Characterizing Directed Path Graphs by Forbidden Asteroids

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Abstract: An asteroidal triple is a stable set of three vertices such that each pair is connected by a path avoiding the neighborhood of the third vertex. Asteroidal triples play a central role in a classical characterization of interval graphs by Lekkerkerker and Boland. Their result says that a chordal graph is an interval graph if and only if it does not contain an asteroidal triple. In this paper, we prove an analogous theorem for directed path graphs which are the intersection graphs of directed paths in a directed tree. For this purpose, we introduce the notion of a special connection. Two non-adjacent vertices are linked by a special connection if either they

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have a common neighbor or they are the endpoints of two vertex-disjoint chordless paths satisfying certain conditions. A special asteroidal triple is an asteroidal triple such that each pair is linked by a special connection. We prove that a chordal graph is a directed path graph if and only if it does not contain a special asteroidal triple.

Keywords: *intersection graph; directed path graph; asteroidal triple*

1. INTRODUCTION

A *hole* is a chordless cycle of length at least four. A graph is a *chordal graph* if it does not contain a hole as an induced subgraph. Gavril [5] proved that a graph is chordal if and only if it is the intersection graph of a family of subtrees of a tree. In this article, whenever we talk about the intersection of subgraphs of a graph we mean that the *vertex sets* of the subgraphs intersect.

A graph is an *interval graph* if it is the intersection graph of a family of intervals on the real line; or equivalently, the intersection graph of a family of subpaths of a path. An *asteroidal triple* in a graph G is a set of three non-adjacent vertices such that for any two of them, there exists a path between them in G that does not intersect the neighborhood of the third. The graph of Figure 1 is an example of a graph that minimally contains an asteroidal triple; the three vertices forming the asteroidal triple are circled.

The following classical theorem was proved by Lekkerkerker and Boland.

Theorem 1 (Lekkerkerker and Boland [10]). *A chordal graph is an interval graph if and only if it does not contain an asteroidal triple.*

Lekkerkerker and Boland [10] derived from Theorem 1 the list of minimal forbidden subgraphs for interval graphs.

The class of path graphs lies between interval graphs and chordal graphs. A graph is a *path graph* if it is the intersection graph of a family of subpaths of a tree. Lévêque et al. [11] found a characterization of path graphs by minimal forbidden subgraphs.

A variant of path graphs has been defined when the tree is a directed graph. A *directed tree* is a directed graph whose underlying undirected graph is a tree. A *directed subpath* of a directed tree is a subpath whose edges are all directed in the same way. A graph is a *directed path graph* if it is the intersection graph of a family of directed subpaths of a

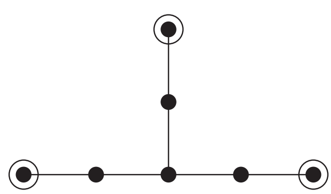

FIGURE 1. Graph containing an asteroidal triple.

