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Contracting chordal graphs and bipartite graphs to paths and trees [1]

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Abstract

Some of the most well studied problems in algorithmic graph theory deal with modifying a graph into an acyclic graph or into a path, using as few operations as possible. In Feedback Vertex Set and Longest Induced Path, the only allowed operation is vertex deletion, and in Spanning Tree and Longest Path, only edge deletions are permitted. We study the edge contraction variant of these problems: given a graph G and an integer k, decide whether G can be transformed into an acyclic graph or into a path using at most k edge contractions. Both problems are known to be NP-complete in general. We show that on chordal graphs these problems can be solved in $O(n+m)$ and $O(nm)$ time, respectively. On the negative side, both problems remain NP-complete when restricted to bipartite input graphs.

Keywords: Edge contractions, chordal graphs, polynomial time algorithms.

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1 Introduction

Graph modification problems play a central role in algorithmic graph theory, not in the least because they can be used to model many graph theoretical problems that appear in practical applications [11,12,13]. The input of a graph modification problem is a graph G and an integer k, and the question is whether G can be modified in such a way that it satisfies some prescribed property, using at most k operations of a given type. Many classical problems, like Clique and Independent Set, can be formulated as graph modification problems. Some important and well studied problems of this kind ask whether a graph can be modified into an *acyclic graph* or into a *path*, using at most k operations. If the only allowed operation is vertex deletion, these problems are widely known as Feedback Vertex Set and Longest Induced Path, respectively. When only edge deletions are permitted, the problems are called Spanning Tree and Longest Path. With the exception of Spanning Tree, all these problems are known to be NP-complete on general graphs [5].

We study the problems of modifying a graph into an acyclic graph or into a path using only *edge contractions*. The edge contraction operation plays a key role in graph minor theory, and it also has applications in Hamiltonian graph theory, computer graphics, and cluster analysis [10]. The problem of contracting an input graph G to a fixed target graph H has recently attracted a considerable amount of interest, and several results exist for this problem when G or H belong to special graph classes [3,8,9,10]. Rather surprisingly, hardly any results seem to be known on the problem of contracting a graph to *some* graph belonging to a specified graph class, rather than to a single *fixed* target graph. The two problems we study in this paper, which we call Tree-Contractibility and Path-Contractibility, take as input a connected graph G and an integer k, and the question is whether G can be contracted to a tree or to a path, respectively, using at most k edge contractions. Note that contracting a connected graph to a tree is equivalent to contracting it to an acyclic graph, as edge contractions preserve connectivity. Previous results easily imply that both problems are NP-complete in general [1,3].

We show that the problems Tree-Contractibility and Path-Contractibility can be solved on chordal graphs in $O(n+m)$ and $O(nm)$ time, respectively. It is known that Tree-Contractibility is NP-complete on bipartite graphs [7], and we show that the same holds for Path-Contractibility. To relate our results to previous work, we would like to mention that Feedback Vertex Set and Longest Induced Path can be solved in polynomial time on chordal graphs [4,14]. However, it is easy to find examples that show that the set of trees and paths that can be obtained from a chordal graph G by at most k edge contractions might be completely different from the set of trees and paths that can be obtained from G by at most k vertex deletions. As an interesting contrast, LONGEST PATH remains NP-complete on chordal graphs [6].

2 Definitions and notation

All the graphs considered in this paper are undirected, finite and simple. We use n and m to denote the number of vertices and edges of the *input* graph of the problem or the algorithm under consideration. For any set $S \subseteq V(G)$, we write $N_G(S) = \cup_{u \in S} N_G(u) \setminus S$ and $N_G[S] = N_G(S) \cup S$. A vertex v is called *simplicial* if the set $N_G[v]$ is a clique. If H is a subgraph of G and $v \in N_G(V(H))$, then we refer to the vertices in $N_G(v) \cap V(H)$ as the *H-neighbors* of v. The *distance* $d_G(u, v)$ between two vertices u and v in G is the number of edges in a shortest path between u and v, and $diam(G) = \max_{u,v \in V(G)} d_G(u, v)$. For any two vertices u and v of a path P in G, we write uPv to denote the subpath of P from u to v in G. We write P_ℓ and C_ℓ to denote the chordless path and the chordless cycle on ℓ vertices, respectively.

The *contraction* of edge $e = uv$ in G removes u and v from G, and replaces them by a new vertex, which is made adjacent to precisely those vertices that were adjacent to at least one of the vertices u and v. Instead of speaking of the contraction of edge uv, we sometimes say that a vertex u is *contracted on v*, in which case we use v to denote the new vertex resulting from the contraction. Let $S \subseteq V(G)$ be a connected set. If we repeatedly contract a vertex of $G[S]$ on one of its neighbors in $G[S]$ until only one vertex of $G[S]$ remains, we say that we *contract S into a single vertex*. We say that a graph G can be *k-contracted* to a graph H, with $k \leq n - 1$, if H can be obtained from G by a sequence of k edge contractions. Note that if G can be k-contracted to H, then H has exactly k fewer vertices than G has. We simply say that a graph G can be *contracted* to H if it can be k-contracted to H for some $k \geq 0$. Let H be a graph with vertex set $\{h_1, \ldots, h_{|V(H)|}\}$. Saying that a graph G can be contracted to H is equivalent to saying that G has a so-called *H-witness structure* $\mathcal{W}$, which is a partition of $V(G)$ into *witness sets* $W(h_1), \ldots, W(h_{|V(H)|})$ such that each witness set induces a connected subgraph of G, and such that for every two $h_i, h_j \in V(H)$, witness sets $W(h_i)$ and $W(h_j)$ are adjacent in G if and only if h_i and h_j are adjacent in H. By contracting each of the witness sets into a single vertex, which can be done due to the connectivity of the witness sets, we obtain the graph H. An H-witness structure of G is, in general, not uniquely defined (see Fig. 1).

