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Asteroids in rooted and directed path graphs [1]

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Abstract

An asteroidal triple is a stable set of three vertices such that each pair is connected by a path avoiding the neighborhood of the third vertex. Asteroidal triples play a central role in a classical characterization of interval graphs by Lekkerkerker and Boland. Their result says that a chordal graph is an interval graph if and only if it contains no asteroidal triple. In this paper, we prove an analogous theorem for directed path graphs which are the intersection graphs of directed paths in a directed tree. For this purpose, we introduce the notion of a strong path. Two non-adjacent vertices are linked by a strong path if either they have a common neighbor or they are the endpoints of two vertex-disjoint chordless paths satisfying certain conditions. A strong asteroidal triple is an asteroidal triple such that each pair is linked by a strong path. We prove that a chordal graph is a directed path graph if and only if it contains no strong asteroidal triple. We also introduce a related notion of asteroidal quadruple, and conjecture a characterization of rooted path graphs which are the intersection graphs of directed paths in a rooted tree.

Keywords: intersection graph, directed path graph, asteroidal triple.

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1 Introduction

A *hole* is a chordless cycle of length at least four. A graph is a *chordal graph* if it contains no hole as an induced subgraph. Gavril [1] proved that a graph is chordal if and only if it is the intersection graph of a family of subtrees of a tree. In this paper, whenever we talk about the intersection of subgraphs of a graph we mean that the *vertex sets* of the subgraphs intersect.

A graph is an *interval graph* if it is the intersection graph of a family of intervals on the real line; or equivalently, the intersection graph of a family of subpaths of a path. An *asteroidal triple* in a graph G is a set of three non-adjacent vertices such that for any two of them, there exists a path between them in G that does not intersect the neighborhood of the third. The graph of Figure 1 is an example of a graph that minimally contains an asteroidal triple; the three vertices forming the asteroidal triple are circled.

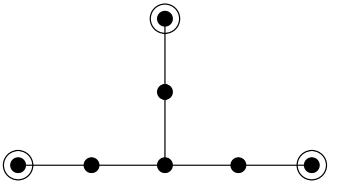

Fig. 1. Graph containing an asteroidal triple

The following classical theorem was proved by Lekkerkerker and Boland.

Theorem 1.1 ([2]) *A chordal graph is an interval graph if and only if it contains no asteroidal triple.*

Lekkerkerker and Boland [2] derived from Theorem 1.1 the list of minimal forbidden subgraphs for interval graphs (see Figure 8).

The class of path graphs lies between interval graphs and chordal graphs. A graph is a *path graph* if it is the intersection graph of a family of subpaths of a tree. Lévêque, Maffray and Preissman [3] found a characterization of path graphs by forbidden subgraphs (see Figure 6).

Two variants of path graphs have been defined when the tree is a directed graph. A *directed tree* is a directed graph whose underlying undirected graph is a tree. A graph is a *directed path graph* if it is the intersection graph of a family of directed subpaths of a directed tree. Panda [4] found a characterization of directed path graphs by forbidden subgraphs (see Figure 7). A *rooted tree* is a directed tree in which the path from a particular vertex r to every other vertex is a directed path; vertex r is called the *root*. A graph is a *rooted path graph* if it is the intersection graph of a family of directed subpaths of a rooted tree.

The problem of finding a characterization of rooted path graphs by forbidden subgraphs is still open.

Clearly, we have the following inclusions between the classes considered :

$$\text{interval} \subset \text{rooted path} \subset \text{directed path} \subset \text{path} \subset \text{chordal}$$

In this paper, we study directed path graphs and rooted path graphs. Our main result is a characterization of directed path graphs analogous to the theorem of Lekkerkerker and Boland. For this purpose, we introduce the notion of a strong path. Two non-adjacent vertices u and v are linked by a strong path if either they have a common neighbor or they are the endpoints of two vertex-disjoint chordless paths satisfying certain technical conditions. (The complete definition is given in Section 2.) A *strong asteroidal triple* in a graph G is an asteroidal triple such that each pair of vertices of the triple is linked by a strong path in G.

Our main result is the following theorem.

Theorem 1.2 *A chordal graph is a directed path graph if and only if it contains no strong asteroidal triple.*

2 Asteroidal triples in directed path graphs

The graph of Figure 1 is a directed path graph that is minimally not an interval graph; in fact, it is a rooted path graph. (It is the graph F_{18} of Figure 8.) So, directed path graphs may contain asteroidal triples. But one can define a particular type of asteroidal triple that is forbidden in directed path graphs.