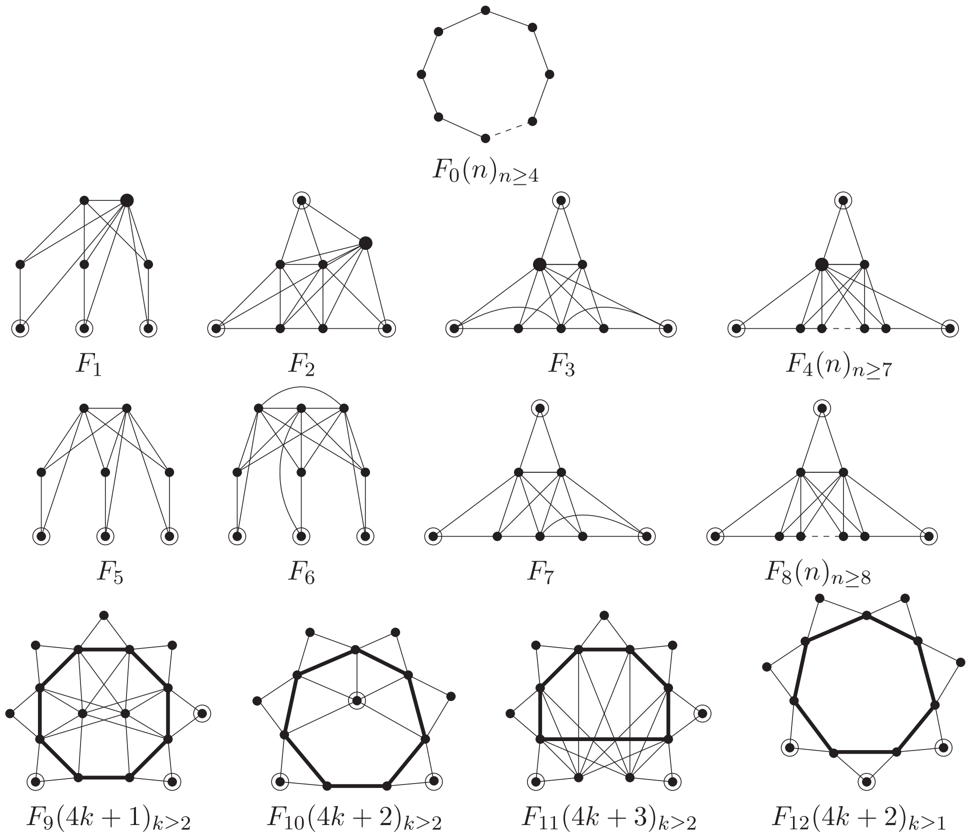

FIGURE 2. Minimal forbidden induced subgraphs for directed path graphs (the vertices in the cycle marked by bold edges form a clique).

directed tree. Panda [15] found a characterization of directed path graphs by forbidden subgraphs (see Fig. 2 where the vertices in the cycle marked by bold edges form a clique and the parameter n in $F_i(n)$ denotes the number of vertices).

Theorem 2 (Panda [15]). *A graph is a directed path graph if and only if it does not contain* $F_0(n)_{n\geq 4}$, F_1, F_2, F_3, $F_4(n)_{n\geq 7}$, F_5, F_6, F_7, $F_8(n)_{n\geq 8}$, $F_9(4k+1)_{k\geq 2}$, $F_{10}(4k+2)_{k\geq 2}$, $F_{11}(4k+3)_{k\geq 2}$ *and* $F_{12}(4k+2)_{k\geq 1}$.

The following inclusions hold by definition:

$$\text{interval graphs} \subset \text{directed path graphs} \subset \text{path graphs} \subset \text{chordal graphs}$$

and these inclusions are strict (see Figs. 4, 5, 6).

In this article, we study directed path graphs. Our result is a characterization of directed path graphs analogous to the theorem of Lekkerkerker and Boland. For this purpose, we introduce the notion of a special connection. Two non-adjacent vertices u and v are linked by a special connection if either they have a common neighbor or they are the endpoints of two vertex-disjoint chordless paths of length three satisfying certain technical conditions. (The complete definition is given in Section 3.) A *special asteroidal triple* in a graph G is an asteroidal triple such that each pair of vertices of the triple is linked by a special connection in G.

Our main result is the following theorem.

Theorem 3. *A chordal graph is a directed path graph if and only if it does not contain a special asteroidal triple.*

In Section 2, we give the definitions and background results needed to prove our main result. In Section 3, we define special connections and establish a property of special connections in clique directed path trees (which are defined in Section 2). In Section 4, we give a proof of our main result using the results of Section 3. Finally, in Section 5, we discuss new problems arising from our work.

