A Computational Geometry-based Local Search Algorithm for Planar Location Problems*

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Abstract. Constraint-based local search is an important paradigm in the field of constraint programming, particularly when considering very large optimisation problems. We are motivated by applications in areas such as telecommunications network design, warehouse location and other problems in which we wish to select an optimal set of locations from a two dimensional plane. The problems we are interested in are so large that they are ideal candidates for constraint-based local search methods. Maintaining the objective function incrementally is often a key element for efficient local search algorithms. In the case of two dimensional plane problems, we can often achieve incrementality by exploiting computational geometry. In this paper we present a novel approach to solving a class of placement problems for which Voronoi cell computation can provide an efficient form of incrementality. We present empirical results demonstrating the utility of our approach against the current state of the art.

1 Introduction

We are motivated by applications in areas such as telecommunications network design, warehouse location and other problems in which we wish to select an optimal set of locations from a two dimensional plane. Local search algorithms have proved very efficient in this area and we are interested in the design of efficient incremental mechanisms involving closest point problems. We investigate a restriction of this class of problems in which the cost of allocating a client to a facility is assumed to be proportional to Euclidean distance. This assumption initially comes from a practical problem encountered by the authors and focused on the design of resilient long-reach passive optical networks [10]. This restriction is however relevant in other application domains, in particular spatial clustering problems such as the one tackled in [12]. The core problem in this application[1] is referred to as the Single Coverage Problem and defined as follows:

* This work is supported by Science Foundation Ireland Grant No. 10/CE/I1853.

[1] We simply present the core problem (which remains NP-complete even if the other constraints of the real application are not considered) to facilitate the process of establishing connections between our problem and problems studied in the literature.

Definition 1 (The Single Coverage problem). *Given a constant κ and a set $S = \{p_i | 1 \leq i \leq m\}$ of points in a two dimensional space where each point p_i is associated with a weight b_i, the Single Coverage problem (SCP) is to decide whether there exists a set $W \subset S$ of cardinality p such that the weighted sum of the distances from the points in $S - W$ to their closest points in W is less than or equal to κ, i.e., $\sum_{p_i \in S-W} b_i \times \min_{p_j \in W} d_{ij} \leq \kappa$.*

S is referred to as the set of clients and W as the set of facilities. Moreover, d_{ij} denotes the Euclidean distance between the points p_i and p_j. The single coverage problem is strongly related to location problems in a plane such as the uncapacitated warehouse location problem (UWLP) [5]. Typically, the SCP differs from the UWLP in three respects: the transportation cost is proportional to the Euclidean distance (although this is usually the case in practice, UWLP does not make any assumption about the transportation costs); there is no fixed cost of opening a warehouse; and the number of opened warehouses is bounded by p. Another, and even closer, problem is the p-median problem [8]. This latter problem only differs from SCP because the similarity or dissimilarity measure used by p-median problems is not restricted to the Euclidean distance. A vast literature deals with location problems in the plane, the survey [6] presents spatial clustering problems.

State of the art algorithms for solving large-scale p-median (k-medoid) or uncapacitated warehouse location problems rely on local search. Genetic algorithms [9], and tabu search in particular have been very successful [11]. We describe in Sections 2 and 3 a local search algorithm for the single coverage problem directly inspired by this previous work. In particular the incrementality of the neighborhood has been described in detail [11, 16, 15, 1]. In Section 4, we present a novel approach to improving the complexity of the incremental algorithms presented in the previous section when the cost is proportional to Euclidean distance. We present the empirical evaluation in Section 6. The approaches are evaluated using two sets of experiments. In the first set of experiments we investigated the behavior of the proposed algorithms on randomly generated data, particularly focusing on benchmarking the primitive operations. In the second set, we considered eight SCP cases where half of them corresponded to real cases coming from network operators in Ireland and UK and the other half corresponded to cases randomly generated. Our experimental evaluation confirmed the superiority observed in the complexity analysis.

2 The Tabu Search

We denote the set of current facilities by W and the current set of clients/nodes by $C = S - W$. We use i as an index for clients whereas j refers to facilities.

Neighborhood. The neighborhood is defined by moving a facility from one location to another. This move is performed in two steps, a facility is closed first and another one is opened. Closing a facility involves removing a point p_j from W.

Opening a facility involves adding a point p_i from C to W. The objective function is evaluated by ensuring that clients are always connected to their closest facilities. This invariant is maintained throughout the search. This neighborhood is used by the main algorithm for k-medoid, namely the Partitioning Around Medoids (PAM) algorithm [8]. However, PAM does not include an incremental evaluation of the moves. Therefore we distinguish the two steps (opening and closing) to develop an incremental scheme using [11] for each of these steps separately. This incremental evaluation of the moves is achieved by maintaining Δ_i^+ (the variation of cost due to adding p_i in W (opening a facility)) and Δ_i^- (the variation of cost due to removing p_i from W (closing a facility)).

Search. The initial p facilities are chosen randomly. The tabu mechanism is very simple. It prevents a point that was a facility in the last t iterations, where t is the length of the tabu-list, from becoming a facility again. The tabu-list is denoted T in Algorithm 1. The first improving move found is performed. If no improving move exists, the facility to close is chosen randomly and the new best location for this facility is opened. After a number of non-improving iterations, the search is restarted from p random facilities.

