how great the concentration of edges in a same area can be. For example, stating that $\mathrm{mad}(G)$ has to be smaller than 2 means that G cannot be anything but a forest. Euler's formula links girth and maximum average degree in the case of planar graphs.

Lemma 1 (Folklore). *For every planar graph G,* $\mathrm{mad}(G) < \frac{2g(G)}{g(G)-2}$.

Dolama and Sopena [11] used this measure of sparseness and proved the following result:

Theorem 3 (Dolama and Sopena [11]). *Every graph with* $\Delta(G) \geq 4$ *and* $\mathrm{mad}(G) < \frac{16}{7}$ *satisfies* $\chi^2(G) = \Delta(G) + 1$.

A consequence of Lemma 1 is that we can transpose any theorem holding for an upper-bound on $\mathrm{mad}(G)$ into a theorem holding for planar graphs with lower-bounded girth, as presented in Table I.

TABLE I. Mad/girth correspondence when G is a planar graph

If G is planar and $g(G) \geq$	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Then mad$(G) <$	6	4	$\frac{10}{3}$	3	$\frac{14}{5}$	$\frac{8}{3}$	$\frac{18}{7}$	$\frac{5}{2}$	$\frac{22}{9}$	$\frac{12}{5}$	$\frac{26}{11}$	$\frac{7}{3}$	$\frac{30}{13}$	$\frac{16}{7}$

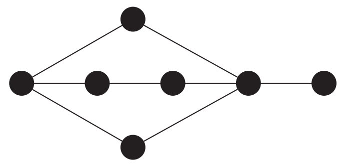

FIGURE 1. A graph G with $\mathrm{mad}(G) = \frac{7}{3}$, $\Delta(G) = 4$ and $\chi^2(G) = 6$.

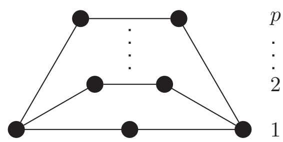

FIGURE 2. A graph G_p with $\Delta(G_p) = p$, $\mathrm{mad}(G_p) = 3 - \frac{5}{2p+1}$ and $\chi^2(G_p) = \Delta(G_p) + 2$.

In particular, Theorem 3 implies that for every planar graph G with $g(G) \geq 16$ and $\Delta(G) \geq 4$, $\chi^2(G) = \Delta(G) + 1$. However, this lower bound on the girth is not as good as the one stated in Theorem 1.(2) by Borodin et al. (16 instead of 15). We present here the following improvement of Theorem 3.

Theorem 4. *Every graph G with $\Delta(G) \geq 4$ and* $\mathrm{mad}(G) < \frac{7}{3}$ *satisfies* $\chi^2(G) = \Delta(G) + 1$.

Theorem 4 and a proof sketch of it have been presented at Eurocomb 2011 [3]. It happens to be optimal, as Montassier [17] pointed out that there exists a graph G with $\mathrm{mad}(G) = \frac{7}{3}$, $\Delta(G) = 4$ and $\chi^2(G) = 6 > \Delta(G) + 1$ (see Fig. 1).

We can transpose it to planar graphs with a lower bound on the girth using Lemma 1.

Corollary 1. *Every planar graph G with $\Delta(G) \geq 4$ and* g(G) ≥ 14 *satisfies* $\chi^2(G) = \Delta(G) + 1$.

It is then an improvement of Theorem 1.(2) (14 instead of 15). However, it is not comparable to the more general result in [9] since we are not considering list-coloring, and is probably not optimal in terms of girth.

The question raised by Conjecture 2 and now solved could be reworded as follows: what is the minimum k such that any graph G with $g(G) \geq k$ and large enough $\Delta(G)$ (depending only on $g(G)$) satisfies $\chi^2_\ell = \Delta(G) + 1$? It is then natural to transpose the question to the maximum average degree, as it is a more refined measure of sparseness. More precisely, what is the supremum M such that any graph G with $\mathrm{mad}(G) < M$ and large enough $\Delta(G)$ (depending only on $\mathrm{mad}(G)$) satisfies $\chi^2_\ell = \Delta(G) + 1$?

We know that $M \leq 3$ due to the family of graphs that appears in [5] (see Fig. 2), which are of increasing maximum degree, of maximum average degree < 3, and are not 2-distance $(\Delta + 1)$-colorable.

We prove here that $\frac{14}{5} \leq M$.

Theorem 5. *There exists a function f such that for a small enough $\epsilon > 0$, every graph with* $\mathrm{mad}(G) < \frac{14}{5} - \epsilon$ *and* $Delta(G) \geq f(\epsilon)$ *admits a list 2-distance* $(\Delta(G) + 1)$*-coloring.*

This answers partially to the transposition of Conjecture 2 to graphs with an upper-bounded maximum average degree. As the maximum average degree is not discrete, it could be expected that a sharper value would be obtained. It is however not the case here, as the theorem does not even match what was already known on planar graphs. Indeed, it only implies that any planar graph of girth at least eight and of large enough maximum degree is list 2-distance $(\Delta + 1)$-colorable, while this holds for planar graphs of girth at least seven.

For specific values of $\Delta(G)$, the following bounds can be obtained.

Theorem 6. *For any graph G, $\chi^2_\ell(G) = \Delta(G) + 1$ in each of the following cases:*

(1) $\Delta(G) \geq 5$ *and* $\mathrm{mad}(G) < \frac{12}{5}$
(2) $\Delta(G) \geq 6$ *and* $\mathrm{mad}(G) < \frac{5}{2}$
(3) $\Delta(G) \geq 8$ *and* $\mathrm{mad}(G) < \frac{18}{7}$

This theorem, once transposed to planar graphs with a lower-bound on the girth, yields the following.

Corollary 2. *If G is a planar graph, then $\chi^2_\ell(G) = \Delta(G) + 1$ in each of the following cases:*

(1) $\Delta(G) \geq 5$ *and* $g(G) \geq 12$
(2) $\Delta(G) \geq 6$ *and* $g(G) \geq 10$
(3) $\Delta(G) \geq 8$ *and* $g(G) \geq 9$

Corollary 2 matches Theorem 2 for $g(G) \geq 12, 10$ and improves it for $g(G) \geq 9$. This seems to support the idea that it is relevant to try to relax the planarity hypothesis when studying the 2-distance colorability of sparse graphs. However, it might be that a difference appears when these theorems are improved to their optimal values, which are yet to be determined. Also, we can prove similarly that $\chi^2_\ell(G) = \Delta(G) + 1$ when $\Delta(G) \geq 14$ and $\mathrm{mad}(G) < \frac{8}{3}$ (this corresponds to a girth lower-bounded by eight for planar graphs), but contrary to the other cases, it is not as good as its planar equivalent in Theorem 2 yet.

More generally, is it possible to get similar results when allowing an additional constant number of colors, as was done by Wang and Lih in [18] for planar graphs? More precisely, what is the supremum N such that any graph G with $\mathrm{mad}(G) < N$ satisfies $\chi^2_\ell(G) \leq \Delta(G) + h(\mathrm{mad}(G))$?

We know that $N \leq 4$ due to the family of graphs presented in Figure 3 (called Shannon's triangle), which are of increasing maximum degree, of maximum average degree < 4 and that need $\frac{3\Delta}{2}$ colors to be list 2-distance colored.

We prove here that N is actually equal to 4.

Theorem 7. *There exists a function h such that every graph G with* $\mathrm{mad}(G) < 4 - \epsilon$ *satisfies* $\chi^2_\ell(G) \leq \Delta(G) + h(\epsilon)$.

In Section 2, we introduce the terminology and notation. In Sections 3, 4, 5, 6, we prove Theorems 4, 5, 6, and 7, respectively, and we justify in Section 7 how they can be transposed to injective colorings. Their proofs all have the same outline, as follows.

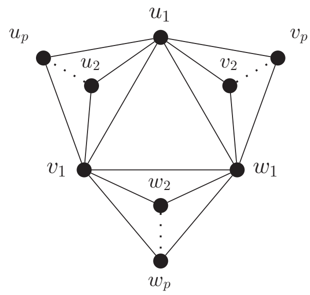

FIGURE 3. A graph G_p with $\Delta(G_p) = 2p$, $\text{mad}(G_p) = 4 - \frac{2}{p}$ and $\chi^2(G) = 3p$.

If the theorem is of the form "Every graph G with $\Delta(G) \geq d$ and $\text{mad}(G) < m$ satisfies $\chi^2(G) \leq f(\Delta(G))$', we prove a stronger version of it by relaxing the constraint on the maximum degree: "For any $k \geq d$, every graph G with $\Delta(G) \leq k$ and $\text{mad}(G) < m$ satisfies $\chi^2(G) \leq f(k)$" so that the property is closed under vertex- or edge-deletion. First, we prove that there are some configurations a minimal counter-example cannot contain, where a graph is a minimal counter-example when it does not satisfy the property but any of its subgraphs does. To that purpose, we assume it contains one of the said configurations, remove some vertices or edges, use the minimality to color the resulting graph, and prove we can extend the coloring to the whole graph, a contradiction. Second, we prove that a graph that does not contain any of those configurations cannot satisfy the hypothesis on the maximum average degree. To that purpose, we assign to each vertex its degree as a weight, introduce discharging rules as to how the weight can be redistributed along the graph (with conservation of the total weight of the graph), and prove that after application of the discharging rules, knowing which configurations are forbidden, every vertex has a final weight of at least m. Since no weight was created nor deleted, this implies that the average degree of the graph is at least m, hence cannot satisfy the hypothesis on the maximum average degree. This completes the proof.

This method of proof is called a discharging method, and was introduced in the beginning of the 20th century. It is notably known for being used to prove the Four Color Theorem ([1] and [2]). When the discharging rules are local (i.e., the weight cannot travel arbitrarily far), as is most commonly, we say the discharging method is *local*. Borodin, Ivanova, and Kostochka introduced in [7] the notion of *global* discharging, which is when there is no bound on the size of the discharging rules (i.e., the weight can travel arbitrarily far along the graph). When it is mixed, that is, the discharging rules are of bounded size but take into account structures of unbounded size in the graph, we say the discharging method is *semiglobal* (see [4] for a first occurrence of such a proof).

Note that some of the configurations presented here are similar to configurations studied in other articles about 2-distance coloring (see e.g., [9, 14]).