2. DEFINITIONS AND BACKGROUND

In a graph G, a *clique* is a set of pairwise adjacent vertices. Let $\mathcal{Q}(G)$ be the set of all (inclusionwise) maximal cliques of G. When there is no ambiguity we will write $\mathcal{Q}$ instead of $\mathcal{Q}(G)$. A vertex in a graph G is called *universal* if it is adjacent to every other vertex of G. Given a vertex v and a set S of vertices, v is called *complete to* S if v is adjacent to every vertex of S. Given two vertices u and v in a graph G, a $\{u,v\}$-*separator* is a set S of vertices of G such that u and v lie in different components of $G\setminus S$ and S is minimal (inclusionwise) with this property. A set is a *separator* if it is a $\{u,v\}$-separator for some u and v in G. Let $\mathcal{S}(G)$ be the set of separators of G. When there is no ambiguity we will write $\mathcal{S}$ instead of $\mathcal{S}(G)$. A classical result [8, 1] (see also [7]) states that, in a chordal graph G, every separator is a clique; moreover, if S is a separator, then there are at least two components of $G\setminus S$ that contain a vertex that is complete to S, and so S is the intersection of two maximal cliques.

A *clique tree* T of a graph G is a tree whose vertices are the members of $\mathcal{Q}$ and such that, for each vertex v of G, those members of $\mathcal{Q}$ that contain v induce a subtree of T, which we will denote by T^v. A classical result of Gavril [5] states that a graph is chordal if and only if it has a clique tree. A *clique path tree* T of G is a clique tree of G such that, for each vertex v of G, T^v is a path. Gavril [6] proved that a graph is a path graph if and only if it has a clique path tree. A *clique directed path tree* T of G is a directed tree such that the underlying undirected tree is a clique path tree of G and for each vertex v of G, the subpath T^v is a directed path. Monma and Wei [14] proved that a graph is a directed path graph if and only if it has a clique directed path tree. A *clique path* T of G is a clique tree of G such that T is a path. A graph is an interval graph if and only if it has a clique path [4]. These results allow us to restrict our attention to intersection models that are clique trees when studying the properties of these graph classes.

For a clique tree T, the *separator of an edge* QQ' of T is defined as $S_{QQ'}=Q\cap Q'$. Note that every edge QQ' satisfies $S_{QQ'}\in\mathcal{S}$; indeed, there exist vertices $v\in Q\setminus Q'$ and $v'\in Q'\setminus Q$ such that the set $S_{QQ'}$ is a $\{v,v'\}$-separator.

If T is a clique tree of G, and Q, Q' are maximal cliques of G then $T[Q,Q']$ denotes the subpath of T of minimum size whose vertices contain Q and Q'.

Given a set $z_1,\ldots,z_r$, $r \geq 2$, of pairwise non-adjacent vertices of G, and a clique tree T of G, the subtrees T^{z_i}, $1 \leq i \leq r$, are disjoint and we can define $T(z_1,\ldots,z_r)$ to be the subtree of T of minimum size that contains at least one vertex of each T^{z_i}. Clearly, the number of leaves of $T(z_1,\ldots,z_r)$ is at most r. Moreover, if $T(z_1,\ldots,z_r)$ has exactly r leaves, then they can be denoted by Q_i, $1 \leq i \leq r$, with $Q_i \in \mathcal{Q}$ and $Q_i \cap \{z_1,\ldots,z_r\} = \{z_i\}$.

We will need the following lemma which is folklore (for example, see [12]).

Lemma 1. *Let G be a chordal graph and z_1, z_2, z_3 three vertices that form an asteroidal triple, then for every clique tree T of G, the subtree $T(z_1, z_2, z_3)$ has exactly 3 leaves.*

For more information about clique trees and chordal graphs, see [2, 7, 13].

3. SPECIAL CONNECTIONS AND CLIQUE DIRECTED PATH TREES

A *special connection* linking u and v is an induced subgraph of one of the following forms (see Fig. 3).