Algorithm 1 presents the general scheme of the tabu search. It assumes that two methods are available for opening and closing a facility (resp. OpenFacility and CloseFacility) while incrementally maintaining the value of the objective function (denoted *obj*) and Δ^+. It is not useful to maintain Δ^- incrementally for this specific neighborhood, where all opened nodes are closed to evaluate each move (Line 8 is only using Δ^+). Δ^- would be useful in a more general context when the closing and opening operations can be done independently, e.g., warehouse location. This algorithm is very like the PAM algorithm; the only difference would be that PAM is selecting the best move rather than the first improving one. However our algorithm is enhanced with the incremental mechanisms, and the tabu metaheuristic, introduced in warehouse location for a similar neighborhood. We believe it is the best starting point for our application and purpose.

3 Incremental Neighborhood

Maintaining the objective function incrementally is a key element for efficient local search algorithms [11]. When moving a facility from one location to another, only a small subset of the clients are reallocated: clients that have lost their current closest facilities and clients that have gained a new closest facility. The cost is not affected by the other clients. Optimal incremental algorithms have been published for opening and closing operations in the context of warehouse location [16, 11, 15]. We present [11] in detail as we will build upon it. The data structures needed to develop the incremental approach and maintain Δ^+ and Δ^- are the following: a_i^1 is the closest facility to client i so that $a_i^1 = \arg\min_{p_j \in W} d_{ij}$; a_i^2 is the second closest facility to client i so that $a_i^2 = \arg\min_{p_j \in W | p_j \neq a_i^1} d_{ij}$; Q_i is a priority queue storing all the current facilities W ordered by increasing distance from i. Consequently a_i^1 and a_i^2 are the first two elements of this queue.

Algorithm 1 TabuSearch()

```
1. Initialize W randomly, C = S − W,
3. While (end condition not reached)
4.    p*_j = −1, bestDelta = ∞, cobj = obj
5.    For each p_j ∈ W − T and as long as bestDelta > 0
6.      CloseFacility(p_j) // updates obj and all Δ+_i incrementally
7.      p_ibest = arg min_{p_i∈C−T}(Δ+_i)
8.      If (Δ+_ibest + (cobj − obj)) < bestDelta
9.        p*_j = p_j, bestDelta = Δ+_ibest + (cobj − obj)
10.     OpenFacility(p_j) // updates obj and all Δ+_i incrementally
11.   If (bestDelta > 0)
12.     p*_j = a random point in W − T
13.   CloseFacility(p*_j)
14.   OpenFacility (arg min_{p_i∈C−T}(Δ+_i))
15.   update tabu list T
```

The variations of the objective function due to closing and opening a facility i are initialized as follows:

$$\Delta_i^- = \sum_{p_j \in S | a_j^1 = p_i} b_j \times (d_{j,a_j^2} - d_{ji}) \tag{1}$$

$$\Delta_i^+ = -\sum_{p_j \in C} b_j \times \max(0, d_{j,a_j^1} - d_{ji}) \tag{2}$$

When closing a facility i, we need to add to the objective function the cost of disconnecting each point connected to p_i and re-connecting them to their second closest facility. Therefore, we add d_{j,a_j^2} and remove d_{ji}. Similarly when opening a facility i, each point p_j of C that is closer to this new facility than to its current closest facility ($d_{j,a_j^1} > d_{ji}$) needs to be disconnected and re-connected decreasing the objective function by the quantity $d_{j,a_j^1} - d_{ji}$. Notice that Δ_i^+ is at most zero (opening never degrades the overall objective function) and Δ_i^- is at least zero (closing always degrades the overall objective function). In what follows we will refer to $d_{j,a_j^2} - d_{ji}$ as the contribution of p_j by which Δ_i^- increases. Similarly, we will say that $d_{j,a_j^1} - d_{ji}$ is the contribution of p_j by which Δ_i^+ decreases. For the sake of clarity we will assume that all the b_i are equal to 1 in the following; the algorithms presented remain identical with general weights. It is simply a matter of multiplying the distance by the weight.

3.1 Closing a Facility

Algorithm 2 presents the incremental maintenance of the data structures, in particular Δ^+ and Δ^-, when closing a given facility p_j.

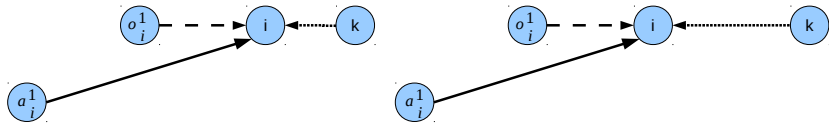
For each client p_i of C, the priority queue, Q_i, is maintained (Line 2). The previous values of the closest and second closest facilities of p_i are saved in o_i^1

Algorithm 2 CLOSEFACILITY(p_j)