2. TERMINOLOGY AND NOTATIONS

In the figures, we draw in black a vertex that has no other neighbor than the ones already represented, in white a vertex that might have other neighbors than the ones represented. When there is a label inside a white vertex, it is an indication on the number of neighbors it has. The label "i" means "exactly i neighbors", the label "i^+" (resp. "i^-") means that

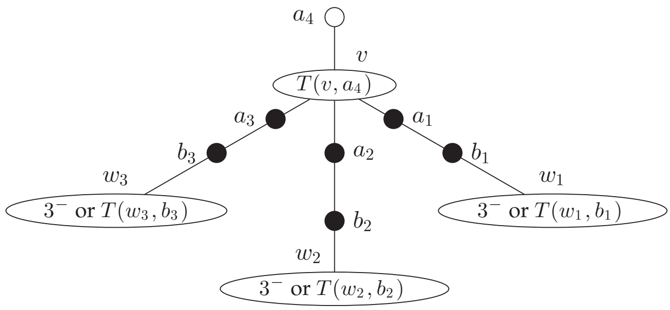

FIGURE 4. A $T(v, a_4)$.

it has at least (resp. at most) i neighbors. Note that the white vertices may coincide with other vertices.

A *constraint* of a vertex u is an already colored vertex that is adjacent to or has a common neighbor with u. Two constraints with the same color count as one.

While proving that a given configuration is forbidden in a minimal counter-example, we may *recolor* vertices. This is when a vertex is already assigned a color in the current coloring of the graph (which corresponds to a valid 2-distance coloring of a given subgraph), but we assign it another available color in order to have a coloring compatible with a valid 2-distance coloring of the whole graph. This is useful when, for example, two vertices that are adjacent or have a common neighbor in the graph are not and do not in the considered subgraph.

Given a vertex u, the *neighborhood* $N(u)$ is the set of vertices that are adjacent to u. A *p-link* ($p \geq 1$) $x - a_1 - ... - a_p - y$ between x and y is a path such that $d(a_1) = ... = d(a_p) = 2$. When a p-link exists between two vertices x and y, we say they are *p-linked*. We define a *branch* of v as a p-link from v to another vertex, for some $p \geq 0$. A graph is *minimal* for a property if it satisfies this property but none of its subgraphs does.

3. PROOF OF THEOREM 4

We prove that every graph G with $\Delta(G) \geq 4$ and $mad(G) < \frac{7}{3}$ satisfies $\chi^2(G) = \Delta(G) + 1$. In the figures of this proof, a label $T(v, a_4)$ inside a vertex means that $T(v, a_4)$ exists, as defined below.

A configuration $T(v, a_4)$ (see Fig. 4) is inductively defined as a vertex v of degree 4 with neighbors a_1, a_2, a_3, a_4, where for $i \in \{1, 2, 3\}$, vertex v is 2-linked by a path v-a_i-b_i-w_i either to a vertex w_i of degree at most 3 or to a configuration $T(w_i, b_i)$. By abusing the notation, $T(v, a_4)$ is also the set that contains v, a_i and b_i, and that includes $T(w_i, b_i)$ if it exists (for $i \in \{1, 2, 3\}$).

Let $k \geq 4$.

A. Forbidden Configurations

We define configurations $(\boldsymbol{C_1})$ to $(\boldsymbol{C_5})$ (see Fig. 5).

- $(\boldsymbol{C_1})$ is a vertex of degree 0 or 1.
- $(\boldsymbol{C_2})$ is a vertex 3-linked to a vertex of degree $\leq k - 1$.

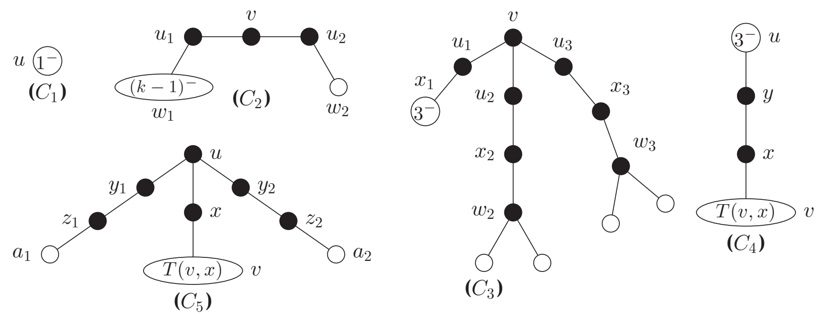

FIGURE 5. Forbidden configurations for Theorem 4.

- ***(C₃)*** is a vertex of degree 3 that is 2-linked to two vertices of degree 3, and 1-linked to a vertex of degree ≤ 3.
- ***(C₄)*** is a vertex u of degree ≤ 3 that is 2-linked by a path u-y-x-v to a vertex v such that $T(v, x)$ exists.
- ***(C₅)*** is a vertex u of degree 3 that is 2-linked to two vertices, and 1-linked by a path u-x-v to a vertex v such that $T(v, x)$ exists.

Lemma 2. *If G is a minimal graph such that $\Delta(G) \leq k$ and G admits no 2-distance $(k+1)$-coloring, G does not contain any of Configurations **(C_1)** to **(C_5)**.*

Proof. We prove Lemma 2 by assuming G contains one of the configurations **(C_1)** to **(C_5)**, using the minimality of G to color one of its subgraphs, and extending the coloring to the whole graph, hence obtaining a contradiction.

We follow the notations introduced on Figure 5. ■

Claim 1. *G cannot contain **(C_1)**.*

Proof. Using the minimality of G, we color $G \setminus \{u\}$. Since $d(u) \leq 1$, vertex u has at most $\Delta(G)$ constraints. There are at least $\Delta(G) + 1$ colors, so the coloring of $G \setminus \{u\}$ can be extended to G. ■

Claim 2. *G cannot contain **(C_2)**.*

Proof. Using the minimality of G, we color $G \setminus \{v, u_1\}$. Vertex u_1 has at most $|\{w_1, u_2\}| + d(w_1) - 1 = d(w_1) + 1 \leq \Delta(G)$ constraints. Hence, we can color u_1. Then v has at most four constraints, so we can extend the coloring of $G \setminus \{v, u_1\}$ to G. ■

Claim 3. *G cannot contain **(C_3)**.*

Proof. Using the minimality of G, we color $G \setminus \{v, u_1, u_2, u_3, x_2, x_3\}$. We color x_2 with a different color of that of w_3 (this is possible since x_2 has at most three constraints), u_3 with the same color as x_2 (the only constraint of u_3 is w_3, and x_2 and w_3 do not have the same color), u_1 (at most four constraints), x_3 (at most four constraints), v (at most four constraints), u_2 (at most four constraints). Thus, we can extend the coloring of $G \setminus \{v, u_1, u_2, u_3, x_2, x_3\}$ to G. ■

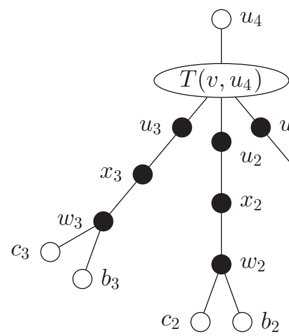

FIGURE 6. v is the only vertex of degree 4 in $T(v, u_4)$.

Claim 4. *If H is a graph that contains some $T(v, u_4)$ and does not contain Configuration **(C_2)**, any partial 2-distance 5-coloring α of H that is not defined on $T(v, u_4)$ nor on $N(u_4)$ (but that may be defined on u_4) can be extended to be also defined on $T(v, u_4)$.*

In the description of a coloring procedure, we note "$x \leftarrow c$" as a shortcut for "We assign color c to x," "$a_4 \overset{T:z}{\longleftarrow} c$" as a shortcut for "We color a_4 with c and apply Claim 4 to color $T(z, a_4)$," and we note also "x" as a shortcut for "We color arbitrarily x with any of the available colors."

Proof. We prove this claim by induction on the size of $T(v, u_4)$.

We name $v - u_i - x_i - w_i$ the vertices along a branch (u_i and x_i are of degree 2), and b_i and c_i the two other neighbors of w_i (if $d(w_i) = 3$) (see Fig. 6). All along this proof, a denotes $\alpha(u_4)$. According to the hypothesis, α is not defined on v, u_i, x_i, and $T(w_i, x_i)$ if it exists, for $i \in \{1, 2, 3\}$.

Assume for the moment that the u_is and x_is are pairwise distinct, and distinct from any of the w_is.

We are in one of the following four cases depending on the structure of $T(v, u_4)$.

(1) Vertex v is the only vertex of degree 4 in $T(v, u_4)$.
We deal with the worst-case situation, that is, the three branches from v end with a vertex of degree 3 (if we can extend the coloring in that case, then we would be able to do the same if one or more were of degree only 2), and α is defined on w_i, b_i, and c_i.
Since we have only five colors, we are always in one of the following five cases (up to symmetry):

- $\alpha(w_1) = \alpha(w_2) = \alpha(w_3)$, $\alpha(b_1) = \alpha(b_2)$.
  - $a = \alpha(w_1)$. Then apply: $v \leftarrow \alpha(b_1), x_3, u_3, u_2, u_1, x_2, x_1$.
  - $a \neq \alpha(w_1)$. Then apply: $v \leftarrow \alpha(w_1), u_1, u_2, u_3, x_1, x_2, x_3$.

- $\alpha(w_1) = \alpha(w_2) \neq \alpha(w_3)$, $\alpha(b_1) = \alpha(w_3)$.
  - $a = \alpha(w_1)$. Then apply: $v \leftarrow \alpha(w_3), x_2, u_2, u_1, x_1, u_3, x_3$.
  - $a \neq \alpha(w_1)$. Then apply: $v \leftarrow \alpha(w_1), x_3, u_3, u_2, u_1, x_2, x_1$.

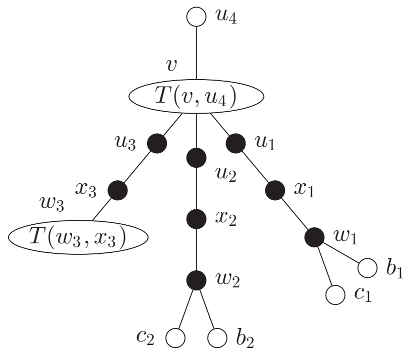

FIGURE 7. v is 2-linked to exactly one vertex of degree 4 in $T(v, u_4)$.

- $\alpha(w_1) = \alpha(w_2) \neq \alpha(w_3), \alpha(b_1) = \alpha(b_2)$ (and $\alpha(w_3) \notin \{\alpha(b_1), \alpha(c_1)\}$).
  - $a = \alpha(w_1)$. Then apply: $v \leftarrow \alpha(w_3), u_2 \leftarrow \alpha(b_2), u_1 \leftarrow \alpha(c_1), x_1, x_2, u_3, x_3$.
  - $a \neq \alpha(w_1)$. Then apply: $v \leftarrow \alpha(w_1), x_3, u_3, u_2, u_1, x_2, x_1$.

- $\alpha(w_1), \alpha(w_2)$, and $\alpha(w_3)$ are pairwise different, $\alpha(w_1) = \alpha(b_2)$.
  - $a = \alpha(w_1)$. Then apply: $v \leftarrow \alpha(w_3), x_2, u_2, x_1, u_1, u_3, x_3$.
  - $a \neq \alpha(w_1)$. Then apply: $v \leftarrow \alpha(w_1), x_3, u_3, u_2, x_2, u_1, x_1$.