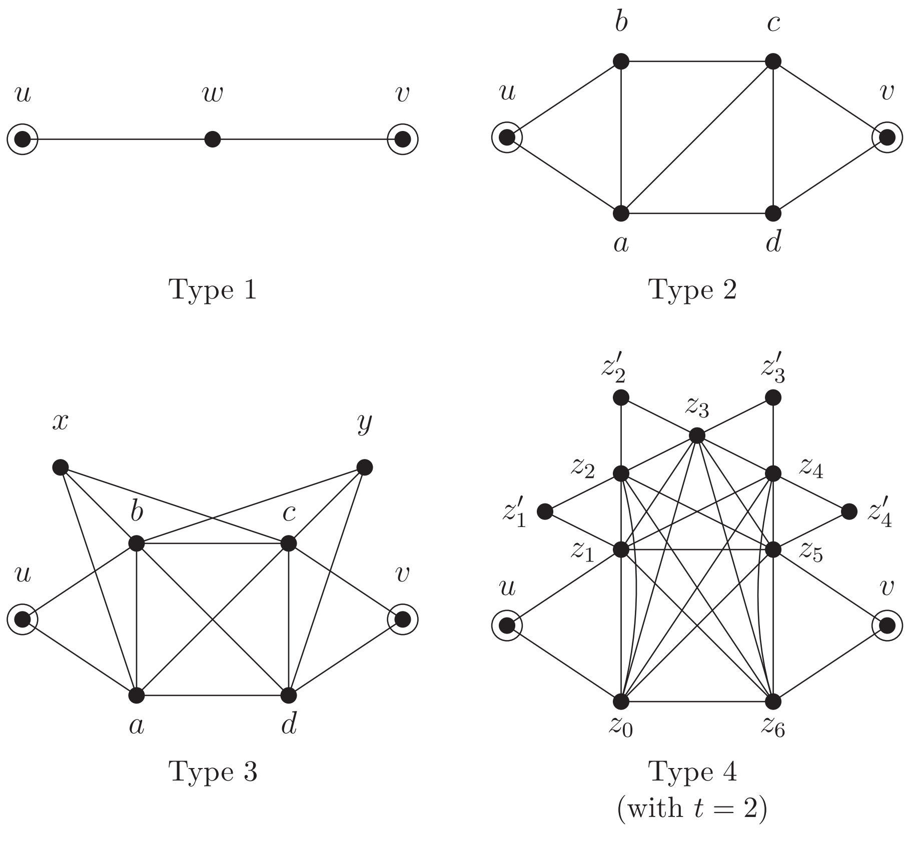

FIGURE 3. The four types of special connections.

- *Type 1*: vertices u,v,w and edges uw,vw.
- *Type 2*: vertices u, v, a, b, c, d and edges ua, ub, vc, vd, ab, bc, cd, da, ac.
- *Type 3*: vertices u, v, a, b, c, d, x, y where vertices a, b, c, d form a clique and there are also edges ua, ub, vc, vd, xa, xb, xc, yb, yc, yd.
- *Type 4*: vertices u, v, $z_0,\dots,z_{2t+2},z'_1,\dots,z'_{2t}$ $(t\geq 1)$ where vertices $z_0,\dots,z_{2t+2}$ form a clique and vertex z'_k $(0\leq k\leq 2t+1$, with $z'_0=u$ and $z'_{2t+1}=v)$ is adjacent to z_k,z_{k+1}.

Note that in special connections of types 2, 3 and 4, vertices u and v are the endpoints of two vertex-disjoint chordless paths of length three.

Special connections are interesting when considering directed path graphs because if u and v are linked by a special connection, then in any directed path graph model T, the subpaths of T corresponding to the vertices forming the special connection have to overlap and they force T to be completely directed in one direction between u and v. More formally:

Lemma 2. *Let G be a directed path graph and let u and v be two non-adjacent vertices that are linked by a special connection. Then, for every clique directed path tree T of G, the subpath $T(u,v)$ is a directed path.*

Proof. Suppose there is a special connection of type 1 between u and v. Let w be a vertex adjacent to both u and v. Then w is in a maximal clique with u and in a maximal clique with v. As T^w is connected, we have $T(u,v)\subseteq T^w$. The subpath T^w is directed by definition of clique directed path tree and so $T(u,v)$ is a directed path.