1. **For each** $p_i \in S$ **do**
2. remove p_j from Q_i
3. $o_i^1 = a_i^1,\ o_i^2 = a_i^2$
4. $a_i^1 = Q_i.getFirst(),\ a_i^2 = Q_i.getSecond()$
5. **If** $(o_i^1 \neq a_i^1) \vee (o_i^2 \neq a_i^2)$ **do**
6. $\Delta^-_{o_i^1} = \Delta^-_{o_i^1} - (d_{i,o_i^2} - d_{i,o_i^1})$
7. $\Delta^-_{a_i^1} = \Delta^-_{a_i^1} + (d_{i,a_i^2} - d_{i,a_i^1})$
8. **If** $(o_i^1 \neq a_i^1)$ **do**
9. **For each** $p_k \in S$ such that $d_{i,k} < d_{i,a_i^1}$ **do**
10. **If** $(d_{i,k} < d_{i,o_i^1})\ \Delta_k^+ = \Delta_k^+ - (d_{i,a_i^1} - d_{i,o_i^1})$
11. **Else** $\Delta_k^+ = \Delta_k^+ - (d_{i,a_i^1} - d_{i,k})$
12. $W = W - \{p_j\},\ C = C \cup \{p_j\}$

and o_i^2 respectively (Line 3). The closest and second closest facilities of p_i are then updated in a_i^1 and a_i^2 using Q_i respectively (Line 4). Lines 5 to 11 deal with the update of Δ^+ and Δ^-. When a facility p_j is closed either the closest facility of p_i can change, or the second closest facility of p_i can change, or none of them changes. Only the points p_i which have a new closest or second closest facility can trigger the changes of the values of Δ^-. Line 6 simply removes from Δ^- the previous contribution of p_i to its old closest facility and Line 7 adds the new contribution of p_i to its new closest facility.

Lines (8–11) update Δ_k^+ with respect to the contribution of i. From Equation (2) recall that the contribution of i for Δ_k^+ can change only when the closest facility of i changes, i.e., when $o_i^1 \neq a_i^1$ (Line 8) and when $d_{i,k} < d_{i,a_i^1}$ (Line 9). Therefore, the iteration is performed on a pre-computed list of points k sorted by distance from i as long as the criteria $d_{i,k} < d_{i,a_i^1}$ holds. If k is closer to i than o_i^1 (i.e., $d_{i,k} < d_{i,o_i^1}$), as shown in Figure 1(a), then it follows that the contribution of i to Δ_k^+ is non-zero. Therefore, the previous contribution, $d_{i,o_i^1} - d_{i,k}$, should be replaced by the new contribution $d_{i,a_i^1} - d_{i,k}$, which is effectively the difference between $d_{i,a_i^1} - d_{i,o_i^1}$ (Line 10). If k is not closer to i than o_i^1 as shown in Figure 1(b) then the contribution of i to Δ_k^+ is 0. Therefore, Δ_k^+ is updated with the new contribution of i (Line 11).



(a) i was already contributing to Δ_k^+ (b) i was not contributing to Δ_k^+

Fig. 1. The two scenarios for a node i that contributes to Δ_k^+ in Algorithm 2. The old association is drawn with a dashed line, the new one is drawn with a continuous line and the potential association is drawn with a dotted line.

Algorithm 3 OpenFacility(p_j)

```
1.       For each p_i ∈ S do
2.         add p_j to Q_i
3 → 7.     identical to Algorithm 2
8.         If (o_i^1 ≠ a_i^1) do
9.           For each p_k ∈ S such that d_{i,k} < d_{i,o_i^1} do
10.            If (d_{i,k} < d_{i,a_i^1}) Δ_k^+ = Δ_k^+ + (d_{i,o_i^1} − d_{i,a_i^1})
11.            Else Δ_k^+ = Δ_k^+ + (d_{i,o_i^1} − d_{i,k})
12.      W = W ∪ {p_j}, C = C − {p_j}
```

We now consider the complexity of Algorithm 2 for closing a facility. Updating one priority queue is done in $\mathcal{O}(\log(p))$, using an implementation based on heaps, and this has to be done for all points thus Lines 1-2 imply a $\mathcal{O}(m \log(p))$ complexity. Updating Δ^- is then done in constant time whereas updating Δ^+ is achieved in time linear in the number of points p_i whose closest facility has changed. This complexity is optimal as it is necessary to consider all the updated points, and they cannot cancel out since d_{i,a_i^1} is always increasing ($d_{i,a_i^1} \geq d_{i,o_i^1}$). The pre-computed lists of points sorted by distance from any other points (Line 9) requires $O(m^2)$ space which can be an issue when solving very large problems. However, in practice the cost is dominated by the update of the priority queues [11]. The update of Δ^+ is costly but only done on a small subset of S whereas the priority queues have to be updated for the $m-p$ points.

3.2 Opening a Facility

The update of Δ^- is identical to Algorithm 2. The update of Δ^+ is very similar. As mentioned above, the contribution of p_i to Δ^+ only needs to be updated when d_{i,a_i^1} is updated, i.e., when $(o_i^1 \neq a_i^1)$. However, in this case the contribution of p_i to a given Δ_k^+ is reduced either partially or completely since a node is being opened. Line 10 refers to the case where p_i remains as a contributor. In this case we just update its contribution by taking into account that a_i^1 is $d_{i,o_i^1} - d_{i,a_i^1}$ closer than d_{i,o_i^1}. In Line 11 we remove the contribution of p_i completely. Finally, in Line 12, W and C are updated accordingly.