- $\alpha(w_1)$, $\alpha(w_2)$, and $\alpha(w_3)$ are pairwise different, $\alpha(b_1) = \alpha(b_2) = \alpha(b_3), \alpha(c_1) = \alpha(c_2) = \alpha(c_3)$.
  - $a = \alpha(w_1)$ (up to permutation). Then apply: $v \leftarrow \alpha(b_1), u_2 \leftarrow \alpha(w_3), u_3 \leftarrow \alpha(w_2), x_2, x_3, u_1, x_1$.
  - $a = \alpha(b_1)$. Then apply: $v \leftarrow \alpha(c_1), u_1 \leftarrow \alpha(w_2), u_2 \leftarrow \alpha(w_3), u_3 \leftarrow \alpha(w_1), x_1, x_2, x_3$.
  - $a = \alpha(c_1)$. Then apply: $v \leftarrow \alpha(b_1), u_1 \leftarrow \alpha(w_2), u_2 \leftarrow \alpha(w_3), u_3 \leftarrow \alpha(w_1), x_1, x_2, x_3$.

(2) Vertex v is 2-linked to exactly one vertex w_3 of degree 4 in $T(v, u_4)$ (see Fig. 7). Again, we deal with the worst-case situation. So, in this drawing, α is defined only on the w_i, b_i, and c_i, for $i = 1$ or 2. Again, because there are only five colors, we are in one of the following three cases;

- $\alpha(w_1) = \alpha(w_2)$.
  - $a = \alpha(w_1)$. Then apply: $x_3 \xleftarrow{T:w_3} a, v \leftarrow \alpha(b_1), u_3, x_2, u_2, u_1, x_1$.
  - $a \neq \alpha(w_1)$. Then apply: $x_3 \xleftarrow{T:w_3} a, v \leftarrow \alpha(w_1), u_3, u_1, u_2, x_1, x_2$.

- $\alpha(w_1) = \alpha(b_2)$.
  - $a = \alpha(w_1)$. Then apply: $x_3 \xleftarrow{T:w_3} a, v \leftarrow \alpha(w_2), u_3, x_1, u_1, u_2, x_2$.
  - $a \neq \alpha(w_1)$. Then apply: $x_3 \xleftarrow{T:w_3} a, v \leftarrow \alpha(w_1), u_3, x_2, u_2, u_1, x_1$.

- $\alpha(b_1) = \alpha(b_2)$ (and $\alpha(w_1) \neq \alpha(w_2)$).
  - $a = \alpha(b_1)$. Then apply: $x_3 \xleftarrow{T:w_3} a, v \leftarrow \alpha(w_1), x_2, u_2, u_3, u_1, x_1$.

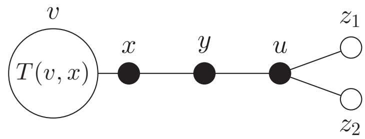

FIGURE 8. Worst case for claim (5).

- $a \neq \alpha(b_1)$. Then apply: $x_3 \overset{T:w_3}{\longleftarrow} c \notin \{\alpha(b_1), \alpha(w_1), \alpha(w_2)\}$, $v \leftarrow \alpha(b_1)$. Assume $\alpha(w_2)$ differs from the coloring of w_3 (true up to permutation as $\alpha(w_1) \neq \alpha(w_2)$). Then apply: $u_3 \leftarrow \alpha(w_2), u_1, u_2, x_1, x_2$.

(3) Vertex v is 2-linked in $T(v, u_4)$ to exactly two vertices w_2 and w_3 of degree 4. If $a = \alpha(w_1)$, then apply: $v \leftarrow \alpha(b_1)$, if $a \neq \alpha(w_1)$, then apply: $v \leftarrow \alpha(w_1)$. In both cases, we then color $x_3 \overset{T:w_3}{\longleftarrow} a$, $x_2 \overset{T:w_2}{\longleftarrow} a, u_2, u_3, u_1, x_1$.
(4) Vertex v is 2-linked in $T(v, u_4)$ to three vertices w_1, w_2 and w_3 of degree 4. Apply: $x_1 \overset{T:w_1}{\longleftarrow} a$, $x_2 \overset{T:w_2}{\longleftarrow} a$, $x_3 \overset{T:w_3}{\longleftarrow} a$, $v \leftarrow$ (the color of w_1), u_2, u_3, u_1.

Let us now justify our assumptions. Since **(C_2)** is forbidden, there is no (i, j) such that $w_i = x_j$ and $w_j = x_i$, nor such that $x_i = x_j$. However, there can be a couple (i, j) such that $w_i = u_j$, $w_j = u_i$, $x_i = x_j$. In that case, we pretend they are distinct (we assign arbitrary colors to the virtual w_i and w_j and their alleged other neighbors b_i, c_i, b_j, c_j), apply the procedure described above, get a coloring α of the resulting graph, then derive from it a coloring of the initial graph by matching α on every common vertex except x_i and then coloring x_i in one of the available colors (indeed $\alpha(u_i) \neq \alpha(u_j)$ as v is a common neighbor, and x_i has exactly three vertices at distance 2 or less). Therefore we can assume without loss of generality that no vertex u_i, x_i superposes with another. The case $w_i = w_j$ is not a problem in the above procedure (note that $w_i = w_j$ can only happen if $d(w_i) \leq 3$).

Also, if α is not defined on all w_i, b_i, and c_i, we can apply the procedure with arbitrary colors for them (it is always possible to assign colors in such a way that w_i, b_i, and c_i receive different colors). ■

Note that this claim implies that the same holds when the number of colors is $k + 1$ instead of just 5: we pick five colors among the range of $k + 1$, this induces a partial 2-distance 5-coloring on the graph. Then we apply the claim and the extended coloring is compatible with the initial coloring.

Claim 5. *G cannot contain* ***(C_4)****.*

Proof. We deal with the worst-case situation, that is, $d(u) = 3$: see Figure 8 for notation.

Using the minimality of G, there exists a coloring α of $G \setminus (T(v, x) \cup \{x, y\})$. We use Claim 4 to extend it to G through: $x \overset{T:v}{\longleftarrow} \alpha(z_1), y$. ■

Claim 6. *G cannot contain* ***(C_5)****.*

Proof. Using the minimality of G, there exists a coloring α of $G \setminus (T(v, x) \cup \{x, u, y_1, y_2\})$. Then we extend the coloring to G as follows:

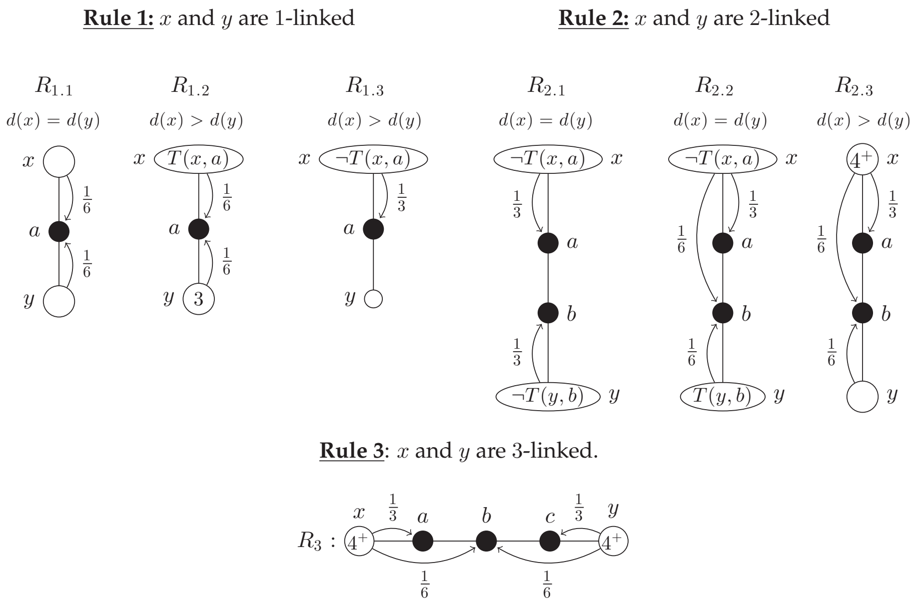

FIGURE 9. Discharging rules R_1, R_2, R_3 for Theorem 4.

- $\alpha(z_1) = \alpha(a_2)$. Then apply: $x \overset{T:v}{\longleftarrow} \alpha(z_1), u, y_1, y_2$.
- $\alpha(z_1) \neq \alpha(a_2)$. Then apply: $x \overset{T:v}{\longleftarrow} \alpha(z_1), u \leftarrow \alpha(a_2), y_1, y_2$. ■

This concludes the proof of Lemma 2.

The following lemma will ensure that the discharging rules we introduce later are well-defined.

Lemma 3. *In a graph H where **(C_4)** is forbidden, and x and y are two vertices of degree 4 such that a path x-a-b-y (with a and b of degree 2) exists, $T(x, a)$ and $T(y, b)$ cannot both exist.*

Proof. Assume by contradiction that there is a path $x - a - b - y$ such that both $T(x, a)$ and $T(y, b)$ exist. We consider without loss of generality that x and y are chosen such that $|T(y, b)|$ is minimum (i.e., there exists no $T(y_2, b_2) \subsetneq T(y, b)$ such that $y_2 - b_2 - a_2 - x_2$ is a 2-link and $T(x_2, a_2)$ exists). Let b' be a neighbor of y, $b' \neq b$. Then $T(y, b')$ exists (by definition, using the existence of $T(y, b)$ and of $T(x, a)$). If there is no vertex w of degree 4 that is 2-linked (with $w - c - b' - y$) to y, then the existence of $T(y, b')$ implies that the graph contains **(C_4)**, a contradiction. If such a w exists, $T(w, c)$ exists, $T(w, c) \subsetneq T(y, b)$ (by definition, using the existence of $T(y, b)$). Consequently, $y - b' - c - w$ is a path such that $T(y, b')$ and $T(w, c)$ exist, with $T(w, c) \subsetneq T(y, b)$, a contradiction. ■

B. Discharging Rules

Let R_1, R_2, R_3 be three discharging rules (see Fig. 9). We use them in the proof of Lemma 4, where the initial weight of a vertex equals its degree, and its final weight is shown to be at least $\frac{7}{3}$. For any two vertices x and y of degree at least 3, with $d(x) \geq d(y)$,

- Rule R_1 is when x and y are 1-linked by a path $x - a - y$.

  - ($R_{1.1}$) If $d(x) = d(y)$, then both x and y give $\frac{1}{6}$ to a.
  - ($R_{1.2}$) If $d(x) > d(y)$ and $T(x, a)$ exists, then both x and y give $\frac{1}{6}$ to a.
  - ($R_{1.3}$) If $d(x) > d(y)$ and $T(x, a)$ does not exist, then x gives $\frac{1}{3}$ to a.