We can now assume that there is a special connection of type 2, 3 or 4 between u and v. Let a, b, c, d be the four vertices as in the definitions of special connections of type 2 and 3. For special connection of type 4, let $a=z_0$, $b=z_1$, $c=z_{2t+1}$ and $d=z_{2t+2}$. Thus, there are two chordless path u-b-c-v and u-a-d-v.

Let Q_u and Q_v be the two extremities of $T(u,v)$ with $u\in Q_u$ and $v\in Q_v$. The separator S of the edge of $T(u,v)$ incident to Q_u contains at least one vertex of $\{b,c\}$, otherwise u and v are in two different components of $G\backslash S$, contradicting the fact that u-b-c-v is a path. Vertex u is not adjacent to c, so $c\notin Q_u$. Then $c\notin S$ and so $b\in S$. Thus, $Q_u\cap\{b,c\}=\{b\}$. Similarly, we obtain $Q_u\cap\{a,b,c,d\}=\{a,b\}$ and $Q_v\cap\{a,b,c,d\}=\{c,d\}$.

When $T[Q,Q']$ is a directed path of T of length one or more, directed from Q to Q', we write $Q\rightsquigarrow Q'$

Suppose the connection is of type 2. Let Q_b be a maximal clique containing $\{a,b,c\}$ and Q_d a maximal clique containing $\{a,c,d\}$. As b and d are not adjacent, we have $Q_b\cap\{a,b,c,d\}=\{a,b,c\}$ and $Q_d\cap\{a,b,c,d\}=\{a,c,d\}$. Thus, Q_u, Q_v, Q_b, Q_d are distinct. As $c\notin Q_u$, we have $Q_u\notin T[Q_b,Q_d]$. As $b\notin Q_d$, we have $Q_d\notin T[Q_u,Q_b]$. So Q_u, Q_b and Q_d appear in this order along T^a. Similarly, Q_b, Q_d and Q_v appear in this order along T^c. Suppose, by symmetry, that T^a is directed from Q_u to Q_d, i.e. $Q_u\rightsquigarrow Q_b\rightsquigarrow Q_d$. As $Q_b\rightsquigarrow Q_d$ and T^c is directed, we have $Q_d\rightsquigarrow Q_v$ and thus $T(u,v)$ is a directed path.

Suppose the connection is of type 3. Let Q be a maximal clique containing $\{a,b,c,d\}$. Let Q_x be a maximal clique containing $\{a,b,c,x\}$. Let Q_y be a maximal clique containing $\{b,c,d,y\}$. As x is not adjacent to d we have $Q_x\cap\{a,b,c,d\}=\{a,b,c\}$. As y is not adjacent to a we have $Q_x\cap\{a,b,c,d\}=\{b,c,d\}$. Thus Q_u, Q_v, Q_x, Q_y, Q are all distinct.

As $d \notin Q_x$, we have $Q_x \notin T[Q, Q_y]$. As $a \notin Q_y$, we have $Q_y \notin T[Q, Q_x]$. So Q_x, Q and Q_y appear in this order along T^b. As $c \notin Q_u$, we have $Q_u \notin T[Q_x, Q_y]$. As $a \notin Q_y$, we have $Q_y \notin T[Q_x, Q]$. So Q_u, Q_x, Q and Q_y appear in this order along T^b. Similarly, Q_x, Q, Q_y and Q_v appear in this order along T^c. Suppose, by symmetry, that T^b is directed from Q_u to Q_y, i.e. $Q_u \rightsquigarrow Q_x \rightsquigarrow Q_y$. As $Q_x \rightsquigarrow Q_y$ and T^c is directed, we have $Q_y \rightsquigarrow Q_v$ and thus $T(u,v)$ is a directed path.