4 A New Incremental Algorithm

The incremental algorithm presented in the previous section is dominated by the $O(m \log(p))$ cost of updating the priority queues. In practice very few points of S are likely to have a new closest or second closest facility. The left part of Figure 2 shows an example of opening a new facility p_j. Facilities are indicated by plain circles and points by crosses. The points for which p_j is the new closest facility are shown in squares whereas the points for which p_j is the new second closest facility are shown in circles. Only a very small number of points of the

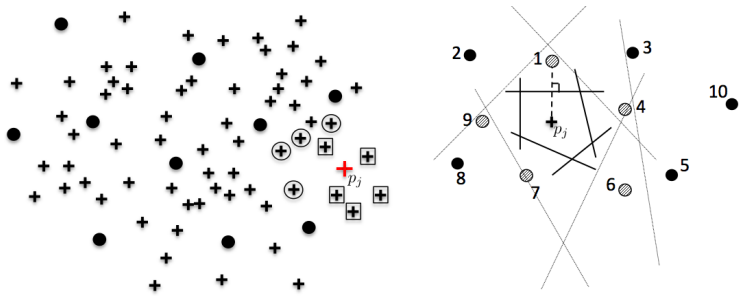


Fig. 2. Example of opening a facility p_j on left. Facilities are shown as plain circles, points as crosses and the points having p_j as their closest (resp. second closest) facility are shown in a square (resp. a circle). Example of the Voronoi cell of p_j ($\mathcal{V}(p_j)$) on right. The boundary of the cell is indicated by the dashed nodes so $\mathcal{B}(p_j) = \{1, 4, 6, 7, 9\}$.

m points of S are affected. In this paper we focus on approaches that do not maintain the priority queues Q_i. The set of points for which a_i^1 and a_i^2 need to be maintained, is computed directly using computational geometry techniques.

We begin with a simple approach. We define the radius r_j of a facility j as the maximum distance between the facility and any of its points that it covers. The radius of each facility can be maintained easily. If a new facility j is opened then the closest and the second closest of only those points i that are within the reach of $\max_{j \in W}(r_j)$ may change. Using the sorted list of nodes i by increasing distance from node j, we only have to iterate over those points i for which $d_{i,j} \leq \max_{j \in W}(r_j)$ rather the complete set S.

This approach already takes advantage of Euclidean distance and we will see below how the space and time complexities of Algorithms 2 and 3 can be improved by exploiting computational geometry techniques. Closest point problems are common in computational geometry [14, 3]. A strongly related work is [13], which relies on triangulation to speed up the PAM algorithm but does not present complexity results and ignores the optimal incremental schemes that have been developed to improve PAM [11, 16, 15, 1]. A more relevant reference is [7] which proposes to improve the k-means algorithm by using geometric reasoning based on kd-trees to speed-up the allocation of each point to its closest cluster. Our work is specific to the p-median/warehouse location problem rather than k-means and the proposed method tries to build upon known optimal incremental algorithms by improving them in the context of Euclidean distances.

4.1 The Closest Points to a New Facility

Firstly we focus on updating a_i^1 when opening a new facility. The question we would like to answer efficiently is: *determine the points in S which are closer to a given point p_j (the facility we would like to open) than to any other points of a given specific set (the facilities W).* This set is precisely characterized by

the *Voronoi cell* [3] of p_j regarding W denoted $\mathcal{V}(p_j)$. A point q lies in $\mathcal{V}(p_j)$ if and only if $dist(q, p_j) < dist(q, p_k)$ for all $p_k \in W$. The right part of Figure 2 shows how a Voronoi cell is built. For any two points p_j and p_k we can define the bisector (see [3], Chapter 7) as the perpendicular bisector of the line segment $\overline{p_j p_k}$. This bisector splits the plane into two half-planes: one containing the point p_j and another containing the point p_k. $\mathcal{V}(p_j)$ can be seen as the intersection of the half-planes containing p_j obtained by bisecting $\overline{p_j p_k}$ for all $p_k \in W$.

Definition 2 (Boundary of p_j). *The boundary of p_j, $\mathcal{B}(p_j)$, is the set of facilities p_k such that the bisector of $\overline{p_j p_k}$ coincides with one of the line segments of the Voronoi cell of p_j.*

Computing one Voronoi cell is based on computing the intersection of p half-planes which can be done in $O(p\,log(p))$ [3]. This, however, does not give us the actual points of S contained in the cell. We propose two approaches to compute the actual points of S, the first one is very simple but requires $O(m^2)$ space while the second one remains in $O(m)$ space.

Approach based on the radius. The first approach does not require any special data structure. It is based on the upper bound on the distance between the newly opened facility j and the nodes which will have j as their facility. The Voronoi cell of p_j is a convex polygon which is associated with a set of (corner) points. The minimum and the maximum distances between j and any of the corner points of the Voronoi cell is denoted by r_{min} and r_{max} respectively. Any point whose distance from j is less than r_{min} will definitely have j as its new facility. Any point whose distance from j is more than r_{max} will not be affected by the new facility. Any point whose distance from j is between r_{max} and r_{min} could possibly be affected. Therefore one has to iterate over all the points i whose distance from j is less than or equal to r_{max}. This is easy if we have access to the points sorted in the increasing distance from j which requires $O(m^2)$ space.