- Rule R_2 is when x and y are 2-linked by a path $x - a - b - y$.

  - ($R_{2.1}$) If $d(x) = d(y)$ and neither $T(x, a)$ nor $T(y, b)$ exist, then x (resp. y) gives $\frac{1}{3}$ to a (resp. b).
  - ($R_{2.2}$) If $d(x) = d(y)$ and $T(y, b)$ exists, then x gives $\frac{1}{3}$ to a and both x and y give $\frac{1}{6}$ to b.
  - ($R_{2.3}$) If $d(x) > d(y)$, then x gives $\frac{1}{3}$ to a and both x and y give $\frac{1}{6}$ to b.

- Rule R_3 is when x and y, both of degree at least 4, are 3-linked by a path $x - a - b - c - y$. Then x gives $\frac{1}{3}$ to a and $\frac{1}{6}$ to b, and symmetrically for y.

We use these discharging rules to prove the following lemma:

Lemma 4. *A graph H that does not contain Configurations **(C_1)** to **(C_5)** satisfies* $\text{mad}(G) \geq \frac{7}{3}$.

Proof. We attribute to each vertex a weight equal to its degree, and apply discharging rules R_1, R_2, and R_3. We show that all the vertices have a weight of at least $\frac{7}{3}$ in the end.
There are no vertices of degree 0 or 1 in the graph, due to the fact that **(C_1)** is forbidden, so we study only the vertices of degree 2 or more. ∎

Claim 7. *All the vertices of degree* 2 *have a weight of at least* $\frac{7}{3}$ *after application of the rules.*

Proof. Consider any maximal p-link $x - s_1 - \cdots - s_p - y$, with $d(x), d(y) \geq 3$. There is no Configuration **(C_2)**, so $p \leq 3$, and every vertex of degree 2 belongs to such a p-link. According to the discharging rules, a vertex of degree 2 never gives away any weight. We prove that it receives at least $\frac{1}{3}$. There are three cases depending on the value of p, each corresponding to Rule R_p:

- If $p = 1$, then Rule R_1 applies to $x - s_1 - y$, and s_1 receives $\frac{1}{3}$.
- If $p = 2$, then Rule R_2 applies to $x - s_1 - s_2 - y$, and both s_1 and s_2 receive $\frac{1}{3}$. Indeed, Lemma 3 ensures that all the cases for x and y are dealt with in Rule R_2.
- If $p = 3$, then since G does not contain Configuration **(C_2)**, $d(x), d(y) \geq 4$. Then, Rule R_3 applies to $x - s_1 - s_2 - s_3 - y$, and s_1, s_2, s_3 receive $\frac{1}{3}$ each.

Consequently, each vertex of degree 2 starts with a weight of $\frac{6}{3}$, gives nothing away and receives at least $\frac{1}{3}$ during the discharging, which makes it end with a weight of at least $\frac{7}{3}$. ∎

Claim 8. *All the vertices of degree* 3 *have a weight of at least* $\frac{7}{3}$ *after application of the rules.*

Proof. We prove that a vertex v of degree 3 never gives away more than $\frac{2}{3}$. To each branch, it gives either $\frac{1}{3}$ [Rule $R_{2.1}$] or $\frac{1}{6}$ [Rules $R_{1.1}, R_{1.2}, R_{2.3}$] (or nothing). We prove that if v gives $\frac{1}{3}$ to two branches, then it gives nothing to the third. Assume, by contradiction, that v gives $\frac{1}{3}$ to two branches, and that the third one receives something from v. Since $R_{2.1}$ is the only rule that makes v give $\frac{1}{3}$ to a branch, it is applied twice. Then the third branch has to induce:

- A configuration for which $R_{1.1}$, $R_{2.1,}$ or $R_{2.3}$ applies, that is, a vertex of degree 2 followed by a vertex of degree at most 3. But then the graph contains **(C_3)**, a contradiction.
- A configuration for which $R_{1.2}$ applies. Then the graph contains **(C_5)**, a contradiction.

If v gives $\frac{1}{3}$ at most once, then v gives at most $\frac{2}{3}$ on the whole. So, in all cases, vertex v starts with a weight of $\frac{9}{3}$, and gives at most $\frac{2}{3}$ away, so it still has a weight of at least $\frac{7}{3}$ after application of the rules. ■

Claim 9. *All the vertices of degree* 4 *have a weight of at least* $\frac{7}{3}$ *after application of the rules.*

Proof. We prove that a vertex v of degree 4 never gives more than $\frac{5}{3}$ away. To each branch, it gives either $\frac{1}{2}$ [Rules $R_{2.2}, R_{2.3}, R_3$], $\frac{1}{3}$ [Rules $R_{1.3}, R_{2.1}$] or $\frac{1}{6}$ [Rules $R_{1.1}, R_{1.2}, R_{2.2}, R_{2.3}$] (or nothing). We prove that if v gives $\frac{1}{2}$ three times, then it gives at most $\frac{1}{6}$ to the fourth branch. Assume that v gives $\frac{1}{2}$ to three branches. The only case when v gives three times $\frac{1}{2}$ is when for u_4 the fourth neighbor of v, $T(v, u_4)$ exists. Indeed, we applied $R_{2.2}$, $R_{2.3,}$ or R_3 on each of the three branches, which means that for $i \in \{1, 2, 3\}$, v is 2-linked through $v - u_i - x_i - w_i$ to a vertex w_i such that $d(w_i) \leq 3$ or $T(w_i, x_i)$ exists, hence $T(v, u_4)$ exists. Let us enumerate the cases for the branch starting from u_4:

- Vertex u_4 is of degree $d(u_4) \geq 3$. No rule applies, so v does not give anything to this branch.
- Vertex v is 1-linked to a vertex u of degree $d(u) \geq 3$. If $d(u) \geq 4$, then $R_{1.1}$ or $R_{1.3}$ applies. If $d(u) = 3$, then $R_{1.2}$ applies. In both cases, v does not give more than $\frac{1}{6}$.
- Vertex v is 2-linked to a vertex of degree at most 3. Then the graph contains **(C_4)**. Hence, this case never occurs.
- Vertex v is 2-linked to a vertex u of degree $d(u) \geq 4$. If $d(u) \geq 5$, then $R_{2.3}$ is applied. If $d(u) = 4$, then, because of Lemma 3, $R_{2.2}$ is applied. In both cases, v does not give away more than $\frac{1}{6}$.

If v does not give $\frac{1}{2}$ more than twice, then v gives at most $\frac{5}{3}$ on the whole. So, in all cases, v starts with a weight of $\frac{12}{3}$, and gives at most $\frac{5}{3}$ away, so it still has a weight of at least $\frac{7}{3}$ after application of the rules. ■

Claim 10. *All the vertices of degree* ≥ 5 *have a weight of at least* $\frac{7}{3}$ *after application of the rules.*

Proof. Each vertex gives at most $\frac{1}{2}$ to each branch. Hence, a vertex v gives at most $d(v) \times \frac{1}{2}$ on the whole. And for $d(v) \geq 5$, we have $d(v) - \frac{1}{2} \times d(v) \geq \frac{7}{3}$. ■

Hence, every vertex of G has a weight of at least $\frac{7}{3}$ after application of the discharging rules. Consequently, $\frac{\sum_{v \in V} d(v)}{|V|} \geq \frac{7}{3}$, which implies $\text{mad}(G) \geq \frac{7}{3}$. This concludes the proof of Lemma 4.

C. Conclusion

Proof of Theorem 4. We prove by contradiction that $\forall k \geq 4$, every graph G with $\Delta(G) \leq k$ and $\text{mad}(G) < \frac{7}{3}$ satisfies $\chi^2(G) \leq k+1$. For $k \geq 4$, let G be a minimal graph with $\Delta(G) \leq k$ and $\text{mad}(G) < \frac{7}{3}$ that does not admit a 2-distance $(k+1)$-coloring. Graph G is also a minimal graph with $\Delta(G) \leq k$ that does not admit a 2-distance $(k+1)$-coloring (all its subgraphs have a maximum average degree $< \frac{7}{3}$). Lemma 2 implies that G does not contain the configurations (C_1), (C_2), (C_3), (C_4), (C_5). Lemma 4 implies that $\text{mad}(G) \geq \frac{7}{3}$, a contradiction. ■

The limitation in transposing the above proof to list-coloring lies in Configurations (C_4) and (C_5): while proving these configurations are forbidden, we often affect the same color to different vertices in order to complete the coloring, which is hard to transpose to list-coloring as the color lists can differ. Configurations (C_1), (C_2), and (C_3) can be forbidden in the case of list-coloring.

4. PROOF OF THEOREM 5

We prove that there exists a function f such that for a small enough $\epsilon > 0$, every graph G with $\text{mad}(G) < \frac{14}{5} - \epsilon$ and $\Delta(G) \geq f(\epsilon)$ satisfies $\chi^2(G) = \Delta(G) + 1$. We choose to present a proof that yields a function f far from optimal, but that is as simple as possible and contains the decisive ideas. It is however easy to improve it, though the proof increases accordingly in complexity and is not of great interest.

Let $\frac{1}{20} \geq \epsilon > 0$, and k be a constant integer, $k \geq f(\epsilon) = \frac{16}{5\epsilon} + 2$. It holds that $k \geq 66$ since $\epsilon \leq \frac{1}{20}$ (this upper-bound was chosen to that purpose).

A. Forbidden Configurations

We define configurations (C_1) to (C_4) (*see Fig. 10*).

- (C_1) is a vertex of degree 0 or 1.
- (C_2) is a vertex of degree at most $k-1$ that is 2-linked to a vertex of degree at most $k-2$.
- (C_3) is a vertex of degree at most 6, that is, 1-linked to a vertex of degree at most 13, and such that the sum of the degrees of its other neighbors is at most $k-1$.
- (C_4) is a set of vertices $\{a_i\}_{0 \leq i \leq p-1}$, $p \geq 3$, such that $\forall i$ (i taken modulo p), a_i is 3-linked (through a path $a_i - b_{2i} - c_i - b_{2i+1} - a_{i+1}$) to a_{i+1}.

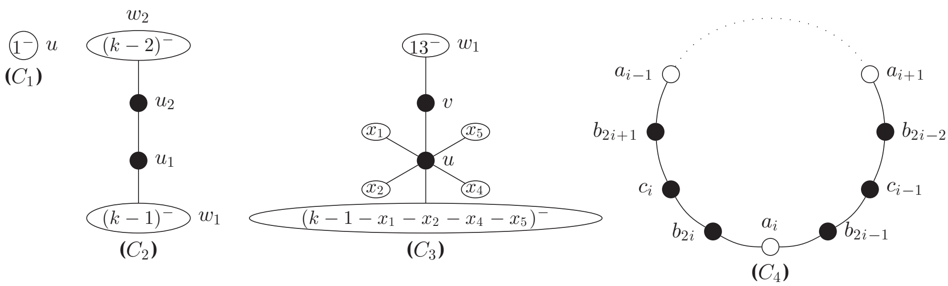

FIGURE 10. Forbidden configurations for Theorem 5.