Suppose the connection is of type 4. Recall that $a = z_0$, $b = z_1$, $c = z_{2t+1}$ and $d = z_{2t+2}$. Let Q be a maximal clique containing the clique $z_0, \ldots, z_{2t+2}$. For $1 \leq k \leq 2t$, let Q_k be a maximal clique containing z'_k, z_k and z_{k+1}. Let $Q_0 = Q_u$ and $Q_{2t+1} = Q_v$. For every k, with $0 \leq k \leq 2t$, as $z_{k+2} \notin Q_k$, we have $Q_{k+1} \notin T[Q_k, Q]$ and as $z_k \notin Q_{k+1}$, we have $Q_k \notin T[Q_{k+1}, Q]$, so vertices Q_k, Q and Q_{k+1} appear in this order along $T^{z_{k+1}}$. We can assume, by symmetry, that T^{z_1} is directed from Q_0 to Q_1, i.e. $Q_0 \rightsquigarrow Q \rightsquigarrow Q_1$. As $Q \rightsquigarrow Q_1$ and T^{z_2} is directed, we have $Q_2 \rightsquigarrow Q$. And so on, for $2 \leq k \leq 2t+1$, the subpath T^{z_k} is directed, so $Q \rightsquigarrow Q_k$ when k is odd and $Q_k \rightsquigarrow Q$ when k is even. So $Q_u = Q_0 \rightsquigarrow Q \rightsquigarrow Q_{2t+1} = Q_v$ and thus $T(u,v)$ is a directed path. ■

4. ASTEROIDAL TRIPLES IN DIRECTED PATH GRAPHS

The graph of Figure 1 is a directed path graph that is minimally not an interval graph. So, directed path graphs may contain asteroidal triples. But one can define a particular type of asteroidal triple that is forbidden in directed path graphs. Recall from Section 1 that a *special asteroidal triple* in a graph G is an asteroidal triple such that each pair of vertices of the triple is linked by a special connection in G. The graph of Figure 4 is an example of a graph that minimally contains a special asteroidal triple. This graph is a path graph which is minimally not a directed path graph. (It is the graph $F_{12}(6)$ of Fig. 2.)

The graph of Figure 5 is another example of a graph that minimally contains a special asteroidal triple. This graph is interesting as it shows that sometimes the path between two vertices of the asteroidal triple that avoids the neighborhood of the third must contain some vertices outside the special connection. The only special connection linking vertices 2 and 3 is $\{a, b, c, d\}$ (a special connection of type 2) and the only path between 2 and 3 that avoids the neighborhood of 1 is 2-b-e-d-3. This graph

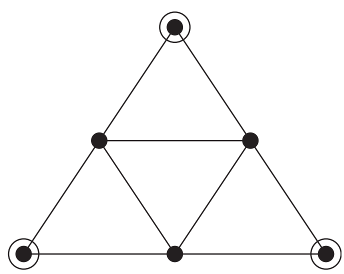

FIGURE 4. A path graph which is minimally not a directed path graph.

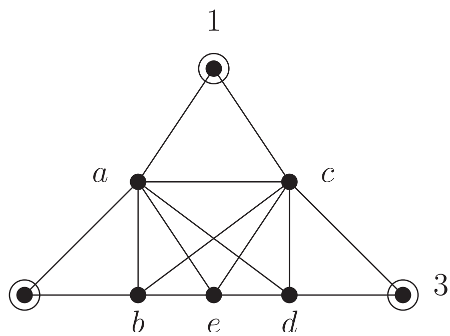

FIGURE 5. A chordal graph which is minimally not a path graph.

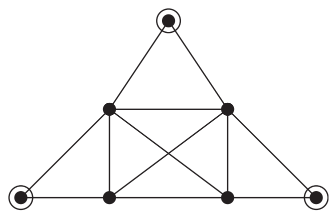

FIGURE 6. A directed path graph which is minimally not an interval graph.

is a chordal graph which is minimally not a path graph. (It is the graph $F_8(8)$ of Figure 2.)