Approach based on a kd-tree. The second approach is based on the use of a common data structure in computational geometry, namely, a kd-tree [2]. A kd-tree for two dimensional data points is a binary tree data structure where at each node the space is partitioned horizontally or vertically. The two children of each node correspond to two regions of the space.

The kd-tree is built once and contains the points of S, the subdivision of the space is made alternatively on the x and y coordinates of a point of S and continues until all regions contain no more than one point. A balanced kd-tree (a tree such that each leaf node is more or less at the same distance to the root) can be easily built by inserting the points in a specific order, simply by choosing at each step the median of the points sorted on the corresponding coordinate. In a balanced kd-tree, obtaining one point of S contained in a rectangular area (a range query parallel to the axis used for partitioning the space in the kd-tree) can be done in $O(\sqrt{m})$ and finding all points of S contained in a rectangular area costs $O(\sqrt{m}+k)$, where k is the number of points in the corresponding area [3].

The tree is traversed as long as the region of a node intersects the area. When a leaf is reached, the corresponding point is added in the list of points to return. Similarly, when the area fully contains the region of a node, all points contained in the subtree rooted at this node are added in the answer to the query.

When the area of the Voronoi cell is not a rectangle but an arbitrary convex polygone, checking the intersection with the rectangular region of a node can be done in $O(h)$ time where h is the size of the boundary of the cell, i.e., $h = |\mathcal{B}(p_j)|$. However, in this latter case, the $O(\sqrt{m})$ is not guaranteed. In order to express the complexity, we consider the enclosing rectangle of the Voronoi cell as the query. Let k be the number of points in the cell. The Voronoi cell can be enclosed in a rectangle containing $k^{'}$ points ($k^{'} > k$) in which case the overall complexity is $O(plog(p) + \sqrt{m} + k^{'})$. In practice we apply the algorithm using the Voronoi cell itself to obtain more pruning in the tree.

4.2 Updating the Two Closest Points when Opening a Facility

We now focus on updating a_i^1 and a_i^2 when opening a new facility. We extend the previous idea to find the set of points that have either a new closest or a new second closest facility. The question we would like to answer efficiently is: *determine the points in S for which a given point, p_j (the facility we would like to open), is one of their two closest neighbors regarding a given specific set (the facilities W).* Determining such a set exactly is slightly harder since the points of the set may not necessarily be enclosed in a convex polygon. Characterizing such a set involves the computation of the Voronoi cell of each facility of $\mathcal{B}(j)$, which will increase the complexity.

We generalize the previous ideas so that the same scheme applies by replacing the concept of Voronoi cell with a set $\mathcal{V}^{'}(p_j)$ containing the set of points q for which p_j is closer than their second closest neighbor in W. In order to do so we suggest a simple convex approximation based on the concept of *Extended Voronoi cell*.

Definition 3 (Extended Voronoi cell). *Given a point p_j, the extended Voronoi cell $\mathcal{V}_2(p_j)$ is defined as the Voronoi cell of p_j associated with the set of facilities $W - \mathcal{B}(p_j)$.*

Figure 3 illustrates an extended Voronoi cell. Similarly the concept of boundary can be extended and we will denote $\mathcal{B}_2(p_j)$ the boundary of the extended Voronoi cell of p_j.

Lemma 1 ($\mathcal{V}^{'}(p_j) \subseteq \mathcal{V}_2(p_j)$)**.**

Proof. Consider a point q outside of $\mathcal{V}_2(p_j)$. q is closer to a facility $p_k \in W - \mathcal{B}(p_j)$ than to p_j because $\mathcal{V}_2(p_j)$ is the Voronoi cell regarding $W - \mathcal{B}(p_j)$. q is also necessarily closer to a point of $\mathcal{B}(p_j)$ than to p_j since q does not belong to $\mathcal{V}(p_j)$. Thus p_j cannot be one of the two closest neighbors of q. □

Notice that $\mathcal{V}^{'}(p_j) \neq \mathcal{V}_2(p_j)$. For example in Figure 3 (left), the area paved with squares within $\mathcal{V}_2(p_j)$ contains points that are closer to 4 and 6 than to p_j.

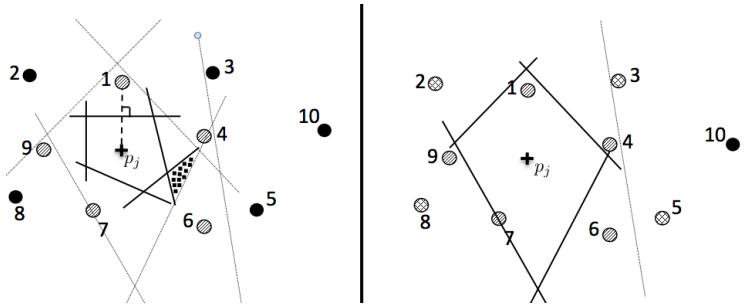


Fig. 3. On the left : Example of $\mathcal{V}(p_j)$ the Voronoi cell of p_j, the boundary of cell is defined by the dashed nodes so $\mathcal{B}(p_j) = \{1, 4, 6, 7, 9\}$. On the right: Example of $\mathcal{V}_2(p_j)$ the extended Voronoi cell of p_j whose boundary $\mathcal{B}_2(p_j) = \{2, 3, 5, 8\}$.