Lemma 5. *If G is a minimal graph such that $\Delta(G) \leq k$ and G admits no 2-distance $(k+1)$-list-coloring, then G cannot contain any of Configurations **(C_1)** to **(C_4)**.*

Proof. We assume G contains the configuration, apply the minimality to color a subgraph of G, and prove this coloring can be extended to the whole graph, a contradiction. ■

Claim 11. *G cannot contain **(C_1)***

Proof. Using the minimality of G, we color $G \setminus \{u\}$. Since $\Delta(G) \leq k$, and $d(u) \leq 1$, vertex u has at most k constraints. There are $k+1$ colors, so the coloring of $G \setminus \{u\}$ can be extended to G. ■

Claim 12. *G cannot contain **(C_2)***

Proof. Using the minimality of G, we color $G \setminus \{u_1, u_2\}$. Vertex u_1 has at most $|\{w_2\}| + d(w_1) \leq 1 + (k-1) \leq k$ constraints. Hence, we can color u_1. Then u_2 has at most $|\{w_1, u_1\}| + d(w_2) \leq 2 + (k-2) \leq k$ constraints, so we can extend the coloring of $G \setminus \{u_1, u_2\}$ to G. ■

Claim 13. *G cannot contain **(C_3)***

Proof. Using the minimality of G, we color $G \setminus \{v\}$. Vertex u has at most $k-1+1$ constraints, hence we can recolor u. Then v has at most $13+5+1 \leq 66 \leq k$ constraints, so we can extend the coloring of $G \setminus \{v\}$ to G. ■

Claim 14. *G cannot contain **(C_4)***

Proof. Using the minimality of G, we color $G \setminus \{b_1, \ldots, b_{2p-1}, c_1, \ldots, c_p\}$. For every j, b_j has at most $k-1$ constraints, hence it has at least two colors available. So coloring the set $\{b_1, \ldots, b_{2p-1}\}$ is equivalent to 2-list-coloring an even cycle. Then every c_i has at most $4 \leq 66 \leq k$ constraints, so we can extend the coloring of $G \setminus \{b_1, \ldots, b_{2p-1}, c_1, \ldots, c_p\}$ to G. ■

This concludes the proof of Lemma 5.

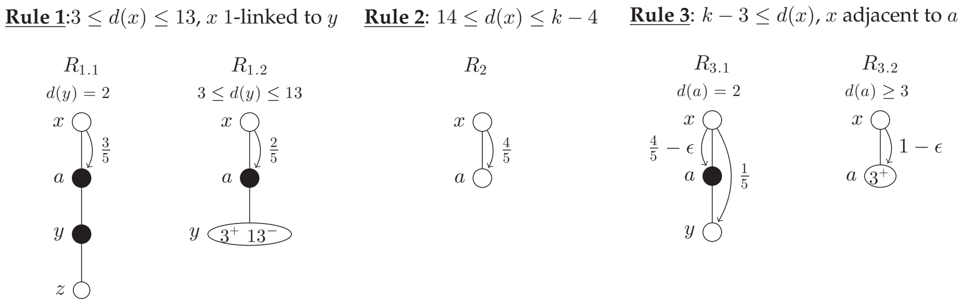

FIGURE 11. Discharging rules R_1, R_2, and R_3 for Theorem 5.

B. Discharging Rules

Let R_1, R_2, R_3, and R_g ("g" stands for "global") be four discharging rules (see Fig. 11): for any vertex x of degree at least 3,

- Rule R_1 is when $3 \le d(x) \le 13$. If x has a neighbor a of degree 2 whose other neighbor is y,
  - Rule $R_{1.1}$ is when $d(y) = 2$. Then x gives $\frac{3}{5}$ to a.
  - Rule $R_{1.2}$ is when $3 \le d(y) \le 13$. then x gives $\frac{2}{5}$ to a.
- Rule R_2 is when $14 \le d(x) \le k - 4$. Then x gives $\frac{4}{5}$ to each of its neighbors.
- Rule R_3 is when $k - 3 \le d(x)$. Let a be a neighbor of x.
  - Rule $R_{3.1}$ is when $d(a) = 2$. Then, for y the other neighbor of a, x gives $\frac{4}{5} - \epsilon$ to a and $\frac{1}{5}$ to y.
  - Rule $R_{3.2}$ is when $d(a) \ge 3$. Then x gives $1 - \epsilon$ to a.
- Rule R_g states that every vertex of degree k gives an additional $\frac{2}{5}$ to a common pot, and every vertex of degree 2, which is adjacent to two vertices of degree 2 receives $\frac{2}{5}$ from this pot.

We use these discharging rules to prove the following lemma:

Lemma 6. *A graph G that does not contain Configurations **(C_1)–(C_4)** satisfies* $\mathrm{mad}(G) \ge \frac{14}{5} - \epsilon$.

Proof. We attribute to each vertex a weight equal to its degree, and apply discharging rules R_1, R_2, R_3, and R_g. We show that all the vertices have a weight of at least $\frac{14}{5} - \epsilon$ in the end.

Since **(C_4)** is forbidden, if we consider the structure A induced in G by the paths $a_1, \ldots, a_5$ where $d(a_2) = d(a_3) = d(a_4) = 2$ (**(C_2)** implies that $d(a_1) = d(a_5) = k$), A is a forest. This means that in G, there are less vertices of degree 2 adjacent to two vertices of degree 2 than vertices of degree k: hence Rule R_g is valid.

- There are no vertices of degree 0 or 1.
- Let s be a maximal p-link of vertices of degree 2 (maximal in the sense that it does not admit a vertex of degree 2 as a neighbor). The p-link s cannot be a cycle as

(C_2) is forbidden. According to the discharging rules, a vertex of degree 2 never gives away any weight. We prove that it receives at least $\frac{4}{5} - \epsilon$. There are three cases depending on p (due to Configuration **(C_2)**, $p \leq 3$):

- $p = 1$. Let a be the only vertex in s.
  * a has a neighbor x of degree at least 14: then it receives at least $\frac{4}{5} - \epsilon$ from it, according to Rule R_2 or R_3.
  * a has two neighbors x_1 and x_2 of degree less than 13: then it receives $\frac{2}{5}$ from each, according to Rule $R_{1.2}$.
- $p = 2$. Let a and b be the vertices of s, and x (resp. y) the other neighbor of a (resp. b), with $d(x) \geq d(y)$. Due to Configuration **(C_2)**, $d(x) \geq k - 1$. Then a receives $\frac{4}{5} - \epsilon$ from x (Rule $R_{3.1}$), and b receives $\frac{1}{5}$ from x (Rule $R_{3.1}$), and at least $\frac{3}{5}$ from y (Rules $R_{1.1}$, R_2, and R_3).
- $p = 3$. Due to Configuration **(C_2)**, for $a_2 - a_3 - a_4$ the vertices of s and a_1 (resp. a_5) the other neighbor of a_2 (resp. a_4), $d(a_1) = d(a_5) = k$. Then Rules $R_{3.1}$ and R_g apply: a_2 (resp. a_4) receives $\frac{4}{5} - \epsilon$ from a_1 (resp. a_5), and a_3 receives $\frac{1}{5}$ from a_1 and a_5, and $\frac{2}{5}$ from R_g.

- Let x be a vertex with $d(x) = 3$. We prove that x loses a weight of at most $\frac{1}{5} + \epsilon$.

  - If x is adjacent to two vertices of degree 2 whose other neighbor is of degree at most 13, then, according to Configuration **(C_3)**, its third neighbor is of degree at least $k - 3$, hence x receives $1 - \epsilon$, due to Rule $R_{3.2}$, and gives at most $2 \times \frac{3}{5}$ away, according to Rule R_1, so it loses at most $\frac{1}{5} + \epsilon$.
  - If x is adjacent to exactly one vertex of degree 2 whose other neighbor is of degree at most 13, then, according to Configuration **(C_3)**, x has a neighbor of degree at least 14, hence it receives at least $\frac{4}{5}$ (according to Rule R_2), and gives at most $\frac{3}{5}$ away (according to Rule R_1), so it loses less than $\frac{1}{5} + \epsilon$.
  - If x is adjacent to no vertex of degree 2 whose other neighbor is of degree at most 13, then it gives nothing away as no rule applies.

- Let x be a vertex with $4 \leq d(x) \leq 6$.

  - If x is adjacent to a vertex of degree 2 whose other neighbor is of degree at most 13, due to Configuration **(C_3)**, we know that x has a neighbor y that is of degree at least 14. Indeed, $5 \times 13 = 65 \leq k - 1$. This means that x receives at least $\frac{4}{5}$ (by Rules R_2 and R_3), and gives at most $(d(x) - 1) \times \frac{3}{5}$ (by Rule R_1). Its final weight is at least $d(x) + \frac{4}{5} - (d(x) - 1)(\frac{3}{5})$, which is at least $\frac{14}{5}$ since $d(x) \geq 4$.
  - If x is adjacent to no vertex of degree 2 whose other neighbor is of degree at most 13, then its final weight is at least $d(x) \geq 4 \geq \frac{14}{5}$.

- Let x be a vertex with $7 \leq d(x) \leq 13$. It gives at most $d(x) \times \frac{3}{5}$ away (due to Rule R_1), which means it has at least a weight of $\frac{14}{5}$ at the end since $d(x) \geq 7$.
- Let x be a vertex with $14 \leq d(x) \leq k - 4$. It gives at most $d(x) \times \frac{4}{5}$ away (due to Rule R_2), which means it has at least a weight of $\frac{14}{5}$ at the end since $d(x) \geq 14$.

- Let x be a vertex with $k-3 \leq d(x)$. It gives at most $d(x) \times (1-\epsilon) + \frac{2}{5}$ away (due to Rules R_3 and R_g), which means it has at least a weight of $\frac{14}{5} - \epsilon$ in the end since $d(x) \geq \frac{16}{5\epsilon} + 2$.