Two non-adjacent vertices u and v are linked by a *strong path* if either they have a common neighbor, or there exist three sets of distinct vertices $X = \{x_1, \dots, x_r\}, Y = \{y_1, \dots, y_s\}, Z$, $(r, s \geq 2,\ |Z| \geq 0)$, such that u-x_1-$\cdots$-x_r-v and u-y_1-$\cdots$-y_s-v are two chordless paths where if $x_i, x_{i+1}, y_j, y_{j+1}$ $(1 \leq i < r,\ 1 \leq j < s)$ is a clique of size four, one of the following is satisfied with $\{l_1, l_2\} = \{x_i, y_j\}$ and $\{r_1, r_2\} = \{x_{i+1}, y_{j+1}\}$:

- *Attachment of type 1 on* $\{l_1, l_2\}, \{r_1, r_2\}$ *:* There exist two non-adjacent vertices z, z' of Z such that vertex z is adjacent to l_1, l_2, r_1 and not r_2 and vertex z' is adjacent to r_1, r_2, l_1 and not l_2.
- *Attachment of type 2 on* $\{l_1, l_2\}, \{r_1, r_2\}$ *:* There exist $4t+3$ $(t \geq 0)$ vertices $z_1, \dots, z_{2t+1}, z'_1, \dots, z'_{2t+2}$ of Z such that vertices $l_1, l_2, r_1, r_2, z_1, \dots, z_{2k+1}$ form a clique Q, vertices $z_1, \dots, z_{2t+1}$ are adjacent to exactly l_1, l_2, r_1, r_2 on $X \cup Y \cup \{u, v\}$, vertices $z'_1, \dots, z'_{2t+2}$ form a stable set, and vertex z'_k $(1 \leq k \leq 2t+2)$ is adjacent to exactly z_{k-1}, z_k on $Q \cup X \cup Y \cup \{u, v\}$ (with

$z_0 = l_1$ and $z_{2t+2} = r_1$).

Recall from Section 1 that a *strong asteroidal triple* in a graph G is an asteroidal triple such that each pair of vertices of the triple are linked by a strong path in G. The graph of Figure 2 is example of a graph that minimally contains a strong asteroidal triple. This graph is a path graph which is minimally not a directed path graph. (It is the graph $F_{17}(6)$ of Figure 7, and also $F_{21}(6)$ of Figure 8.)

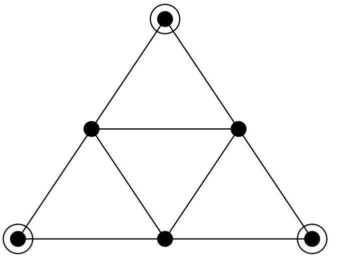

Fig. 2. Graph containing a strong asteroidal triple

The graph of Figure 3 is another example of a graph that minimally contains a strong asteroidal triple. This graph is interesting as it shows that sometimes the path between two vertices of the asteroidal triple that avoids the neighborhood of the third must contain some vertices outside the strong path. The only strong path linking 2 and 3 is $X = \{x_1, x_2\}, Y = \{y_1, y_2\}, Z = \emptyset$ and the only path between 2 and 3 that avoids the neighborhood of 1 is y_1-t-x_2. This graph is a chordal graph which is minimally not a path graph. (It is the graph $F_{10}(8)$ of Figures 6, 7 and $F_{21}(8)$ of Figure 8.)

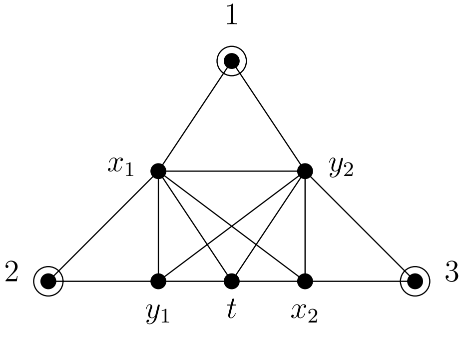

Fig. 3. Graph containing a strong asteroidal triple

The graph of Figure 1 is an example of a graph that contains an asteroidal triple that is not strong as for two vertices of the asteroidal triple, there is no common neighbor and no pair of disjoint paths between them. The graph of

Figure 4 is another example of a graph that contains an asteroidal triple that is not strong. In this graph, there exist two disjoint paths $\{x_1, x_2\}, \{y_1, y_2\}$ between 2 and 3 but x_1, x_2, y_1, y_2 is a clique of size four and there are no vertices that can play the role of Z in the definition of strong path. This graph is a rooted path graph which is minimally not an interval graph. (It is the graph $F_{21}(7)$ of Figure 8.)