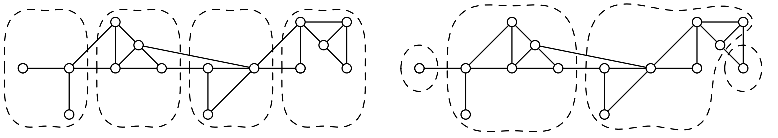

Fig. 1. Two P_4-witness structures of a graph.

3 Contracting chordal graphs

A graph is *chordal* if it does not contain a chordless cycle on at least four vertices as an induced subgraph. It is easy to show that the class of chordal graphs is closed under edge contractions, and we use this observation throughout this section. Let $\mathcal{W}$ be an H-witness structure of a graph G. Levin, Paulusma and Woeginger [10] observed that if u and v are two vertices of G and x and y are two vertices of H such that $u \in W(x)$ and $v \in W(y)$, then $d_G(u,v) \geq d_H(x,y)$. This observation immediately implies that a graph G cannot be contracted to a chordless path of length more than $diam(G)$. We show that if G is chordal, then G can be contracted to a chordless path of length $diam(G)$. Note that this is not the case for every graph: for example, the graph C_ℓ has diameter $\lfloor \ell/2 \rfloor$, but cannot be contracted to a chordless path of length more than 1.

Theorem 3.1 *Every connected chordal graph G can be contracted to a chordless path of length $diam(G)$.*

Proof (Sketch). Let u and v be two vertices of a connected chordal graph G such that $d_G(u,v) = diam(G)$, and let P be a shortest path from u to v. Let $R = G - N_G[V(P)]$. We can repeatedly contract a vertex in $V(R)$ on one of its neighbors such that the resulting graph G' is isomorphic to $G[N_G[V(P)]]$. It can be shown that every vertex in $Q = N_G(V(P))$ has exactly one, two, or three P-neighbors, and that these neighbors are consecutive vertices of P. We repeatedly contract every vertex of Q with exactly one P-neighbor on this P-neighbor. Then, as long as there is a vertex in Q with exactly three P-neighbors, we contract such a vertex on its middle P-neighbor. If all the vertices in Q have exactly two P-neighbors, we arbitrarily pick a vertex in Q and contract it on either one of its P-neighbors. If this causes vertices in Q to have three P-neighbors, we deal with those vertices first. It can be shown that, after each of the contractions described above, P is still a shortest path from u to v. After all vertices of Q have been contracted on one of their P-neighbors, we end up with P, which has length $diam(G)$. $\square$

Corollary 3.2 *A connected chordal graph G can be k-contracted to a chordless path if and only if $k \geq n - diam(G)$.*

Corollary 3.3 Path-Contractibility *can be solved in* $O(nm)$ *time on chordal graphs.*

We now turn our attention to Tree-Contractibility on chordal graphs. We say that a tree T is *optimal* for G if G can be contracted to T, but cannot be contracted to any tree with strictly more vertices than T.

Lemma 3.4 *If a connected graph* G *has a simplicial vertex* v*, then* G *has a* T*-witness structure* $\mathcal{W}$ *for some optimal tree* T*, such that* $W(x) = \{v\}$ *for some vertex* x *of* T*.*

Lemma 3.4 naturally suggests an algorithm for contracting a connected chordal graph to an optimal tree.

Theorem 3.5 Tree-Contractibility *can be solved in* $O(n+m)$ *time on chordal graphs.*

Proof (Sketch). We repeatedly find a simplicial vertex v, contract its neighborhood into a single vertex, and remove v from the graph. We continue this process until we have removed all vertices. By applying all the edge contractions that have been performed during this procedure to the original graph G, we find an optimal tree for G. It is clear that this algorithm runs in $O(nm)$ time. Using an implementation of clique trees using *difference sets* [2], we can show that our algorithm runs in linear time. □

Note that the problem of contracting a chordal graph to a tree is equivalent to the problem of contracting a chordal graph to a bipartite graph.

4 Contracting bipartite graphs

The following result was implicitly proved by Heggernes et al. [7].

Theorem 4.1 Tree-Contractibility *is NP-complete on bipartite graphs.*

Using a reduction from the NP-complete problem Hypergraph 2-Colorability, we can show that the problem of deciding whether a bipartite graph can be contracted to a chordless path on ℓ vertices is NP-complete for any fixed $\ell \geq 6$. This, together with the observation that a graph G can be k-contracted to a path if and only if G can be contracted to P_{n-k}, yields the following result.

Theorem 4.2 Path-Contractibility *is NP-complete on bipartite graphs.*

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