Consequently, after application of the discharging rules, every vertex v of G has a weight of at least $\frac{14}{5} - \epsilon$, meaning that $\sum_{v \in G} d(v) \geq \sum_{v \in G} (\frac{14}{5} - \epsilon)$. Therefore, $\text{mad}(G) \geq \frac{14}{5} - \epsilon$. This completes the proof of Lemma 6. ■

C. Conclusion

Proof of Theorem 5. We prove by contradiction that there exists a function f such that for a small enough $\epsilon > 0$, for any $k \geq f(\epsilon)$, every graph G with $\text{mad}(G) < \frac{14}{5} - \epsilon$ and $\Delta(G) \leq k$ satisfies $\chi^2_\ell(G) \leq k+1$. For $\frac{1}{20} \geq \epsilon > 0$, $k \geq \frac{16}{5\epsilon} + 2$, let G be a minimal graph such that $\Delta(G) \leq k$, $\text{mad}(G) < \frac{14}{5} - \epsilon$ and G does not admit a list 2-distance $(k+1)$-coloring. Graph G is also a minimal graph such that $\Delta(G) \leq k$ and G does not admit a list 2-distance $(k+1)$-coloring (all its proper subgraphs satisfy $\Delta \leq k$ and mad $< \frac{14}{5} - \epsilon$, so they admit a list 2-distance $(k+1)$-coloring). By Lemmas 5, graph G cannot contain $(\boldsymbol{C_1})$ to $(\boldsymbol{C_4})$. Lemma 6 implies that $\text{mad}(G) \geq \frac{14}{5} - \epsilon$, a contradiction. ■

The limit in transposing it to a greater upper-bound on the mad lies in the case where a vertex of degree 3 is 2-linked to two vertices of degree k, and is adjacent to a vertex of degree k. Assume we prove, for some a, that every vertex has a weight of at least $2 + a$ after application of the discharging rules. Then, on the whole, the vertices of degree 2 and 3 in the configuration need to receive at least $4 \times a - (1-a)$. It means that if $a \geq \frac{4}{5}$, the vertices of degree k have to give at least 1 when they are adjacent to such a configuration. However, we cannot forbid this configuration, nor a cycle of them, so we were not able to prove that the vertices of degree k can afford to do that and still have a weight of at least $2 + a$ at the end.

5. PROOF OF THEOREM 6

We prove that every graph G with $\Delta(G) \geq 5$ (resp. 6, 8) and $\text{mad}(G) < \frac{12}{5}$ (resp. $\frac{5}{2}$, $\frac{18}{7}$) satisfies $\chi^2_\ell(G) = \Delta(G) + 1$.

It is a refined version of the proof of Theorem 5 when restricted to particular cases ($\text{mad}(G) < \frac{12}{5}, \frac{5}{2}, \frac{18}{7}$).

Let k be a constant integer, $k \geq 5$.

A. Forbidden Configurations

We define configurations $(\boldsymbol{C_1})$ to $(\boldsymbol{C_8})$ (see Fig. 12).

- $(\boldsymbol{C_1})$ is a vertex u of degree 0 or 1.
- $(\boldsymbol{C_2})$ is a vertex w_1 of degree at most $k-1$ that is 2-linked (through a path $w_1 - v_1 - v_2 - w_2$) to a vertex w_2 of degree at most $k-2$.

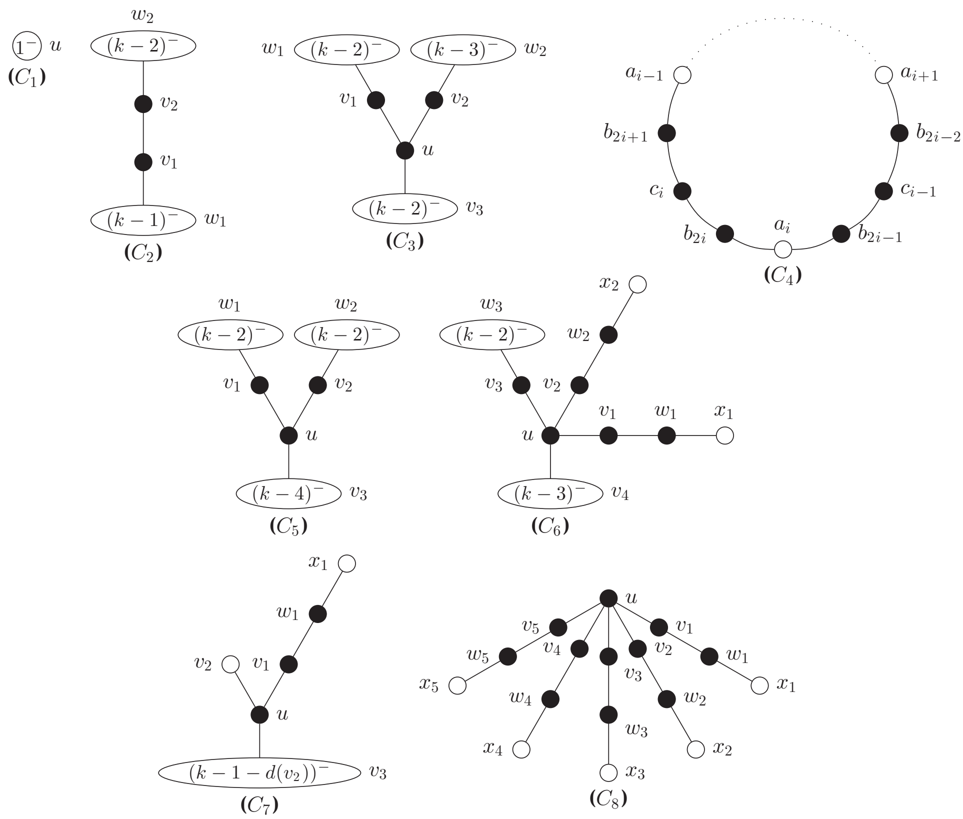

FIGURE 12. Forbidden configurations for Theorem 6.

- **(C_3)** is a vertex u of degree 3 that is 1-linked (through a path $u - v_1 - w_1$) to a vertex w_1 of degree at most $k-2$, 1-linked (through a path $u - v_2 - w_2$) to a vertex w_2 of degree at most $k-3$, and whose third neighbor v_3 is of degree at most $k-2$.
- **(C_4)** is a set of vertices $\{a_i\}_{0\le i\le p-1}$, $p \geq 3$, such that $\forall i$ (i taken modulo p), a_i is 3-linked (through a path $a_i - b_{2i} - c_i - b_{2i+1} - a_{i+1}$) to a_{i+1}.
- **(C_5)** is a vertex u of degree 3 that is 1-linked (through two paths $u - v_1 - w_1$ and $u - v_2 - w_2$) to two vertices w_1 and w_2 of degree at most $k-2$, and whose third neighbor v_3 is of degree at most $k-4$.
- **(C_6)** is a vertex u of degree 4 that is 2-linked (through two paths $u - v_1 - w_1 - x_1$ and $u - v_2 - w_2 - x_2$) to two vertices x_1 et x_2, 1-linked (through a path $u - v_3 - w_3$) to a vertex w_3 of degree at most $k-2$, and whose fourth neighbor v_4 is of degree at most $k-3$.
- **(C_7)** is a vertex u of degree 3 that is 2-linked (through a path $u - v_1 - w_1 - x_1$) to a vertex x_1, and such that the sum of the degrees of its two other neighbors is at most $k-1$.
- **(C_8)** is a vertex u of degree 5 that is 2-linked (through a path $u - v_i - w_i - x_i$, $i \in \{1, 2, \ldots, 5\}$) to five vertices $x_1, \ldots, x_5$.

Lemma 7. *If G is a minimal graph such that $\Delta(G) \leq k$ and G admits no list 2-distance $(k+1)$-coloring, and if $i \leq k$, then G cannot contain Configuration **(C_i)**.*

Proof. We assume G contains the configuration, apply the minimality to color a subgraph of G, and prove this coloring can be extended to the whole graph, a contradiction. ■

Claim 15. *G cannot contain **(C_1)***

Proof. Using the minimality of G, we color $G \setminus \{u\}$. Since $\Delta(G) \leq k$, and $d(u) \leq 1$, vertex u has at most k constraints. There are $k+1$ colors, so the coloring of $G \setminus \{u\}$ can be extended to G. ■

Claim 16. *G cannot contain **(C_2)***

Proof. Using the minimality of G, we color $G \setminus \{v_1, v_2\}$. Vertex v_1 has at most $|\{w_2\}| + d(w_1) \leq 1 + (k-1) \leq k$ constraints. Hence, we can color v_1. Then v_2 has at most $|\{w_1, v_1\}| + d(w_2) \leq 2 + (k-2) \leq k$ constraints, so we can extend the coloring of $G \setminus \{v_1, v_2\}$ to G. ■

Claim 17. *G cannot contain **(C_3)***

Proof. Using the minimality of G, we color $G \setminus \{u, v_1, v_2\}$. Vertex u has at most $k-2+1+1 \leq k$ constraints. Hence, we can color u. Then we color v_1 (at most $k-2+2 \leq k$ constraints), and v_2 (at most $k-3+3 \leq k$ constraints), so we can extend the coloring of $G \setminus \{u, v_1, v_2\}$ to G. ■

Claim 18. *G cannot contain **(C_4)***

Proof. Using the minimality of G, we color $G \setminus \{b_1, \ldots, b_{2p-1}, c_1, \ldots, c_p\}$. For every j, b_j has at most $k-2$ constraints, hence it has at least two colors available. So coloring the set $\{b_1, \ldots, b_{2p-1}\}$ is equivalent to list 2-coloring an even cycle. Then every c_i has at most $4 \leq k$ constraints, so we can extend the coloring of $G \setminus \{b_1, \ldots, b_{2p-1}, c_1, \ldots, c_p\}$ to G. ■

Claim 19. *G cannot contain **(C_5)***

Proof. Using the minimality of G, we color $G \setminus \{u, v_1, v_2\}$. We can color successively v_1 (at most $k-2+1 \leq k$ constraints), v_2 (at most $k-2+2 \leq k$ constraints), and u (at most $k-4+2+2 \leq k$ constraints), so we can extend the coloring of $G \setminus \{u, v_1, v_2\}$ to G. ■

Claim 20. *If $k \geq 6$, G cannot contain **(C_6)***

Proof. Using the minimality of G, we color $G \setminus \{u, v_1, v_2, v_3\}$. Vertex u has at most $k-3+1+1+1 \leq k$ constraints. Hence, we can color u. Then we color v_3 (at most $k-2+2 \leq k$ constraints), v_2 (at most $5 \leq k$ constraints), and v_2 (at most $6 \leq k$ constraints), so we can extend the coloring of $G \setminus \{u, v_1, v_2\}$ to G. ■

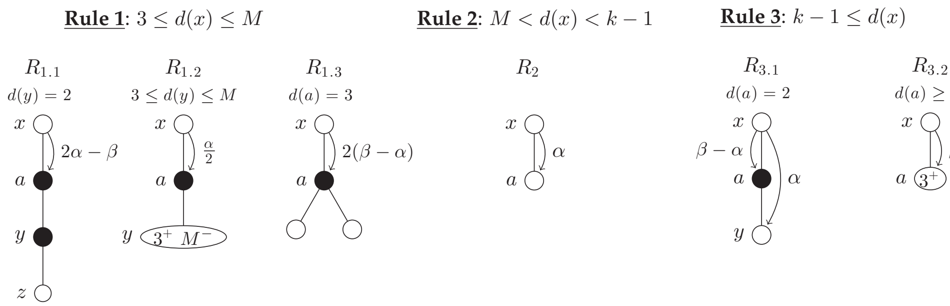

FIGURE 13. Discharging rules R_1, R_2, and R_3 for Theorem 6.