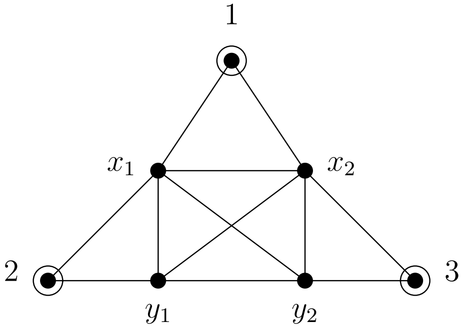

Fig. 4. Graph containing an asteroidal triple that is not strong

A corollary of Theorem 1.2 and [4] is the following.

Corollary 2.1 *The chordal graphs that minimally contain a strong asteroidal triple are the graphs* F_1, F_3, F_4, $F_5(n)_{n\geq 7}$, F_6, F_7, F_9, $F_{10}(n)_{n\geq 8}$, $F_{13}(4k+1)_{k\geq 2}$, $F_{15}(4k+2)_{k\geq 2}$, $F_{16}(4k+3)_{k\geq 2}$, $F_{17}(4k+2)_{k\geq 1}$.

3 Asteroidal quadruples in rooted path graphs

The notion of asteroidal triple can be generalized to four vertices. An *asteroidal quadruple* in a graph G is a set of four vertices such that any three of them is an asteroidal triple. The graph of Figure 5 is an example of a graph that minimally contains an asteroidal quadruple.

The graph of Figure 5 is a rooted path graph, so a rooted path graph may contain asteroidal quadruples. But one can define a particular type of asteroidal quadruple that is forbidden in rooted path graphs.

One can try to use the notion of strong asteroidal triple to define a *strong asteroidal quadruple* as a set of four vertices such that any three of them is an strong asteroidal triple. This is not interesting for our purpose because then every graph that contains a strong asteroidal quadruple also contains a strong asteroidal triple. And, by Theorem 1.2, we already know that directed path graphs and thus rooted path graphs contain no strong asteroidal triple.

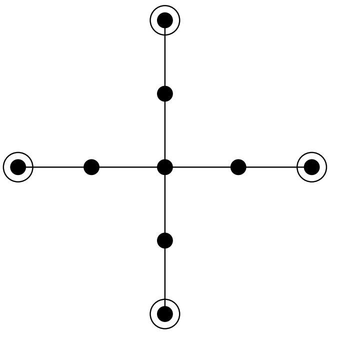

Fig. 5. Graph containing an asteroidal quadruple

One can define another four-vertex variant of asteroidal triple that will be useful. A *weak asteroidal triple* in a graph G is an asteroidal triple such that two vertices of the asteroidal triple are linked by a strong path in G. The difference from the definition of a strong asteroidal triple is that we do not expect that there is a strong path linking any two of the three vertices but just linking two of them.

Now we can generalize this notion to four vertices. A *weak asteroidal quadruple* is a set of four vertices such that any three of them is a weak asteroidal triple. Weak asteroidal quadruples are interesting when considering directed path graphs because of the following theorem.

Theorem 3.1 *A rooted path graph contains no weak asteroidal quadruple.*

By Theorems 1.2 and 3.1, we know that a rooted path graph contains no hole, no strong asteroidal triple and no weak asteroidal quadruple. We conjecture that the converse is also true.

Conjecture 3.2 *A chordal graph is a rooted path graph if and only if it contains no strong asteroidal triple and no weak asteroidal quadruple.*

If Conjecture 3.2 is true, it will give a characterization of rooted path graphs analogous to our Theorem 1.2 on directed path graphs and to Lekkerkerker and Boland's characterization of interval graphs.