The graph of Figure 6 is an example of a graph that contains an asteroidal triple that is not special. This graph is a directed path graph which is minimally not an interval graph.

We will now prove Theorem 3 which gives a characterization of directed path graphs by forbidden asteroids.

Proof of Theorem 3. ($\Longrightarrow$) Suppose that G is a directed path graph and z_1, z_2, z_3 is a special asteroidal triple of G. Let T be a clique directed path tree of G. By Lemma 1, $T(z_1,z_2,z_3)$ has exactly 3 leaves $Q_i \in \mathcal{Q}$, $1 \le i \le 3$, and further, $Q_i \cap \{z_1,z_2,z_3\} = \{z_i\}$. By Lemma 2, $T(z_1,z_2)$, $T(z_2,z_3)$, $T(z_3,z_1)$ are directed paths of $T(z_1,z_2,z_3)$. Suppose, by symmetry, that $T(z_1,z_2)$ is directed from Q_1 to Q_2. Then $T(z_1,z_3)$ is directed from Q_1 to Q_3, but then $T(z_2,z_3)$ is not a directed path, a contradiction.

($\Longleftarrow$) All chordal graphs of Figure 2 contain a special asteroidal triple; the three vertices forming the asteroidal triple are circled. The graphs F_1, F_2, F_3 and $F_4(n)_{n\ge 7}$ are obtained from a graph containing an asteroidal triple by adding a universal vertex; this universal vertex forms a special connection of type 1 linking each pair of vertices of the asteroidal triple. In the graphs F_5, F_6, F_7 and $F_8(n)_{n\ge 8}$, the special connections are of type 1 or 2. In the graphs $F_9(4k+1)_{k\ge 2}$, $F_{10}(4k+2)_{k\ge 2}$, $F_{11}(4k+3)_{k\ge 2}$ and $F_{12}(4k+2)_{k\ge 1}$, the special connections are of type 1, 2, 3 or 4. So if G is a chordal graph containing no special asteroidal triple, it does not contain $F_1, \ldots, F_{12}$, and so it is a directed path graph by Theorem 2. ■

TABLE I. References to results and open problems.

	Forbidden subgraphs	Forbidden asteroids
Path graphs	[11]	?
Directed path graphs	[15]	Theorem 3
Rooted path graphs	?	?
Interval graphs	[10]	[10]

In the proof of Theorem 3, we use the list of forbidden subgraphs obtained by Panda [15]. It would be nice to find a simple proof of this result, similar to the proof of Theorem 1 presented in [9].

A corollary of Theorems 2 and 3 is the following.

Corollary 1. *The chordal graphs that minimally contain a special asteroidal triple are the graphs* $F_1, F_2, F_3, F_4(n)_{n\geq 7}, F_5, F_6, F_7, F_8(n)_{n\geq 8}, F_9(4k+1)_{k\geq 2}, F_{10}(4k+2)_{k\geq 2}, F_{11}(4k+3)_{k\geq 2}$ *and* $F_{12}(4k+2)_{k\geq 1}$.

5. CONCLUSION

We have defined a particular type of asteroidal triple to obtain a characterization of directed path graphs by forbidden asteroids. One can also try to prove a similar result for path graphs. Path graphs are a superclass of directed path graphs that may contain some special asteroidal triples (as path graphs can contain $F_{12}(4k+2)_{k\geq 1}$). Can one define a particular type of special asteroidal triple that will give a characterization of path graphs by forbidden asteroids?

A *rooted tree* is a directed tree in which the path from a particular vertex r to every other vertex is a directed path; vertex r is called the *root*. A graph is a *rooted path graph* if it is the intersection graph of a family of directed subpaths of a rooted tree. The problem of finding a characterization of rooted path graphs by forbidden subgraphs is still open. In the extended abstract [3], we gave a more general definition of special connection (where types 2, 3 and 4 can be longer) to study both directed path graphs and rooted path graphs at the same time. Here we give a simplier definition of special connection to simplify the characterization theorem for directed path graphs.

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