Claim 21. *If $k \geq 7$, G cannot contain* ***(C_7)***

Proof. Using the minimality of G, we color $G \setminus \{v_1\}$. We recolor u (at most $k - 1 + 1 \leq k$ constraints), then we color v_1 (at most $5 \leq k$ constraints), so we can transform the coloring of $G \setminus \{v_1\}$ into a coloring of G. ■

Claim 22. *If $k \geq 8$, G cannot contain* ***(C_8)***

Proof. Using the minimality of G, we color $G \setminus \{u, v_1, v_2, v_3, v_4, v_5\}$. We color vertices $u, v_1, v_2, v_3, v_4, v_5$ (each has at most seven constraints), so we can extend the coloring of $G \setminus \{u, v_1, v_2, v_3, v_4, v_5\}$ to G.

This concludes the proof of Lemma 7. ■

B. Discharging Rules

Let α, β ($1 > \beta \geq \alpha > 0$), M ($k - 1 \geq M \geq 3$) be parameters that we will assign later. Let R_1, R_2, R_3, and R_g be four discharging rules (see Fig. 13): for any vertex x of degree at least 3,

- Rule R_1 is when $3 \leq d(x) \leq M$.
  - If x has a neighbor a of degree 2 whose other neighbor is y,
    * Rule $R_{1.1}$ is when $d(y) = 2$. Then x gives $2\alpha - \beta$ to a.
    * Rule $R_{1.2}$ is when $3 \leq d(y) \leq M$. Then x gives $\frac{\alpha}{2}$ to a.
  - Rule $R_{1.3}$ is when $d(x) \geq 4$ and x has a neighbor a of degree 3. Then x gives $2(\beta - \alpha)$ to a.
- Rule R_2 is when $M < d(x) < k - 1$. Then x gives α to each of its neighbors.
- Rule R_3 is when $k - 1 \leq d(x)$. Let a be a neighbor of x.
  - Rule $R_{3.1}$ is when $d(a) = 2$. Then, for y the other neighbor of a, x gives α to a and $\beta - \alpha$ to y.
  - Rule $R_{3.2}$ is when $d(a) \geq 3$. Then x gives β to a.
- Rule R_g states that every vertex of degree k gives an additional $3\alpha - 2\beta$ to a common pot, and every vertex of degree 2 which is adjacent to two vertices of degree 2 receives $3\alpha - 2\beta$ from this pot.

We use these discharging rules to prove the following lemma:

Lemma 8. *A graph G with $\Delta(G) \leq k$ that does not contain Configuration* **(C_i)** *for $i \leq k$ satisfies* $\text{mad}(G) \geq 2 + \alpha$, *where* $\alpha = \frac{2}{5}$ *if* $k = 5$, $\frac{1}{2}$ *if* $7 \geq k \geq 6$, $\frac{4}{7}$ *if* $k \geq 8$.

Proof. If $k \leq 7$, we choose $M = k - 2$ and $\beta = \alpha$. If $k \geq 8$, we choose $M = k - 3$ and $\beta = \alpha + \frac{1}{21}$.

We attribute to each vertex a weight equal to its degree, and apply discharging rules R_1, R_2, R_3, and R_g. We show that all the vertices have a weight of at least $2 + \alpha$ in the end.

Since **(C_4)** is forbidden, if we consider the structure A induced in G by the paths $a_1, \ldots, a_5$ where $d(a_2) = d(a_3) = d(a_4) = 2$ (**(C_2)** implies that $d(a_1) = d(a_5) = k$), A is a forest. This means that in G, there are less vertices of degree 2 adjacent to two vertices of degree 2 than vertices of degree k: hence Rule R_g is valid.

- There are no vertices of degree 0 or 1.
- Let s be a maximal path of vertices of degree 2 (maximal in the sense that it does not admit a vertex of degree 2 as a neighbor; every vertex of degree 2 belongs to such a path as Configuration **(C_2)** is forbidden). According to the discharging rules, a vertex of degree 2 never gives away weight. We prove that it receives at least α. There are three cases depending on the size of s (s can't be of size greater than 3 due to Configuration **(C_2)**):

  - $|s| = 1$. Let a be the only vertex in s.
    * a has a neighbor x of degree at least $M + 1$: then it receives at least α from it, according to Rule R_2 or R_3.
    * a has two neighbors x_1 and x_2 of degree at most M: then it receives $\frac{\alpha}{2}$ from each, according to Rule $R_{1.2}$.
  - $|s| = 2$. Let a and b be the vertices of s, and x (resp. y) the other neighbor of a (resp. b), with $d(x) \geq d(y)$. Due to Configuration **(C_2)**, $d(x) \geq k - 1$. Then a receives α from x (Rule $R_{3.1}$), and b receives $\beta - \alpha$ from x (Rule $R_{3.1}$), and at least $2\alpha - \beta$ from y (Rules $R_{1.1}$, R_2, and $R_{3.1}$).
  - $|s| = 3$. Due to Configuration **(C_2)**, for $a_2 - a_3 - a_4$ the vertices of s and a_1 (resp. a_5) the other neighbor of a_2 (resp. a_4), $d(a_1) = d(a_5) = k$. Then Rules $R_{3.1}$ and R_g apply: a_2 (resp. a_4) receives α from a_1 (resp. a_5), and a_3 receives $\beta - \alpha$ from a_1 and a_5, and $3\alpha - 2\beta$ from R_g.

- Let x be a vertex with $d(x) = 3$. We prove that x loses a weight of at most $1 - \alpha$.

  - If x is adjacent to a vertex of degree 2 whose other neighbor is also of degree 2.
    * x has no other neighbor of degree 2 whose other neighbor is of degree at most M. Then, according to Rule $R_{1.1}$, it gives $2\alpha - \beta$, which is less than $1 - \alpha$ if $k \leq 7$, as $\alpha = \beta \leq \frac{1}{2}$. If $k \geq 8$, then according to Configuration **(C_7)**, x has a neighbor of degree at least 4, hence $R_{1.3}$, R_2, or R_3 applies and x receives at least $2(\beta - \alpha)$, and $1 - \alpha + 2(\beta - \alpha) \geq 2\alpha - \beta$ when $\alpha = \frac{4}{7}$ and $\beta = \frac{4}{7} + \frac{1}{21}$.
    * x has a second neighbor of degree 2 whose other neighbor is of degree at most M. Then, according to Configuration **(C_3)**, the third neighbor of x is of degree at least $k - 1$. Then, x receives at least β (Rule $R_{3.2}$), gives at most $2 \times (2\alpha - \beta)$ (Rules $R_{1.1}$ and $R_{1.2}$), and $1 - \alpha + \beta \geq 2 \times (2\alpha - \beta)$.

- If x is adjacent to two vertices of degree 2 whose other neighbor is of degree at least 3 and at most M, then we have three cases:
  * $k = 5$, $\alpha = \frac{2}{5}$. Vertex x gives at most $3 \times \frac{\alpha}{2}$ (Rule $R_{1.2}$), but $1 - \alpha \geq 3 \times \frac{\alpha}{2}$ as $\alpha = \frac{2}{5}$.
  * $7 \geq k \geq 6$, $\alpha = \frac{1}{2}$. According to Configuration **(C_5)**, the third neighbor of x is of degree at least 3, so, according to Rule $R_{1.2}$, x gives $2 \times \frac{\alpha}{2} \leq 1 - \alpha$ as $\alpha = \frac{1}{2}$.
  * $k \geq 8$, $\alpha = \frac{4}{7}$. Then $M = k - 3$. According to Configuration **(C_3)**, the third neighbor of x is of degree at least $k - 1$, hence x receives β (Rule $R_{3.2}$) and gives at most $2 \times \frac{\alpha}{2}$ (Rule $R_{1.2}$) away, so it loses nothing.
- If x is adjacent to exactly one vertex of degree 2 whose other neighbor y is of degree at least 3 and at most 13, then x gives at most $\frac{\alpha}{2}$ (Rule $R_{1.2}$), which is possible as $1 - \alpha \geq \frac{\alpha}{2}$.
- If x is adjacent to no vertex of degree 2 whose other neighbor is of degree at most 13, then it gives nothing away.

• Let x be a vertex with $d(x) = 4$, in the case where $4 \leq M$. Then $k \geq 6$ and $\alpha \geq \frac{1}{2}$. We are in one of the following two cases.

  - If x is 2-linked to two vertices, and 1-linked to a vertex of degree at most M, then, according to Configuration **(C_6)**, its other neighbor y is of degree at least $k - 2$. According to Rules $R_{1.1}$ and $R_{1.2}$, x gives nothing to y, and gives at most $2\alpha - \beta$ to its other three neighbors. If $k \leq 7$ and $\alpha \leq \frac{1}{2}$, on the whole x gives at most $3(2\alpha - \beta) \leq 2 - \alpha$. If $k \geq 8$, $M = k - 3$, hence x receives at least α from y, and $3(2\alpha - \beta) \leq 2 - \alpha + \alpha$.
  - If not, according to R_1, x gives at most $3 \times \frac{\alpha}{2} + 2\alpha - \beta$ or $2 \times (2\alpha - \beta) + 2 \times (2\beta - 2\alpha)$, and in both cases, it has a weight of at least $2 + \alpha$ at the end.

• Let x be a vertex with $d(x) = 5$, in the case where $5 \leq M$. Then $k \geq 7$. If $k = 7$ and $\alpha = \frac{1}{2}$, then it gives at most $5 \times (2\alpha - \beta) \leq 3 - \alpha$. If $k \geq 8$ and $\alpha = \frac{4}{7}$, Configuration **(C_8)** states that x cannot be 2-linked to 5 vertices, hence it gives at most $4 \times (2\alpha - \beta) + \frac{\alpha}{2} \leq 3 - \alpha$.
• Let x be a vertex with $6 \leq d(x) \leq M$. It gives at most $d(x) \times (2\alpha - \beta)$ away, which means it has at least a weight of $2 + \alpha$ at the end since $d(x) \geq 6$.
• Let x be a vertex with $M < d(x) < k - 1$. It gives at most $d(x) \times \alpha$ away (R_2), which means it has at least a weight of $2 + \alpha$ at the end since $d(x) > M$.
• Let x be a vertex with $k - 1 \leq d(x) < k$. It gives at most $d(x) \times \beta$ away (R_3), which means it has at least a weight of $2 + \alpha$ in the end since $d(x) \geq k - 1$.
• Let x be a vertex with $d(x) = k$. It gives at most $d(x) \times \beta + 3\alpha - 2\beta$ away (R_3 and R_g), which means it has at least a weight of $2 + \alpha$ in the end since $d(x) = k$.