4.3 Updating the Two Closest Points when Closing a Facility

We consider how to update a_i^1 and a_i^2 when closing a facility p_j. Similar to the previous case, the set of points that have p_j as their closest or second closest facility can be computed using the extended Voronoi cell of p_j. In this case however we can assume that we maintain the set of points connected to p_j in a dedicated data structure, e.g. a list. When closing p_j, the closest or second closest facility of these points has to be updated. A simple solution would be to iterate over the current opened facilities W to find the two closest. Alternatively, this is exactly a 2-nearest neighbors problem. One seeks to quickly identify the two nearest neighbors in W of the points that were connected to p_j. The k-nearest neighbors is a classic problem in machine learning [2] and efficient implementations rely on the use of kd-trees [2, 4]. Assuming that we maintain a kd-tree for the set W, finding the two nearest neighbors of a given point can be done efficiently in a balanced kd-tree. The worst-case complexity remains $O(p)$ as it is easy to construct examples where all the leaves of the tree will have to be checked. The complexity analysis presented in [4] reports that the expected number of nodes inspected in the tree is in $O(log(p))$.

4.4 Updating Algorithms 2 and 3

The complexity reported for the following algorithms does not include the complexity due to maintaining Δ^- and Δ^+ which is optimal [11] and linear in the number of changes of closest or second closest. We introduce three additional data structures:

1. S_j, corresponding to the list of nodes for which facility p_j is either the closest or second closest facility.
2. KW is a kd-tree of the set W of facilities. KW is therefore dynamic and must be updated when closing/opening facilities.
3. KS is a kd-tree of the set S of nodes. KS is static and pre-computed initially.

Algorithm 4 OpenFacility2(p_j)

```
1.      compute V_2(p_j)
2.      compute the set of points S_2 in V_2(p_j) using the kd-tree KS of S
3.      For each p_i ∈ S_2 do
4.        o_i^1 = a_i^1 , o_i^2 = a_i^2
5.        a_i^1 = arg min_{p_k ∈ {o_i^1, p_j}} d_{i,k} , a_i^2 = arg min_{p_k ∈ {o_i^1, o_i^2, p_j} − {a_i^1}} d_{i,k}
6.        If (o_i^1 ≠ a_i^1 ∨ o_i^2 ≠ a_i^2) S_{p_j} = S_{p_j} ∪ {p_i}, S_{o_i^2} = S_{o_i^2} − {p_i}
7 → 14.   identical to Lines 5-12 Algorithm 3
15.     add p_j in the kd-tree KW of the facilities
```

Algorithm 4 is the new version of Algorithm 3 taking advantage of the ideas based on computational geometry. The extended Voronoi cell of the facility opened is computed first (Line 1) and the points contained in the cell (S_2) are extracted using the kd-tree KS of S (Line 2). The loop over all the points of S is replaced by a loop over the points contained in S_2. The closest or second-closest facility of p_i might now be p_j but this update takes constant time; so does the update of the S_j data structure, which is useful for Algorithm 5 when closing a facility. Finally, the incremental maintenance of Δ^- and Δ^+ remain unchanged.

Line 1 takes $O(plog(p))$, Line 2 (assuming we are using the enclosing rectangle of the Voronoi cell) takes $O(\sqrt{m}+k^{'})$. $k^{'}$ is the number of points in the enclosing rectangle so it is greater than the number of points contained in the cell ($k^{'} \geq k$). Finally Line 5 is performed in $O(1)$ and the update of KW (line 15) is done in $O(log(p))$. The complexity of Algorithm 4 is $O(plog(p) + \sqrt{m} + k^{'})$. We recall that the complexity of the previous incremental algorithm is dominated by the $O(mlog(p))$ factor which involves examining systematically all the m points. Algorithm 4 does not have this drawback as m does not appear directly in the complexity but only in a worst case where $k^{'} = m$. In practice, we expect $k^{'}$ to be much smaller than m.

Algorithm 5 CloseFacility2(p_j)

```
1.      remove p_j from KW
2.      For each p_i ∈ S_j do
3.        o_i^1 = a_i^1 , o_i^2 = a_i^2
4.        update a_i^1, a_i^2 using a 2-nearest neighbors search in KW
5.        S_{a_i^2} = S_{a_i^2} ∪ {p_i}
6 → 13.   identical to Lines 5-12 of Algorithm 2
14.     S_j = ∅
```

Similarly, Algorithm 5 is the new version of Algorithm 2. The list S_j is used to iterate over the points which had p_j as their closest or second closest. The

only difference is that the update of the a_i^1 and a_i^2 is done by using the kd-tree of the facilities KW since we no longer maintain the priority queues. The worst-case complexity of the nearest neighbors search in a balanced kd-tree is $O(p)$ but its expected complexity is $O(log(p))$. Note that KW has to be re-balanced from time to time to guarantee this complexity. The update of the kd-tree KW is done in Line 1 and takes $O(log(p))$ so that the overall expected complexity for closing a facility is $O(klog(p))$ with $|S_j| = k$.