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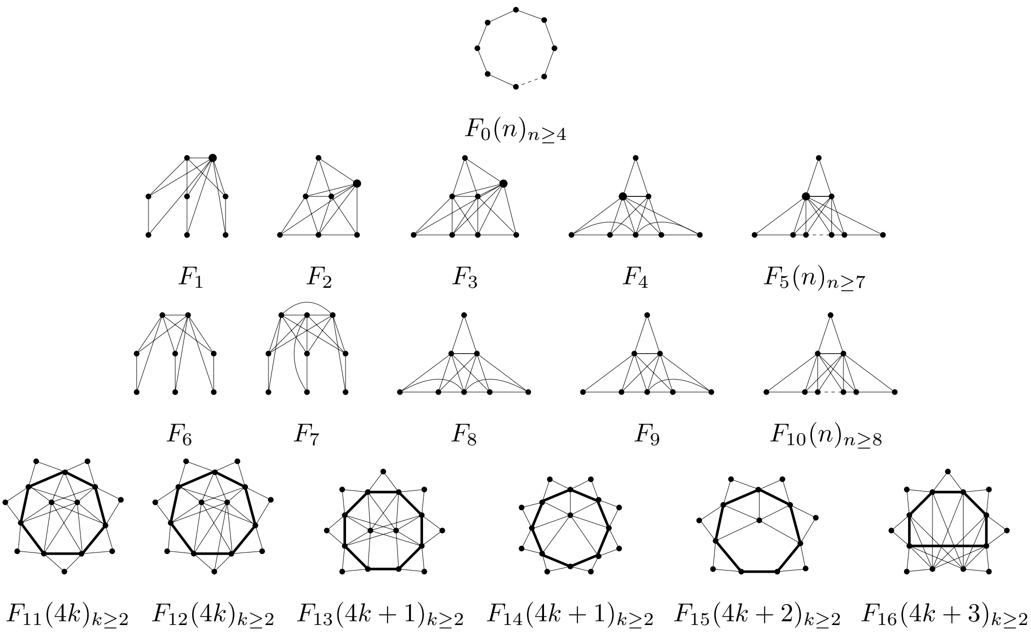

Fig. 6. Minimal forbidden induced subgraphs for path graphs (bold edges form a clique)

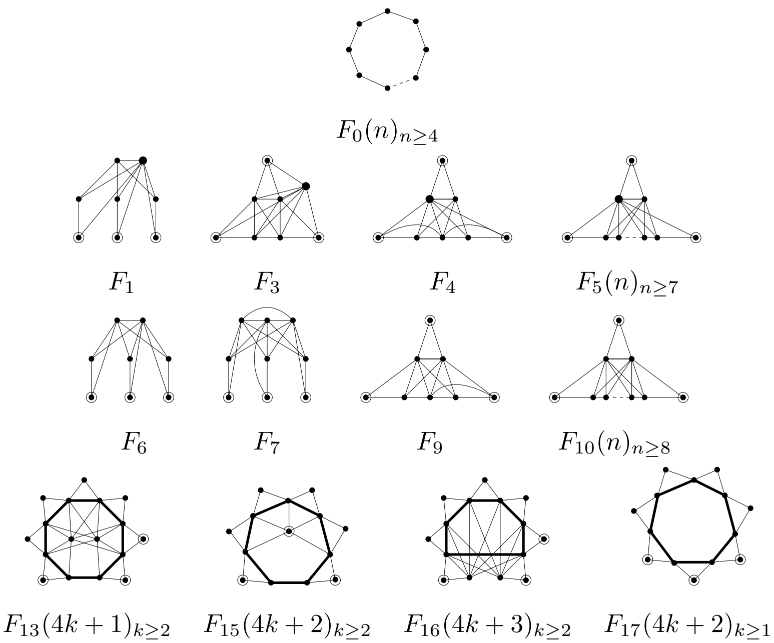

Fig. 7. Minimal forbidden induced subgraphs for directed graphs (bold edges form a clique)

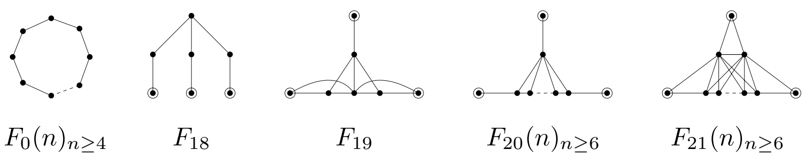

Fig. 8. Minimal forbidden induced subgraphs for interval graphs