Consequently, after application of the discharging rules, every vertex v of G has a weight of at least $2 + \alpha$, meaning that $\sum_{v \in G} d(v) \geq \sum_{v \in G}(2 + \alpha)$. Therefore, $\text{mad}(G) \geq 2 + \alpha$. ■

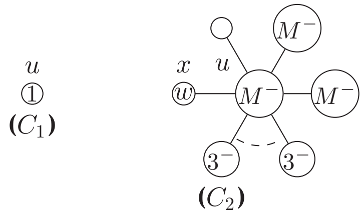

FIGURE 14. Forbidden configurations for Theorem 7.

C. Conclusion

Proof of Theorem 6. We prove by contradiction that $\forall k \geq 5$, every graph G with $\Delta(G) \leq k$ and $\text{mad}(G) < 2 + \alpha$, where $\alpha = \frac{2}{5}$ if $k = 5$, $\frac{1}{2}$ if $7 \geq k \geq 6$, $\frac{4}{7}$ if $k \geq 8$, satisfies $\chi^2_\ell(G) \leq k + 1$. Let G be a minimal graph such that $\Delta(G) \leq k$, $\text{mad}(G) < 2 + \alpha$ and G does not admit a list 2-distance $(k+1)$-coloring. Graph G is also a minimal graph such that $\Delta(G) \leq k$ and G does not admit a list 2-distance $(k+1)$-coloring (all its proper subgraphs satisfy $\Delta \leq k$ and mad $< 2 + \alpha$, so they admit a list 2-distance $(k+1)$-coloring). By Lemma 7, graph G cannot contain Configuration **(C_i)** if $i \leq k$. Lemma 8 implies that $\text{mad}(G) \geq 2 + \alpha$, a contradiction. ■

6. PROOF OF THEOREM 7

Let $1 > \epsilon > 0$, let $M = \frac{8}{\epsilon} - 2$, and $h(\epsilon) = 5M - 6$.

Note that $M - (4 - \epsilon) = M \times (1 - \frac{\epsilon}{2})$, and that $h(\epsilon) \geq 2M + 3$.

Again, we choose to present a simple proof despite the fact that it means the function h is probably not as good as possible. However, it is still optimal up to a constant factor as the graph family presented in Figure 3 shows that it could not be less than $\frac{2}{\epsilon}$. Indeed, the family $(G_p)_{p \in \mathbb{N}^*}$ satisfies $\chi^2_\ell(G_p) \geq \chi^2(G_p) \geq 3p = \Delta(G_p) + \frac{2}{4 - \text{mad}(G_p)}$.

We prove by contradiction that every graph G with $mad(G) < 4 - \epsilon$ admits a 2-distance $(\Delta(G) + h(\epsilon))$-list-coloring.

We call *weak* a vertex of degree 2 or 3 that has at most one neighbor of degree M^+. In the figures, the label "w" means the vertex is weak.

A. Forbidden Configurations

We define Configurations **(C_1)** and **(C_2)** (see Fig. 14). Configuration **(C_1)** is a vertex u of degree 1. Configuration **(C_2)** is a vertex u of degree M^- that has a weak neighbor x, and at most 3 neighbors of degree 4^+, among which at most one is of degree M^+.

Lemma 9. *If G is a minimal graph such that G admits no list 2-distance $(\Delta(G) + h(\epsilon))$-coloring, then G cannot contain Configurations **(C_1)** nor **(C_2)**.*

Proof.

(C_1) We color $G \setminus \{u\}$ using the minimality of G. Vertex u has at most $\Delta(G)$ constraints, so there is a free color for u, a contradiction.

FIGURE 15. Discharging rules R_1, R_2 for Theorem 7.

(C_2) We remove the (u, x) edge, and use the minimality of G to color the resulting graph. We recolor u (at most $\Delta(G) + 2M + 3(M - 3) + 2 = \Delta(G) + 5M - 7$ constraints), and x (at most $\Delta(G) + M + M$ constraints), so we can transform the coloring of $G \setminus \{(u, x)\}$ into a coloring of G. ■

B. Discharging Rules

Let R_1, R_2 be two discharging rules (see Fig. 15). Discharging rule R_1 states that a vertex of degree at least M gives $1 - \frac{\epsilon}{2}$ to each of its neighbors. Discharging rule R_2 states that a vertex of degree less than M gives $1 - \frac{\epsilon}{2}$ to each of its weak neighbors.

We use these discharging rules to prove the following lemma.

Lemma 10. *A graph G that does not contain Configurations **(C_1)** or **(C_2)** satisfies* $\text{mad}(G) \geq 4 - \epsilon$.

Proof. We attribute to each vertex a weight equal to its degree, and apply the two discharging rules R_1, R_2. We show that each vertex of G has a weight of at least $4 - \epsilon$ at the end of the discharging.

Let u be a vertex of G. Since Configuration **(C_1)** is forbidden, we have $d(u) \geq 2$. We make a case analysis whether u gives some weight away or not.

- u gives some weight away.
  - If $d(u) \geq M$, **(R_1)** is applied, and by definition of M, vertex u gives $1 - \frac{\epsilon}{2}$ to each of its neighbors and still has a weight of at least $4 - \epsilon$.
  - If $d(u) < M$, **(R_2)** is applied and u has a weak neighbor x. Since **(C_2)** is forbidden, u is in one of these two situations:
    - * u has at least two neighbors of degree M^+. According to R_1, they each give $1 - \frac{\epsilon}{2}$ to u. Then u has at most $d - 2$ weak neighbors, and $d(u) - (4 - \epsilon) + 2(1 - \frac{\epsilon}{2}) \geq (d(u) - 2)(1 - \frac{\epsilon}{2})$, so u has a weight of at least $4 - \epsilon$ after application of the discharging rules.
    - * u has at least four neighbors of degree 4^+. So u has at most $d - 4$ weak neighbors, and $d(u) - (4 - \epsilon) \geq (d(u) - 4)(1 - \frac{\epsilon}{2})$, hence u has a weight of at least $4 - \epsilon$ after application of the discharging rules.

- u gives no weight away.
  - $d(u) \geq 4$. Then u still has a weight of at least $4 - \epsilon$ after application of the discharging rules.
  - u is a weak vertex. Then, according to **(C_2)**, it cannot be adjacent to another weak vertex, so it gives nothing away and receives $1 - \frac{\epsilon}{2}$ from each of its neighbors. After application of the discharging rules, it has a weight at least $2 + 2 \times (1 - \frac{\epsilon}{2}) = 4 - \epsilon$
  - $d(u) \leq 3$ and u is not weak. Then, u has at least two neighbors of degree at least M, so u receives at least $2 \times (1 - \frac{\epsilon}{2})$. It had initially a weight of at least 2 and

gave nothing away, meaning that it has a weight of at least $4 - \epsilon$ after application of the discharging rules.

Consequently, after application of the discharging rules, every vertex in G has a weight of at least $4 - \epsilon$ after application of the discharging rules, meaning that $\sum_{v \in V} d(v) \geq \sum_{v \in V}(4 - \epsilon)$. Therefore, $\text{mad}(G) \geq 4 - \epsilon$. ■

C. Conclusion

***Proof of Theorem* 7.** We prove by contradiction that $\forall 1 > \epsilon > 0$, every graph G with $\text{mad}(G) < 4 - \epsilon$ satisfies $\chi^2_\ell(G) \leq \Delta(G) + h(\epsilon)$. Let G be a minimal graph with $\text{mad}(G) < 4 - \epsilon$ that does not admit a list 2-distance $(\Delta(G) + h(\epsilon))$-coloring. Graph G is also a minimal graph that does not admit a list 2-distance $\Delta(G) + h(\epsilon)$-coloring. By Lemma 9, G cannot contain Configurations $(\boldsymbol{C_1})$ nor $(\boldsymbol{C_2})$. Lemma 10 implies that $\text{mad}(G) \geq 4 - \epsilon$, a contradiction. ■

7. INJECTIVE COLORING

A *list injective k-coloring* of a graph is a (not necessarily proper) list k-coloring of its vertices such that two vertices with a common neighbor are of different color, or, in other words, such that no vertex has two neighbors with the same color.

Note that the proofs for Theorems 5 to 7 also work, with close to no alteration, for list injective coloring with one color less. Indeed, the discharging part of each proof does not depend on the problem considered, and the configuration part of it can easily be checked to work also for this as, though one less color is available, every critical vertex has at least one less constraint since already colored neighbors do not count anymore. There is no reason to think that this would be the case for any discharging proof about list 2-distance coloring, but it happens to be the case most often.

We thus obtain the following theorem.

Theorem 8. *There exists function f and g such that for a small enough $\epsilon > 0$, any graph G with one of the following properties admits a list injective $(\Delta(G) + 1)$-coloring.*

(1) $\text{mad}(G) < \frac{12}{5}$ *and* $\Delta(G) \geq 5$
(2) $\text{mad}(G) < \frac{5}{2}$ *and* $\Delta(G) \geq 6$
(3) $\text{mad}(G) < \frac{18}{7}$ *and* $\Delta(G) \geq 8$
(4) $mad(G) < \frac{14}{5} - \epsilon$ *and* $Delta(G) \geq f(\epsilon)$

And any graph G with $mad(G) < 4 - \epsilon$ admits a list injective $(\Delta(G) + h(\epsilon))$-coloring.

8. CONCLUSION

We have proved in Theorem 4 and Theorem 6 that some known results on the 2-distance colorability of planar graphs with a lower-bounded girth could be improved by studying graphs with an upper-bounded maximum average degree. Through Theorem 5, we have partially answered the question of how Conjecture 2 could be transposed to graphs with an upper-bounded maximum average degree. With Theorem 7, we have answered what Theorem 5 becomes when an additional constant number of colors is allowed.

It happens that relatively efficient algorithms can be derived from any of those theorems, as we have actually proved in the intermediary lemmas that any graph with a small enough maximum average degree contains one of the given configurations, and that coloring a graph containing one of them comes down to coloring a subgraph. Hence the graph can be colored recursively by spotting a configuration, coloring the corresponding subgraph, and extending the coloring. Besides, applying the discharging rules on the graph speeds up the spotting of the configurations as the weight does not have the expected value near them. The resulting algorithms are at worst cubic in the number of vertices.

As pointed out all along, there are a few open questions remaining. Can the proof of Theorem 4 be extended to list-coloring? What are the optimal values of Theorem 2? What is the optimal upper-bound on the maximum average degree in Theorem 5? In particular, does there exist a family of graphs of increasing maximum degree, of maximum average degree tending to $\frac{14}{5}$ and who are not 2-distance $(\Delta + 1)$-colorable, or can the theorem be improved?

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