5 Time and Space Complexities

We can distinguish three different approaches:

- BL (Base Line): the approach proposed in [11] corresponding to Algorithms 2 and 3 that iterates over all points, updates the priority queues as well as the two closest facilities of each point when needed.
- LIBL (Less Incremental Base Line): this approach is a simple modification of the BL that does not use geometry. It simply ignores the priority queues in Algorithm 3 as the two closest facilities can be updated in constant time when opening a facility and it is based on Algorithm 5 for closing a facility but does not use the kd-tree KW to update the two closest facilities. It simply iterates over W to update the two closest facilities.
- GEO (Geometric): the new approach proposed based on Algorithm 4 and 5.

We summarize the complexity of the three approaches in Tables 1 and 2 where m is the number of nodes, p the number of facilities, k the number of nodes which have p_j as a closest or second closest, and $k^{'}$ is an upper bound on k useful to express the complexity (it is the number of points contained in the enclosing rectangle of the extended Voronoi cell) as we still have $k^{'} \leq m$; ignoring the linear update of Δ^- and Δ^+ which is identical in the three cases.

The space complexity is dominated by the pre-computed lists of points sorted by distance (refered as *proximity lists*) from any other points (Line 9 of Algorithms 3 and 2) which requires $O(m^2)$ space. This is an issue for solving large problems. It can be overcomed in the case of GEO as it is simply a query in the kd-tree KS where we seek all the points contained in a circle, centered in p_i, of radius d_{i,a_i^1} (Algorithm 2) or d_{i,o_i^1} (Algorithm 3). Various mechanisms can be explored for BL and LIBL such as storing only a fixed limited number of points in the lists and relying in an $O(m)$ search in the worst case but we loose the optimal update of Δ^- and Δ^+. So overall the proximity lists require $O(m^2)$ space, the priority queues are consuming $O(mp)$, the various data structures a,Δ and

Table 1. Summary of time complexities of the different schemes

operation	BL	LIBL	GEO
open a facility p_j	$O(mlog(p))$	$O(m)$	$O(plog(p) + \sqrt{m} + k^{'})$
close a facility p_j	$O(mlog(p))$	$O(kp)$	*expected* : $O(klog(p))$, worst-case: $O(kp)$

Table 2. Summary of space complexities of the different schemes

	BL	LIBL	GEO
space complexity	$O(m^2 + mp)$	$O(m^2)$	$O(m + p)$

S are all in $O(m)$ and the two kd-trees, KS and KW, need respectively $O(m)$ and $O(p)$ space. Notice that GEO is the only scheme that is scalable in memory with m and p, in practice.

6 Empirical Study

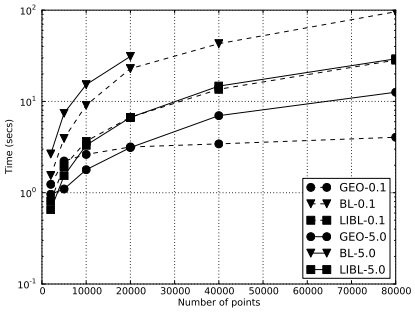
The experiments are organized into two sets.[2] In the first set of experiments we investigate the behavior of the proposed algorithms on randomly generated data. In particular we focus in these experiments on benchmarking the two algorithms openFacility and closeFacility. The algorithms compared are identical regarding the incremental update of Δ^+ and Δ^-. They only differ in the way that the closest and second closest facilities are identified and maintained. Therefore, the time reported when comparing GEO, BL and LIBL are only considering the update of the closest and second closest facilities. Instances are randomly generated by uniformly drawing points in the plane with x and y in $[0, 10000]$. For each m in {2000, 5000, 10000, 20000, 40000, 80000} ten instances were generated[3]. We performed 100 runs for each instance and each value of p. For each m we consider four values of p, which are defined as percentages of m: 0.1% ,1%, 2.5% and 5%. Each run consists of picking p random points out of the m points to define the set W, and a single random extra point: $p_j \notin W$ when the facility is being opened or $p_j \in W$ when it is being closed. In each run we measure the time needed by the different approaches: BL, LIBL and GEO. For each pair (m, p), 1000 queries are performed overall (100 on each of the 10 instances) and the cumulative times are reported.

As shown in Figures 4(a) and 4(b), the time of GEO increases moderately with m in practice. This is particularly true in the closeFacility case (see Figure 4(b)). The increase observed in the openFacility case is mainly due to the increase of points in the cell that need to be processed as m increases. In Figures 4(c) and 4(d), we also observed that LIBL does not seem to be significantly affected by the size of p, which suggests that the advantage of GEO over LIBL reduces when p/m increases. BL cannot scale any further because of memory issues. Notice that the priority lists are consuming $O(mp)$ space, which becomes too big when $m = 80000$.

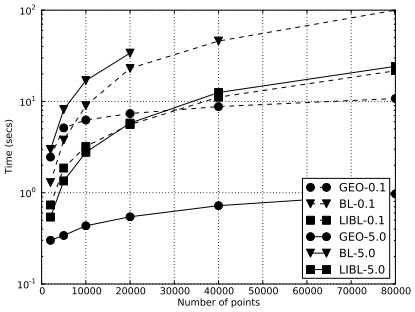
In the second set of experiments we considered eight SCP cases. Table 3 shows the parameters of each case. Half of the cases correspond to real cases

[2] Experiments were run on Linux 2.6.25 x64 on a Dual Quad Core Xeon CPU machine with overall 11.76 GB of RAM and processor speed of 2.66GH.

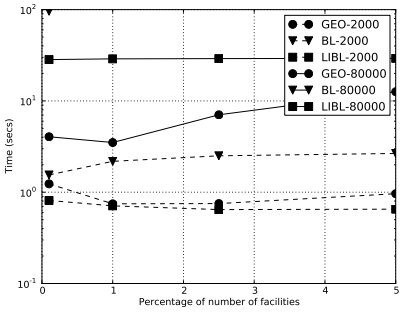
[3] The large sizes considered would be more relevant for p-median problems than for warehouse location problems.



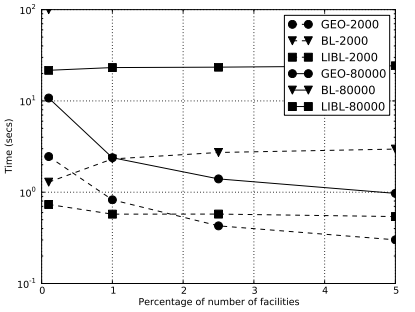
(a) Open - p fixed (0.1 and 5).



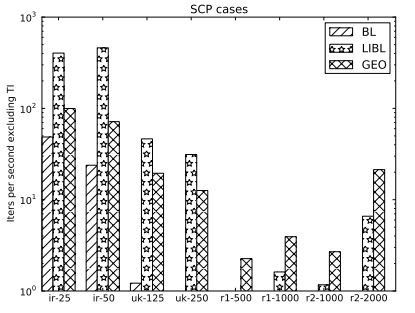
(b) Close - p fixed (0.1 and 5).



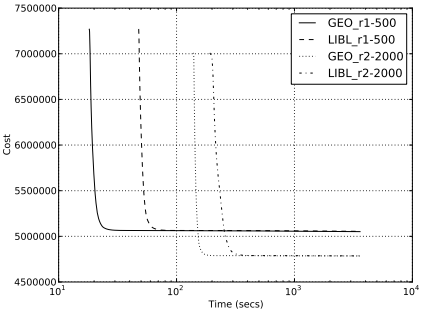
(c) Open - m fixed (2000 and 80000).



(d) Close - m fixed (2000 and 80000).



(e) Number of iterations per second.



(f) Evolution of the quality of the solution.

Fig. 4. A summary of the results from our empirical evaluation

coming from network operators in Ireland and UK. The second half corresponds to cases that were randomly generated. In the real case there is only one instance. In the random cases 10 instances were generated. For each instance we had 10 runs. The values reported in each case correspond to the average on all the instances over all runs. For each run we considered a threshold of 3600

Table 3. SCP cases

	ir-25	ir-50	uk-125	uk-250	r1-500	r1-1000	r2-1000	r2-2000
m	1100	1100	5390	5390	20000	20000	40000	40000
p	25	50	125	250	500	1000	1000	2000
source	real	real	real	real	random	random	random	random

seconds. We observed the approaches spend a significant amount of time in the initialization of the data structures which takes takes place each time we restart our current implementation. We restart if after a specified number of iterations no improvement of the best known solution has been seen. In order to have a fair measure of the speed (i.e., number of iteration per second) of each approach, we discounted the initialization time when computing the speed. Figure 4(e) shows the comparison of the approaches with respect to the speed. We can see that, even though LIBL performs quite well in the medium-size cases, GEO is the faster approach when m is big. When looking at the evolution of the quality of the solution we observe that the approaches tend to converge quite rapidly. In Figure 4(f) we only show random cases since the situation of the real cases is even more extreme. The difference observed between the approaches is mostly due to the time they spend in finding the first solution, which is basically the time spent in the first initialization.

7 Conclusion

We have presented a novel approach to achieve efficient and incremental evaluation of the neighborhood of local search algorithms for facility location problems. These ideas apply when the cost for allocating a client to a facility is proportional to the Euclidean distance. We showed how to use computational geometry to efficiently maintain the closest or second closest client to a facility. We also showed how this can be integrated within existing state of the art local search techniques for this class of problems. Any neighborhood involving the maintenance of the two closest points could benefit from these ideas and the techniques presented make sense for a constraint-based local search framework where this type of incrementality is needed for spatial location problems. We presented empirical results that demosntrated the utility of our approach[4].

Many improvements are possible as computational geometry is a very rich and active domain. For example, the use of range trees [3] instead of kd-trees would lead to a $O(log^2(m)+k)$ complexity (instead of $O(\sqrt{(m)}+k)$ for kd-tree) for a small increase of the space complexity to $O(mlog(m))$ (instead of $O(m)$ for the kd-tree). We will explore the generalization of this work to other metric distances or general distance matrices.

[4] Our focus has been on local search approaches to SCP. However, we are certainly aware of heuristic approaches to tackle SCP [12]. We are currently considering these types of approaches too.

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